Astronomical papers prepared for the use of the American ...

United States Naval Observatory. Nautical ...

ASTRONOMICAL PAPERS

PREPARED FOR THE USE OF THE

AMERICAN EPHEMERIS AND NAUTICAL ALMANAC

PUBLISHED BY AUTHORITY OF CONGRESS

BUREAU OF EQUIPMENT
NAVY DEPARTMENT

VOL. V

WASHINGTON
GOVERNMENT PRINTING OFFICE
1895

CONTENTS.

A DEVELOPMENT

OF THE

PERTURBATIVE FUNCTION

IN

COSINES OF MULTIPLES OF THE MEAN ANOMALIES AND OF ANGLES BETWEEN THE PERIHELIA AND COMMON NODE AND IN POWERS OF THE ECCENTRICITIES AND MUTUAL INCLINATION.

By SIMON NEWCOMB.

CONTENTS OF PART I.

A DEVELOPMENT OF THE PERTURBATIVE FUNCTION IN POWERS OF THE ECCENTRICITIES AND MUTUAL INCLINATION AND IN COSINES OF MULTIPLES OF THE ARGUMENTS.

I.

Preliminary.

The following development of the perturbative function was made and nearly completed about 1880, with the intention of using it in the computation of the periodic perturbations of the four inner planets. It was, however, not made use of for that purpose, owing to a belief that a development depending on the eccentric anomalies would be more convenient. But in researches of any sort having for their object the investigation of laws of change in the planetary system through long intervals of time, a development of the present form is the only one that can be used, because it is the only one in which the function appears as an analytic function of the elements.

I may add in this connection that in my opinion the computation of the periodic perturbations of the planets generally, for any one century, can be made with more facility by the CAUCHY-HANSEN method, as developed by HANSEN in the first part of his *Auseinandersetzung*, than by any method of development in powers of the eccentricities and inclinations. In adopting the latter method, I was influenced largely by the example of HANSEN himself, who, in his last work on the subject of the perturbations of Jupiter, used it as the basis of his investigation. It is also to be remarked that in the problem of determining the numerical coefficient of any one term of high order, the analytic development offers the advantage of enabling us to compute this coefficient independently of the others, while, in the numerical method, the determination of such a coefficient requires us to go through a considerable part of the work of computing all the others. This circumstance, and the consideration that the analytic development is the only one which can lead to results of importance in the general theory of planetary motion, have led to the completion of the work and the publication of the present paper.

The method in which the development is effected is set forth in the *American Journal of Mathematics*, Vol. III. The more complete exposition given in these *Astronomical Papers*, Vol. III, Part I, although there applied only to a development in cosines of multiples of the eccentric anomalies, is founded on substantially the same principles, and the work is effected by a similar series of operations. I therefore deem it sufficient in the present connection to present an explanation of the processes without detailed demonstration. It seems to me that the curious theorem by which we are enabled to pass at once from the separate developments in powers of e and e' to the general development may be susceptible of extension to other cases in celestial mechanics.

The fundamental quantities which enter into the development are the well known functions $b_n^{(i)}$, and their derivatives, the quantity $b_n^{(i)}$ being a function of α defined as the coefficient of $\cos iL$ in the development of

$$(1-2\alpha\cos L+\alpha^2)^{-\frac{n}{2}}=\tfrac{1}{2}\sum_{i=-\infty}^{i=+\infty} b_n^{(i)}\cos iL \qquad (1)$$

The latest and most complete development founded on these functions is that given by LeVerrier in his *Annales de l'Observatoire de Paris*, Vol. I. The present development differs from that of LeVerrier principally in that, instead of the successive derivatives of b as to α, I use the derivatives as to the logarithm of α.

The form also differs from that of LeVerrier in that, for the sake of condensation, I have included all the terms depending on the mutual inclination of the orbits in the values of the quantities represented by A, B, C, etc., defining these symbols as the coefficients of the cosines of the angles which remain in the development when the eccentricities vanish.

I use the symbols D and D′ to represent derivatives as to the logarithms of the mean distances of the inner and outer planets respectively. Since each term of R is of the degree -1 in these mean distances it follows that we have symbolically

$$D+D'+1=0$$

when these symbols are considered as operators upon any term of R. By means of this equation, therefore, the derivatives as to $\log a'$ may be replaced by those as to $\log a$.

As in researches relating to this subject, by others, it has been customary to use the derivatives as to α or a simply, the equations for changing one class into the other are here given for convenient reference:

$$
\begin{aligned}
D &= \alpha D_\alpha \\
D^2 &= \alpha D_\alpha + \alpha D^2_\alpha \\
D^3 &= \alpha D_\alpha + 3\,\alpha^2 D^2_\alpha + \alpha^3 D^3_\alpha \\
D^4 &= \alpha D_\alpha + 7\,\alpha^2 D^2_\alpha + 6\,\alpha^3 D^3_\alpha + \alpha^4 D^4_\alpha \\
D^5 &= \alpha D_\alpha + 15\,\alpha^2 D^2_\alpha + 25\,\alpha^3 D^3_\alpha + 10\,\alpha^4 D^4_\alpha + \alpha^5 D^5_\alpha \\
D^6 &= \alpha D_\alpha + 31\,\alpha^2 D^2_\alpha + 90\,\alpha^3 D^3_\alpha + 65\,\alpha^4 D^4_\alpha + 15\,\alpha^5 D^5_\alpha + \alpha^6 D^6_\alpha \\
D^7 &= \alpha D_\alpha + 63\,\alpha^2 D^2_\alpha + 301\,\alpha^3 D^3_\alpha + 350\,\alpha^4 D^4_\alpha + 140\,\alpha^5 D^5_\alpha + 21\,\alpha^6 D^6_\alpha + \alpha D^7_\alpha \\
D^8 &= \alpha D_\alpha + 127\,\alpha^2 D^2_\alpha + 966\,\alpha^3 D^3_\alpha + 1701\,\alpha^4 D^4_\alpha + 1050\,\alpha^5 D^5_\alpha + 266\,\alpha^6 D^6_\alpha + 28\,\alpha^7 D^7_\alpha + \alpha^8 D^8_\alpha
\end{aligned}
$$

$$
\begin{aligned}
\alpha D_\alpha &= D \\
\alpha^2 D^2_\alpha &= -D + D^2 \\
\alpha^3 D^3_\alpha &= 2D - 3D^2 + D^3 \\
\alpha^4 D^4_\alpha &= -6D + 11D^2 - 6D^3 + D^4 \\
\alpha^5 D^5_\alpha &= 24D - 50D^2 + 35D^3 - 10D^4 + D^5 \\
\alpha^6 D^6_\alpha &= -120D + 274D^2 - 225D^3 + 85D^4 - 15D^5 + D^6 \\
\alpha^7 D^7_\alpha &= 720D - 1764D^2 + 1624D^3 - 735D^4 + 175D^5 - 21D^6 + D^7 \\
\alpha^8 D^8_\alpha &= -5040D + 13068D^2 - 13132D^3 + 6769D^4 - 1960D^5 + 322D^6 - 28D^7 + D^8
\end{aligned}
$$

Since the development of R is almost identical with that of the reciprocal of the mutual distance of the planets, it will suffice, in the present paper, to develop Δ^{-1}, where we put Δ = the linear distance of the planets from each other. Δ is therefore a homogeneous symmetric function of r and r', the radii vectores, of the degree -1.

II.

Development in powers of the mutual inclination.

Firstly, taking the orbits as circular, I put

λ, λ', the distances of the planets from their common node;
γ, the mutual inclination of the planes of their orbits.

Then I assume the function

$$\Delta_0^{-1} = \{a^2 - 2\,aa'\,(\cos\lambda\cos\lambda' + \cos\gamma\sin\lambda\sin\lambda') + a'^2\}^{-\frac{1}{2}}$$
$$= \frac{1}{a'}\left\{1 - 2\alpha\,(\cos\lambda\cos\lambda' + \cos\gamma\sin\lambda\sin\lambda') + \alpha^2\right\}^{-\frac{1}{2}}$$

to be developed in the form

$$\Delta_0^{-1} = \sum_{i=-\infty}^{i=+\infty}\begin{cases}\frac{1}{2}\,A_i\cos(i\,\lambda' - i\,\lambda)\\ +\sigma^2\,B_i\cos\left((i+1)\,\lambda' - (i-1)\,\lambda\right)\\ +\sigma^4\,C_i\cos\left((i+2)\,\lambda' - (i-2)\,\lambda\right)\\ +\ \ldots\ldots\ldots\ldots\end{cases} \qquad (2)$$

where

$$\sigma = \sin\frac{1}{2}\gamma$$

Into the methods of effecting this part of the development I do not enter. I only assume that the coefficients A_i, B_i, C_i, etc., are known functions of a, a' and σ^2, so that each of the quantities

$$a'\,A_i,\ a'\,B_i,\ \text{etc.},$$

is a function of α and σ^2, we having put

$$\alpha = \frac{a}{a'}$$

Among the methods of effecting the development is that in a series proceeding according to the ascending powers of σ. The values of the coefficients, when the development is effected in this way, are shown up to σ^8 in Exhibit IV.

But we are not necessarily confined to a development in powers of σ. TISSERAND has shown how the coefficients may be obtained when the series in powers of σ is not sufficiently convergent. The question of convergence is one of such importance as to demand a special investigation.

III.

Convergence of the development in powers of the inclination.

The question may arise whether, if the development according to powers of σ were continued far enough, the series might not cease to converge after a certain point, and thus come under the class treated by POINCARÉ, which, apparently convergent when only a few terms are considered, ultimately diverge. This question must be settled, because there can be no certainty that the first terms of such a series are really an approximation to the functions which the series is supposed to represent, within the limits of error of the smallest terms.

There is no difficulty in showing that the coefficients of each power of σ in the values of $a' A_i$, $a' B_i$, etc., are developable in a convergent series in powers of α whenever $\alpha < 1$, and are therefore finite and completely defined functions of α. For each of these coefficients is a linear function of GAUSSIAN hypergeometric series which satisfy the conditions of convergence. But it is still possible that the successive values of these coefficients may constantly increase so as to render the development in powers of σ divergent. What we have to do is to find, if possible, definite limits within which all the series which give the values of A_i, B_i, etc., are unconditionally convergent. To reach a satisfactory conclusion we should consider the development in its most general form when the ellipticity of the orbits is included. We therefore start with the well-known expression

$$\Delta^{-1} = (r^2 - 2 r r' \cos V + r'^2)^{-\frac{1}{2}}$$

We have, using the notation of previous papers of this series,

$$\begin{aligned} \cos V &= \cos v \cos v' + \cos \gamma \sin v \sin v' \\ &= \cos (v - v') - 2 \sigma^2 \sin v \sin v' \end{aligned}$$

and we put, assuming $r' > r$

$$\rho = \frac{r}{r'}$$

Then

$$\Delta^{-1} = \frac{1}{r'} (1 - 2 \rho \cos V + \rho^2)^{-\frac{1}{2}}$$

If we put

$$\begin{aligned} f &= 1 - 2 \rho \cos (v - v') + \rho^2 \\ q &= 4 \rho \sigma^2 \sin v \sin v' \end{aligned}$$

we shall have to develop $(f + q)^{\frac{1}{2}}$ in powers of q, and hence of σ. In this development the coefficient of q^μ is

$$(-1)^\mu \frac{1 \cdot 3 \cdot 5 \ldots 2 \mu - 1}{2 \cdot 4 \cdot 6 \ldots 2 \mu} f^{-\mu - \frac{1}{2}}$$

This coefficient of $f^{-\mu-\frac{1}{2}}$ is always less than unity, and if we put

$$n = 2 \mu + 1$$

we shall have

$$f^{-\mu-\frac{1}{2}}=f^{-\frac{n}{2}}=\frac{1}{2}\Sigma\,\beta_n^{(i)}\cos i\,(\mathrm{v}-\mathrm{v}')$$

$$(i=-\infty\quad .\ .\ -2,-1,0,1,2\quad .\ .\ .\ \infty)$$

The coefficients β are well known to be positive and developable in a convergent hypergeometric series in powers of ρ whenever $\rho<1$. Moreover, by putting $\mathrm{v}-\mathrm{v}'=0$ we find for the sum of these coefficients corresponding to any value of n

$$\frac{1}{2}\beta_n^{(0)}+\beta_n^{(1)}+\beta_n^{(2)}+\quad .\ .\ .\quad=(1-\rho)^{-n}$$

The coefficients β being all positive, this expression is a superior limit for the value of any one of them, and the whole expression forms a convergent series.

The terms of the general development being alternately positive and negative, it will be convergent if these terms approach the limit zero for increasing values of μ. For any value of μ the general term, omitting the numerical coefficient, may be written

$$q^\mu f^{-\mu-\frac{1}{2}}=\frac{1}{2}q^\mu\,\Sigma\,\beta_n^{(i)}\cos i\,(\mathrm{v}-\mathrm{v}')$$

The limiting value of the factor under the sign σ being $(1-\rho)^{-n}$ it follows that the term in question will, when μ and n increase indefinitely, converge toward zero for any special values of q and ρ for which

$$\frac{q^\mu}{(1-\rho)^n}=\frac{(4\,\rho\,\sigma^2\sin\mathrm{v}\,\sin\mathrm{v}')^\mu}{(1-\rho)^{2\mu+1}}\tag{A}$$

so converges. Since $\sin\mathrm{v}$ and $\sin\mathrm{v}'$ never exceed unity a sufficient condition of this convergence is

$$\frac{4\,\rho\,\sigma^2}{(1-\rho)^2}<1$$

This gives, for the mutual inclination γ, the condition

$$\sigma=\sin\frac{1}{2}\gamma<\frac{1-\rho}{2\sqrt{\rho}}=\frac{r'-r}{2\sqrt{r\,r'}}\tag{B}$$

This condition is fulfilled for every pair of the eight major planets for all values of r and r' which they can assume. We therefore conclude:—

In the case of the eight major planets the development of $\varDelta^{-1}$ in powers of the mutual inclination is absolutely convergent for every pair of positions which any two of these planets can assume.

The question is still open whether this proposition may not fail when we consider all the terms which enter into the development in a periodic series; that is to say, whether the second member of (A) may not, when thus developed, contain positive and negative terms the absolute value of whose coefficients exceed the maximum value of the expression, which, however, partially annul each other. As μ increases, the μth root of (A), omitting σ^2, approaches the limit

$$\frac{4\,r\sin\mathrm{v}\;\;r'\sin\mathrm{v}'}{(r'-r)^2}\tag{C}$$

Putting λ for the mean value of v this expression may be developed in a periodic series, each term of which is of the form

$$h_{i,i',j,j'} \cos(i\lambda + i'\lambda' + jg + j'g')$$

g and g' being the mean anomalies. Now, CAUCHY has shown that for all values of the eccentricities which any of the planets have, the development of v is convergent. We conclude that the same is true for the above expression. But the FOURIER integral expression for the coefficients h shows that the modulus of every such coefficient must be less than the maximum value of the expression. This maximum corresponds to

$$r' = a'(1 - e');$$
$$r = a\,(1 + e);$$
$$v = v' = 90^\circ$$

The development is therefore certainly convergent when

$$2\sin\frac{1}{2}\gamma < \frac{a'(1-e') - a(1+e)}{\sqrt{aa'(1-e')(1+e)}} = \frac{1 - e' - \alpha(1+e)}{\sqrt{\alpha(1-e')(1+e)}} \qquad \text{(C)}$$

In this form the condition corresponds to the most unfavorable case, namely, that in which the aphelion of the inner planet and the perihelion of the outer one are each 90° distant from the common node. In any case where (C) is not very small the expression (A) will have nearly its maximum value when

$$v = v' = \pm 90^\circ.$$

The two planets will then be in conjunction at a distance of $\pm 90^\circ$ from their common node, and their heliocentric angular distance will be γ. The condition (B) may be written

$$\frac{4rr'\sin^2\frac{1}{2}\gamma}{(r'-r)^2} < 1$$

In the position of the two planets just supposed we shall have

$$4rr'\sin^2\frac{1}{2}\gamma = \Delta^2 - (r'-r)^2$$

Δ being the linear distance of the planets. The condition may therefore be written

$$\frac{\Delta}{r'-r} < \sqrt{2}$$

This expression exceeds the secant of the angle between the inner planet and the sun, as seen from the outer planet, by a quantity which diminishes with the inclination of the orbits. We finally conclude:—

Theorem; the development will always be convergent if, both orbits being movable in their own planes, no point of the inner orbit can ever be seen from any point of the outer one at a greater heliocentric latitude above or below the plane of the outer orbit than 45°.

This condition does not mark the absolute limit of the convergence, but only assures it when fulfilled.

IV.

General form of the development.

It will be seen that the development (2) can be thrown into the general form

$$\frac{a'}{\Delta_0}=\Sigma_\nu\,\Sigma_\mu\,h_{\mu,\,\nu}\cos(\mu\,\lambda+\nu\,\lambda') \tag{3}$$

where μ and ν take all integral values, positive and negative, subject to the condition that, in any one term, both must be even or both odd. The terms in the first line of (2) are then formed by putting

$$\begin{array}{ll} & \mu=-i;\ \nu=i; \\ \text{In the second,} & \mu=-i+1;\ \nu=i+1 \\ \text{In the third,} & \mu=-i+2;\ \nu=i+2 \\ & \quad\text{etc.,} \qquad \text{etc.} \end{array} \tag{3'}$$

It will be remarked that the quantity Δ_0^{-1} with which we are primarily concerned is of the degree -1 in the abstract quantity length. But as it is better to deal, so far as may be, with purely numerical ratios, I consider principally the development of $a'\,\Delta_0^{-1}$. Thus, while the coefficients A_i, B_i, C_i, etc., are functions of the degree -1 in the element of length, or in a and a', the products

$$a'\,A_i,\ a'\,B_i,\ a'\,C_i,\ \text{etc.,}$$

are pure numerical ratios, functions only of α and σ. The same is true of the coefficient h which I use in a general way to represent the products $a'\,A_i$, etc.

In continuing the development of the form (3), I regard the quantities $h_{\mu,\,\nu}$ as functions of log a and of log a'.

The development when the eccentricity e of the inner planet does not vanish, is then formed by substituting in (2) or (3),

$$\begin{array}{ll} \text{for } \lambda;\ \ldots\ldots & \mathrm{v}=\lambda+\mathrm{E} \\ \text{for } \log a;\ \ldots & \rho=\log r \end{array} \tag{4}$$

E being the equation of the centre. Putting g for the mean anomaly, E and $\rho-\log a$ will be the functions of e and g only, developable in powers of e and sines or cosines of multiples of g. The actual developments of v and ρ to terms of the eighth order inclusive, are shown in Exhibit I.

By making the substitution (4) in (3), any term of the latter, say

$$h\cos(\nu\,\lambda'+\mu\,\lambda)=h\cos\mathrm{N}$$

may be developed in the form

$$\sum_{j=-\infty}^{j=+\infty} h_j\cos(\nu\,\lambda'+\mu\,\lambda+j\,g)$$

In the present paper I suppose the coefficient h_j developed in powers of the eccentricity e. The lowest power of e in any term will then be $e^{\pm j}$, the sign of the expo-

nent being always so taken as to make it positive. If we put k for this exponent, any value of h_j may be developed in the form

$$h_j = e^k P_j^{(k)} + e^{k+2} P_j^{(k+2)} + e^{k+4} P_j^{(k+4)} + \ldots$$

in which the P's are functions of σ and α.

It is to be remarked that, in the case of the terms in A_i, when $\mu + \nu = 0$, the terms having negative values of j may be merged with the corresponding ones having positive values.

V.

Formation of the operators Π_j^n.

It is shown in the papers already quoted that each value of P, say $P_j^{(n)}$, may be expressed in the form

$$P_j^{(n)} = \Pi_j^{(n)} h$$

where h represents any one of the coefficients

$$a' A_i; \quad \sigma^2 a' B_i; \quad \sigma^4 a' C_i; \quad \text{etc.}$$

while $\Pi_j^{(n)}$ is an entire function of the degree n of the index μ and of the symbol D. Thus a symbolic expression of the form

$$\left((i)_0 + (i)_1 D + (i)_2 D^2 + \ldots + D^n \right) h$$

is to be interpreted as meaning

$$(i)_0 h + (i)_1 \alpha \frac{dh}{d\alpha} + (i)_2 \alpha \frac{d}{d\alpha} (\alpha \frac{dh}{d\alpha}) + \text{etc.}$$

Each value of Π may be expressed as a linear function of values of lower order in the following way: Let us put

v_i, ρ_i, the coefficients of e^i in the developments of E and of ρ, respectively, so that we have

$$E = e v_1 + e^2 v_2 + e^3 v_3 + \ldots$$
$$\rho - \log a = e \rho_1 + e^2 \rho_2 + e^3 \rho_3 + \ldots$$

Then each v_i and ρ_i will be of the form

$$i\, v_i = \frac{1}{2} \Sigma k_j^{(i)} \sin jg$$

$$i\, \rho_i = \frac{1}{2} \Sigma h_j^{(i)} \cos jg$$

$$(j = i;\ i-2;\ i-4 \ \ldots \ldots \ -i)$$

the k's and h's being rational numerical fractions, subject to the conditions

$$k_j = -k_{-j}; \quad h_j = h_{-j}$$

whose values follow at once from the developments in Exhibit I, and are given in Exhibit II. We shall then have

$$\Pi_0^0 = 1$$

and all other values of Π are found by the following general recurrent formulæ:—

$$
\begin{aligned}
2(n+1)\,\Pi^{n+1}_{n+1} = {} & (k'_1\,\mu + h'_1\,\mathrm{D})\,\Pi^n_n \\
& + (k''_2\,\mu + h''_2\,\mathrm{D})\,\Pi^{n-1}_{n-1} \\
& + (k'''_3\,\mu + h'''_3\,\mathrm{D})\,\Pi^{n-2}_{n-2} \\
& + \cdots\cdots \\
& + (k^{(n+1)}_{(n+1)}\,\mu + h^{(n+1)}_{(n+1)}\,\mathrm{D})\,\Pi^0_0
\end{aligned}
$$

In writing the remaining operators I put, for brevity

$$\mathrm{H}^i_\iota = k^{(i)}_\iota\,\mu + h^{(i)}_\iota\,\mathrm{D}$$

$$
\begin{array}{llll}
2(n+1)\,\Pi^{n+1}_{n-1} = \mathrm{H}^1_1\,\Pi^n_{n-2} & + \mathrm{H}^1_{-1}\,\Pi^n_n & & \\
\qquad + \mathrm{H}^2_2\,\Pi^{n-1}_{n-3} & + \mathrm{H}^2_0\,\Pi^{n-1}_{n-1} & & \\
\qquad + \mathrm{H}^3_3\,\Pi^{n-2}_{n-4} & + \mathrm{H}^3_1\,\Pi^{n-2}_{n-2} & & \\
\qquad + \cdots\cdots & + \cdots\cdots & & \\
\qquad + \mathrm{H}^n_n\,\Pi^1_{-1} & + \mathrm{H}^{n+1}_{n-1}\,\Pi^0_0 & & \\
2(n+1)\,\Pi^{n+1}_{n-3} = \mathrm{H}^1_1\,\Pi^n_{n-4} & + \mathrm{H}^1_{-1}\,\Pi^n_{n-2} & & \\
\qquad + \mathrm{H}^2_2\,\Pi^{n-1}_{n-5} & + \mathrm{H}^2_0\,\Pi^{n-1}_{n-3} & + \mathrm{H}^2_{-2}\,\Pi^{n-1}_{n-1} & \\
\qquad + \mathrm{H}^3_3\,\Pi^{n-2}_{n-6} & + \mathrm{H}^3_1\,\Pi^{n-2}_{n-4} & + \mathrm{H}^3_{-1}\,\Pi^{n-2}_{n-2} & \\
\qquad + \cdots & + \cdots & + \cdots\cdots & \\
\qquad + \mathrm{H}^{n-1}_{n-1}\,\Pi^2_{-2} & + \mathrm{H}^n_{n-2}\,\Pi^1_{-1} & + \mathrm{H}^{n+1}_{n-3}\,\Pi^0_0 & \\
2(n+1)\,\Pi^{n+1}_{n-5} = \mathrm{H}^1_1\,\Pi^n_{n-6} & + \mathrm{H}^1_{-1}\,\Pi^n_{n-4} & & \\
\qquad + \mathrm{H}^2_2\,\Pi^{n-1}_{n-7} & + \mathrm{H}^2_0\,\Pi^{n-1}_{n-5} & + \mathrm{H}^2_{-2}\,\Pi^{n-1}_{n-3} & \\
\qquad + \mathrm{H}^3_3\,\Pi^{n-2}_{n-8} & + \mathrm{H}^3_1\,\Pi^{n-2}_{n-6} & + \mathrm{H}^3_{-1}\,\Pi^{n-2}_{n-4} & + \mathrm{H}^3_{-3}\,\Pi^{n-2}_{n-2} \\
\qquad + \cdots & + \cdots & + \cdots & + \cdots \\
\qquad + \mathrm{H}^{n-2}_{n-2}\,\Pi^3_{-3} & + \mathrm{H}^{n-1}_{n-3}\,\Pi^2_{-2} & + \mathrm{H}^n_{n-4}\,\Pi^1_{-1} & + \mathrm{H}^{n+1}_{n-5}\,\Pi^0_0
\end{array}
$$

These formulæ have been written in such a way that the law of progression of the several operators may be seen by inspection and induction. As we diminish the suffix of Π by successive steps of two units each, the number of columns of products in the second member of the equation increases by one at each step, while the number in each column diminishes by one, in consequence of one of the factors vanishing, whenever a suffix in k, h, or H, exceeds the exponent in absolute value. Since Π^n_{-j} can be formed from Π^n_j by changing the sign of μ, it follows that by continuing the formulæ until the suffix becomes the negative of the exponent we should reproduce the equivalent of the first of the above equations except as to the sign of μ. This is the result which we reach by continuing the equations. In the last equation all the columns will be reduced to their last members, which it will be seen will then be identical with the numbers in the first equation, with the exception of the signs of the suffixes, which will all be changed.

We may also for the negative suffixes apply the following rule:

Each value of Π with j negative may be formed from the corresponding value for j positive by changing the sign of μ and of the suffixes in all the values of Π which enter into it. As an example of the formulæ we have, by putting $n=0$,

$$
\begin{aligned}
2\,\Pi^1_1 &= \mu\,k'_1 + h'_1\,\mathrm{D} = 2\,\mu - \mathrm{D} \\
2\,\Pi^1_{-1} &= \mu\,k'_{-1} + h'_{-1}\,\mathrm{D} = -2\,\mu - \mathrm{D}
\end{aligned}
$$

and then by putting $n=1$

$$4\,\Pi_2^2=(\mu\,k_1'+h_1'\,\mathrm{D})\,\Pi_1^1+(\mu\,k_2''+h_2''\,\mathrm{D})\,\Pi_0^0$$
$$=(2\,\mu-\mathrm{D})\,\Pi_1^1+\frac{5}{2}\,\mu-\frac{3}{2}\mathrm{D}$$
$$4\,\Pi_0^2=(\mu\,k_1'+h_1'\,\mathrm{D})\,\Pi_{-1}^1+(\mu\,k_{-1}'+h_{-1}'\,\mathrm{D})\,\Pi_1^1+h_0''\,\mathrm{D}\,\Pi_0^0$$
$$=(2\,\mu-\mathrm{D})\,\Pi_{-1}^1+(-2\,\mu-\mathrm{D})\,\Pi_1^1+\mathrm{D}$$

in which the values of Π_1^1 and Π_{-1}^1, already found, are to be substituted.

The system of values of Π_j^n, thus found, up to $n=8$, are given in Exhibit V, after making the substitution

$$\mu=-i$$

in order that they may represent at once the values to be used in the development. If we wish the values in terms of μ, or those for negative values of j, we have only to remark that if we put

$$\mathrm{F}\,(i)=\Pi_j^n\ \text{(as given)}$$

we shall have in terms of μ

$$\Pi_j^n\ =\mathrm{F}\,(-\mu)$$
$$\Pi_{-j}^n=\mathrm{F}\,(\mu)$$

VI.

Eccentricity of the outer orbit.

In what precedes we have considered the eccentricity of the inner orbit alone. We have now to pass to that of the outer or accented one. If we express by D′ the derivative as to log a', the coefficients of the development in powers of e' will be found from those in powers of e by replacing μ by ν and D by D′. But because Δ is of the degree -1 in a and a', we have symbolically

$$\mathrm{D}+\mathrm{D}'+1=0$$

As it will be convenient to have all the derivatives taken with respect to the same quantity, we will replace the symbol D′ by its equivalent

$$\mathrm{D}'=-1-\mathrm{D}$$

Thus the symbolic coefficients are found by replacing in each as a function of μ

$$\mathrm{D}\ \text{by}\ -(1+\mathrm{D})$$
$$\mu\ \text{by}\ \nu$$

or if we use the Π's as given

$$i\ \text{by}\ -\nu$$

Actually, instead of expressing the operators for the outer orbit as functions of ν, they are written in terms of i by simply putting

$$i=\nu$$

These operators are represented by symbols of the form $\Pi_{0,\,j}^{0,\,n}$

VII.

Combination of the two eccentricities.

Thus far we have considered the developments only when one or the other of the two eccentricities vanish. Consequently our series contain only powers of e without e', or e' without e. In the papers already quoted we have demonstrated the following theorem:

THEOREM.—*If any term of the development of* $a' \Delta^{-1}$ *in powers of* e *be*

$$e^n \Pi_j^n \, h \cos(\nu \lambda' + \mu \lambda + j g)$$

and the corresponding term in powers of e' *be*

$$e'^{n'} \Pi_{0,\,j'}^{0,\,n'} \, h \cos(\nu \lambda' + \mu \lambda + j' g')$$

then the corresponding term in the development in powers of e *and* e' *will be*

$$e^n e'^{n'} \Pi_j^n \Pi_{0,\,j'}^{0,\,n'} \, h \cos(\nu \lambda' + \mu \lambda + j' g' + j g)$$

The values of the symbolic products

$$\Pi_{j,\,j'}^{n,\,n'} = \Pi_j^n \times \Pi_{0,\,j'}^{0,\,n'}$$

are given so far as required for the development to terms of the seventh order in Exhibit V. We remark, in explanation of them, that, had we adhered to the use of our conventional indices μ and ν, then each $\Pi_{j,\,j'}^{n,\,n'}$ would be an entire function of μ and ν, as well as of D, of the degree of $n+n'$ As the formation and management of these functions would be inconvenient, we use, instead of μ and ν, their values in terms of i, as shown on pp. 11 and 14. The values of Π obtained by the substitutions are distinguished by accents, thus:

$\Pi_{j,\,j'}^{n,\,n'}$ means $\Pi_{j,\,j'}^{n,\,n'}$ when we put $\mu = -i$; $\quad \nu = i$
$\Pi'^{n,\,n'}_{j,\,j'}$ means $\Pi_{j,\,j'}^{n,\,n'}$ when we put $\mu = -i + 1$; $\nu = i + 1$
$\Pi''^{n,\,n'}_{j,\,j'}$ means $\Pi_{j,\,j'}^{n,\,n'}$ when we put $\mu = -i + 2$; $\nu = i + 2$

In the values of Π_j^n as actually written we have put $-i$ for μ. Hence, in forming the products, the substitutions to be made in the factors as written are

For Π'; in Π_j^n change i to $i - 1$; in $\Pi_{0,\,j'}^{0,\,n'}$ change i to $i + 1$
For Π''; in Π_j^n change i to $i - 2$; in $\Pi_{0,\,j'}^{0,\,n'}$ change i to $i + 2$
etc., etc.

The symbolic development, so far as is necessary to show its various terms, is given in Exhibit III.

VIII.

Controls upon the accuracy of the results.

A control upon the accuracy of all the terms, especially of the symbolic operators Π, as computed and printed, is desirable. We shall give three controls, each of which the reader can apply for himself to the printed values of such of the expressions as he may wish to test.

Control First.—When the planets are in perihelia, that is, when

$$g=0°;\ g'=0°$$

the value of $a'\,\Delta^{-1}$ should reduce to

$$\{(1-e')^2-2\,\alpha\,(1-e)\,(1-e')\cos V_0+\alpha^2(1-e)^2\}^{-\frac{1}{2}}$$

where

$$\cos V_0=\cos w\cos w'+\cos\gamma\sin w\sin w'$$

where w and w' are the distances of the perihelia from the common node.

To control the operators Π we may take the case when $e'=0$; $\gamma=0$. Then our development should reduce to that of

$$a'\,\Delta_1^{-1}=\{1-2\,\alpha\,(1-e)\cos(w'-w)+\alpha^2(1-e)^2\}^{-\frac{1}{2}}$$

which may be formed from the development

$$(1-2\,\alpha\cos(w'-w)+\alpha^2)^{-\frac{1}{2}}=\frac{1}{2}\,\Sigma\,b^{(i)}\cos i\,(w'-w)$$

by replacing

$$\alpha \text{ by } \alpha-e\,\alpha$$

or

$$\log\alpha \text{ by } \log\alpha+\log(1-e)$$

Putting for brevity

$$\mathrm{e}=-\log(1-e)$$

the development of this substitution in powers of e will be

$$a'\Delta_1^{-1}=\frac{1}{2}\,\Sigma\,\beta^{(i)}\cos i\,(w'-w)$$

where

$$\beta^{(i)}=(1-\mathrm{e}\,D+\frac{\mathrm{e}^2}{1\,.\,2}\,D^2-\frac{\mathrm{e}^3}{1\,.\,2\,.\,3}\,D^3+\quad.\quad.\quad.\quad)\,b_1^{(i)}$$

We now substitute for e its developed value

$$\mathrm{e}=e+\frac{1}{2}\,e^2+\frac{1}{3}\,e^3+\quad.\quad.\quad.$$

and arrange the result in powers of e. But it is not necessary to form the powers of e, because if we define the operator K_n by the series

$$\beta^{(i)} = (1 + K_1 e + K_2 e^2 + K_3 e^3 + \quad . \quad . \quad . \quad . \quad) \, b_1^{(i)}$$

each value of K after the first may be found by the equation

$$n \, K_n = - D \, K''_{n-1}$$

where K″ is the result of substituting D − 1 for D in K. We thus find

$$\begin{aligned}
K_1 &= - D \\
2 K_2 &= - D + D^2 \\
6 K_3 &= - 2 D + 3 D^2 - D^3 \\
24 K_4 &= - 6 D + 11 D^2 - 6 D^3 + D^4 \\
120 K_5 &= - 24 D + 50 D^2 - 35 D^3 + 10 D^4 - D^5 \\
720 K_6 &= - 120 D + 274 D^2 - 225 D^3 + 85 D^4 - 15 D^5 + D^6 \\
5040 K_7 &= - 720 D + 1764 D^2 - 1624 D^3 + 735 D^4 - 175 D^5 + 21 D^6 - D^7 \\
40320 K_8 &= - 5040 D + 13068 D^2 - 13132 D^3 + 6769 D^4 - 1960 D^5 + 322 D^6 - 28 D^7 + D^8
\end{aligned}$$

The transformation of these quantities into the corresponding ones for the outer planet is made by changing D into − 1 − D. We thus have

$$\begin{aligned}
K'_1 &= 1 + D \\
2 K'_2 &= 2 + 3 D + D^2 \\
6 K'_3 &= 6 + 11 D + 6 D^2 + D^3 \\
24 K'_4 &= 24 + 50 D + 35 D^2 + 10 D^3 + D^4 \\
120 K'_5 &= 120 + 274 D + 225 D^2 + 85 D^3 + 15 D^4 + D^5 \\
720 K'_6 &= 720 + 1764 D + 1624 D^2 + 735 D^3 + 175 D^4 + 21 D^5 + D^6 \\
5040 K'_7 &= 5040 + 13068 D + 13132 D^2 + 6769 D^3 + 1960 D^4 + 322 D^5 + 28 D^6 + D^7 \\
40320 K'_8 &= 40320 + 109584 D + 118124 D^2 + 67284 D^3 + 22449 D^4 + 4536 D^5 + 546 D^6 + 36 D^7 + D^8
\end{aligned}$$

To apply this control to the actual development of R itself we must find what the latter reduces to when we put

$$g = g' = 0; \; e' = 0$$

To make the comparison with the special development complete we must combine each term having a positive value of i with the term for which i is negative. The symbolic coefficient of $a' \, A_i \cos i \, (w - w')$ in $a' \Delta^{-1}$ will then reduce to

$$1 + e \, (\Pi_1^1 + \Pi_{-1}^1) + e^2 \, (\Pi_2^2 + \Pi_0^2 + \Pi_{-2}^2) + \text{etc.},$$

while in the special development the same coefficient reduces to $\beta^{(i)}$. We therefore have for the first control

$$\Pi_1^1+\Pi_{-1}^1=\mathrm{K}_1$$
$$\Pi_2^2+\Pi_0^2\ +\Pi_{-2}^2=\mathrm{K}_2$$
$$\Pi_3^3+\Pi_1^3\ +\Pi_{-1}^3+\Pi_{-3}^3=\mathrm{K}_3$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

in the same way

$$\Pi_{0,1}^{0,1}+\Pi_{0,-1}^{0,1}=\mathrm{K}'_1$$

etc., etc.

Control second.—The preceding operation controls only coefficients of even powers of i. For those of odd powers we examine $\frac{d\varDelta^{-1}}{dg}$ instead of $\varDelta^{-1}$. Taking, as before, what is left of the development of $\varDelta^{-1}$ when σ and e' vanish, we have, when $g=g'=0°$

$$\frac{d\varDelta^{-1}}{dg}=\frac{1}{2}\Sigma i\,\beta^{(i)}\frac{d\mathrm{v}}{dg}\sin i\,(w'-w)$$

where $\frac{d\mathrm{v}}{dg}$ is to have its value at perihelion, namely,

$$\frac{d\mathrm{v}}{dg}=\frac{\sqrt{1-e^2}}{(1-e)^2}$$

$$=1+2e+\frac{5}{2}e^2+3e^3+\frac{27}{8}e^4+\frac{15}{4}e^5+\frac{65}{16}e^6+\frac{35}{8}e^7+\frac{595}{128}e^8+\frac{315}{64}e^9+\ .\ .\ .$$

Multiplying this series, the term 1 excepted, into $\beta^{(i)}$, we find that if we put H_n for the coefficient of e^n in the expression, each term with i negative being merged into the corresponding one with i positive, so that the coefficient will take the value

$$\left(\frac{d\mathrm{v}}{dg}-1\right)i\beta^{(i)}$$

we shall have

$$\begin{array}{rllllllll}
\mathrm{H}_1= & 2 \\
2\,\mathrm{H}_2= & 5- & 4\,\mathrm{D} \\
2\,\mathrm{H}_3= & 6- & 7\,\mathrm{D}+ & 2\,\mathrm{D}^2 \\
24\,\mathrm{H}_4= & 81- & 118\,\mathrm{D}+ & 54\,\mathrm{D}^2- & 8\,\mathrm{D}^3 \\
24\,\mathrm{H}_5= & 90- & 149\,\mathrm{D}+ & 88\,\mathrm{D}^2- & 22\,\mathrm{D}^3+ & 2\,\mathrm{D}^4 \\
240\,\mathrm{H}_6= & 575- & 1791\,\mathrm{D}+ & 1240\,\mathrm{D}^2- & 410\,\mathrm{D}^3+ & 65\,\mathrm{D}^4- & 4\,\mathrm{D}^5 \\
240\,\mathrm{H}_7= & 3150- & 6226\,\mathrm{D}+ & 4853\,\mathrm{D}^2- & 1920\,\mathrm{D}^3+ & 410\,\mathrm{D}^4- & 45\,\mathrm{D}^5+ & 2\,\mathrm{D}^6 \\
40320\,\mathrm{H}_8= & 187425- & 395232\,\mathrm{D}+ & 336854\,\mathrm{D}^2- & 151984\,\mathrm{D}^3+ & 39410\,\mathrm{D}^4- & 5908\,\mathrm{D}^5+ & 476\,\mathrm{D}^6- & 16\,\mathrm{D}^7
\end{array}$$

This expression should be identical with the corresponding coefficient derived from the general development

$$\left(\frac{d\mathrm{v}}{dg}-1\right)i\,\beta^{(i)}$$

If we differentiate the general development as a function of w, w', g and g' with respect to g and then put $g=g'=0°$ we find that the coefficient of $\sin i\,(w'-w)$ becomes

$$
\begin{aligned}
&i\,(1+e^2\,P^2_0+e^4\,P^4_0+\ \ldots\ )\\
&+(i-1)\,(e\,P^1_1+e^3\,P^3_1+\ \ldots\ )\\
&+(i+1)\,(e\,P^1_{-1}+e^3\,P^3_{-1}+\ \ldots\ )\\
&+(i-2)\,(e^2\,P^2_2+\ \ldots\ )\\
&+(i+2)\,(e^2\,P^2_{-2}+\ \ldots\ )\\
&=i\,\beta^{(i)}+e\,(P^1_{-1}-P^1_1)+e^2\,(2\,P^2_{-2}-2\,P^2_2)+e^3\,(3\,P^3_{-3}+P^3_{-1}-P^3_1-3\,P^3_3)+\text{etc.}
\end{aligned}
$$

Comparing the coefficient of e^n in this development with that derived from the special development we have the conditions

$$n\,\Pi^n_{-n}+(n-2)\,\Pi^n_{-n+2}+(n-4)\,\Pi^n_{-n+4}+\ \ldots\ -n\,\Pi^n_n=i\,H_n$$

which affords a complete control on the odd powers of i in the development.

We have, in the same way,

$$n\,\Pi^{0,n}_{0,-n}+(n-2)\,\Pi^{0,n}_{0-n+2}+\ \ldots\ -n\,\Pi^{0,n}_{0,n}=i\,H'_n$$

where the H'_n have the values

$$
\begin{aligned}
H'_1&=2\\
2\,H'_2&=9+4\,D\\
2\,H'_3&=15+11\,D+2\,D^2\\
24\,H'_4&=261+250\,D+78\,D^2+8\,D^3\\
24\,H'_5&=351+399\,D+166\,D^2+30\,D^3+2\,D^4\\
240\,H'_6&=4485+5781\,D+2900\,D^2+710\,D^3+85\,D^4+4\,D^5\\
720\,H'_7&=16605+23568\,D+13553\,D^2+4050\,D^3+665\,D^4+57\,D^5+2\,D^6\\
40320\,H'_8&=1117305+1715040\,D+1095822\,D^2+378784\,D^3+76650\,D^4+9100\,D^5+588\,D^6+16\,D^7
\end{aligned}
$$

The combined products $\Pi^{n,n'}_{j,j'}$ may be almost completely controlled by the condition that

$$\Sigma\,\Pi^{n,n'}_{j,j'}\begin{pmatrix} j=n,\ n-2,n-4\ \ldots\ -n\\ j'=n',n'-2,n'-4\ \ldots\ -n'\end{pmatrix}$$

should come out equal to $K_n\times K'_{n'}$, K'_n being the value of K resulting from the substitution of $-1-D$ for D in K_n.

For, from the equations,

$$
\begin{aligned}
K_n&=\Pi^n_n+\Pi^n_{n-2}+\ \ldots\ +\Pi^n_{-n}\\
K'_{n'}&=\Pi^{0,n'}_{0,n'}+\Pi^{0,n'}_{0,n'-2}+\ \ldots\ +\Pi^{n'}_{-n'}
\end{aligned}
$$

we have

$$K_n\,K'_{n'}=\Sigma_{j,j'}\,\Pi^n_j\times\Pi^{0,n'}_{0,j'}=\Sigma_{j,j'}\,\Pi^{n,n'}_{j,j'}$$

Control third.—This control rests on the principle that Δ is a symmetric function

of a and a'. In consequence, every symbolic operator expressed as a function of i

$$\Pi^{n,\,n'}_{j.\,j'}(i)$$

should be transformed into

$$\Pi^{n'.\,n}_{j'.\,j}(-i)=\Pi^{n'.\,n}_{-j'-j}(i)$$

by the substitution of $-1-\mathrm{D}$ for D.

This control completely guards against errors in the process by which the values of the operators are derived from the fundamental ones Π^{n}_{j}, but would not detect an error arising from the use of a wrong term in the fundamental operators.

The correctness of the latter is, however, very completely assured by controls I and II. We then have, for the products, two controls, namely, III, by which the values of Π may be tested in pairs, and that stated at the end of control II, whereby all the products pertaining to the same values of n and n' may be tested.

Exhibit I.

Developments of the equation of the center and of the logarithm of the radius vector in powers of the eccentricity and in sines and cosines of multiples of the mean anomaly.

$$
\begin{aligned}
E = &\left(2e - \frac{1}{4}e^3 + \frac{5}{96}e^5 + \frac{107}{4608}e^7 + \frac{6217}{368640}e^9\right)\sin g \\
&+\left(\frac{5}{4}e^2 - \frac{11}{24}e^4 + \frac{17}{192}e^6 + \frac{43}{5760}e^8\right)\sin 2g \\
&+\left(\frac{13}{12}e^3 - \frac{43}{64}e^5 + \frac{95}{512}e^7 - \frac{973}{61440}e^9\right)\sin 3g \\
&+\left(\frac{103}{96}e^4 - \frac{451}{480}e^6 + \frac{4123}{11520}e^8\right)\sin 4g \\
&+\left(\frac{1097}{960}e^5 - \frac{5957}{4608}e^7 + \frac{164921}{258048}e^9\right)\sin 5g \\
&+\left(\frac{1223}{960}e^6 - \frac{7913}{4480}e^8\right)\sin 6g \\
&+\left(\frac{47273}{32256}e^7 - \frac{1773271}{737280}e^9\right)\sin 7g \\
&+ \frac{556403}{322560}e^8 \sin 8g \\
&+ \frac{10661993}{5160960}e^9 \sin 9g
\end{aligned}
$$

$$
\begin{aligned}
\log r - \log a = &\frac{1}{4}e^2 + \frac{1}{32}e^4 + \frac{1}{96}e^6 + \frac{5}{1024}e^8 \\
&+\left(-e + \frac{3}{8}e^3 + \frac{1}{64}e^5 + \frac{127}{9216}e^7\right)\cos g \\
&+\left(-\frac{3}{4}e^2 + \frac{11}{24}e^4 - \frac{3}{64}e^6 + \frac{9}{640}e^8\right)\cos 2g \\
&+\left(-\frac{17}{24}e^3 + \frac{77}{128}e^5 - \frac{743}{5120}e^7\right)\cos 3g \\
&+\left(-\frac{71}{96}e^4 + \frac{129}{160}e^6 - \frac{387}{1280}e^8\right)\cos 4g \\
&+\left(-\frac{523}{640}e^5 + \frac{10039}{9216}e^7\right)\cos 5g \\
&+\left(-\frac{899}{960}e^6 + \frac{6617}{4480}e^8\right)\cos 6g \\
&- \frac{355081}{322560}e^7 \cos 7g \\
&- \frac{47259}{35840}e^8 \cos 8g
\end{aligned}
$$

The above developments are given here for convenient reference, and to verify the values of h and k, which are found in Exh. II, and are derived immediately from them. That of E is derived from LeVerrier (*Annales de l'Observatoire*, Vol. I and Vol. X, p. 10). That of $\log r$ is derived from the formula

$$\log r = \int \frac{d\,r}{r\,d\,e}\,d\,e$$

$\frac{1}{r}$ being taken from Cayley's Tables (*Mem. R. A. S.*, Vol. XXIX,) and $\frac{dr}{de}$ derived from LeVerrier. A comparison with Delambre's development (*Tables du Soleil Int., Sig. f,*) shows two typographic errors in the latter, the coefficient of $e^8 \cos 2\,g$ being there given. as $9:240$, and that of $e^8 \cos 8\,g$ as $-47529:35840$, though the terms in e^9 are correct Both developments can be controlled by seeing whether $\log r$ and the derivative of E as to g reduce to the proper values for $g=0$ and $g=180^0$.

EXHIBIT II.

The factors h and k.

k^{I}_{1} $+2$	h^{I}_{1} -1	k^{I}_{-1} -2	h^{I}_{-1} -1	0	0	0	0	0	0
k^{II}_{2} $+\frac{5}{2}$	h^{II}_{2} $-\frac{3}{2}$	k^{II}_{0} 0	h^{II}_{0} $+1$	k^{II}_{-2} $-\frac{5}{2}$	h^{II}_{-2} $-\frac{3}{2}$	0	0	0	0
k^{III}_{3} $+\frac{13}{4}$	h^{III}_{3} $-\frac{17}{8}$	k^{III}_{1} $-\frac{3}{4}$	h^{III}_{1} $+\frac{9}{8}$	k^{III}_{-1} $+\frac{3}{4}$	h^{III}_{-1} $+\frac{9}{8}$	k^{III}_{-3} $-\frac{13}{4}$	h^{III}_{-3} $-\frac{17}{8}$	0	0
k^{IV}_{4} $+\frac{103}{24}$	h^{IV}_{4} $-\frac{71}{24}$	k^{IV}_{2} $-\frac{11}{6}$	h^{IV}_{2} $+\frac{11}{6}$	k^{IV}_{0} 0	h^{IV}_{0} $+\frac{1}{4}$	k^{IV}_{-2} $+\frac{11}{6}$	h^{IV}_{-2} $+\frac{11}{6}$	k^{IV}_{-4} $-\frac{103}{24}$	h^{IV}_{-4} $-\frac{71}{24}$
k^{V}_{5} $+\frac{1097}{192}$	h^{V}_{5} $-\frac{523}{128}$	k^{V}_{3} $-\frac{215}{64}$	h^{V}_{3} $+\frac{385}{128}$	k^{V}_{1} $+\frac{25}{96}$	h^{V}_{1} $+\frac{5}{64}$	k^{V}_{-1} $-\frac{25}{96}$	h^{V}_{-1} $+\frac{5}{64}$	k^{V}_{-3} $+\frac{215}{64}$	h^{V}_{-3} $+\frac{385}{128}$
k^{VI}_{6} $+\frac{1223}{160}$	h^{VI}_{6} $-\frac{899}{160}$	k^{VI}_{4} $-\frac{451}{80}$	h^{VI}_{4} $+\frac{387}{80}$	k^{VI}_{2} $+\frac{17}{32}$	h^{VI}_{2} $-\frac{9}{32}$	k^{VI}_{0} 0	h^{VI}_{0} $+\frac{1}{8}$	k^{VI}_{-2} $-\frac{17}{32}$	h^{VI}_{-2} $-\frac{9}{32}$
k^{VII}_{7} $+\frac{47273}{4608}$	h^{VII}_{7} $-\frac{355081}{46080}$	k^{VII}_{5} $-\frac{41699}{4608}$	h^{VII}_{5} $+\frac{70273}{9216}$	k^{VII}_{3} $+\frac{665}{512}$	h^{VII}_{3} $-\frac{5201}{5120}$	k^{VII}_{1} $+\frac{749}{4608}$	h^{VII}_{1} $+\frac{889}{9216}$	k^{VII}_{-1} $-\frac{749}{4608}$	h^{VII}_{-1} $+\frac{889}{9216}$
k^{VIII}_{8} $+\frac{556403}{40320}$	h^{VIII}_{8} $-\frac{47259}{4480}$	k^{VIII}_{6} $-\frac{7913}{560}$	h^{VIII}_{6} $+\frac{6617}{560}$	k^{VIII}_{4} $+\frac{4123}{1440}$	h^{VIII}_{4} $-\frac{387}{160}$	k^{VIII}_{2} $+\frac{43}{720}$	h^{VIII}_{2} $+\frac{9}{80}$	k^{VIII}_{0} 0	h^{VIII}_{0} $+\frac{5}{64}$

EXHIBIT III.

Symbolic development of the function $\Delta^{-1}=(r^2-2\,r\,r'\,\cos V+r'^2)^{-\frac{1}{2}}$ in powers of the eccentricities and of the sine of half the mutual inclination, and in cosines of multiples of the mean anomalies and distances of perihelia from common node.

Notation: g; mean anomaly

w; distance from common node to perihelion

γ; mutual inclination of orbits

$\lambda=w+g$

$\sigma=\sin\frac{1}{2}\gamma$

$V_i=i\,\lambda'-i\,\lambda$

$V'_i=(i+1)\,\lambda'-(i-1)\,\lambda$

$V''_i=(i+2)\,\lambda'-(i-2)\,\lambda$

Development when $e=e'=0$:

$$\begin{aligned}\Delta_0^{-1}=\sum_{i=-\infty}^{i=+\infty}\Big\{&\tfrac{1}{2}A_i\cos V_i \quad .\;.\;.\;. && \text{(Terms of class 0)}\\ &+\sigma^2 B_i\cos V'_i \quad .\;.\;.\;. && \text{(Terms of class 1)}\\ &+\sigma^4 C_i\cos V''_i \quad .\;.\;.\;. && \text{(Terms of class 2)}\\ &+\;.\;.\;.\;. && \vdots\quad\vdots\;\Big\}\end{aligned}$$

When e and e' are considered the terms of class 0 develop into

$$[\Delta^{-1}]_0=\frac{1}{2}\,\Sigma\, e^n\, e'^{n'}\, P^{n,\,n'}_{j,\,j'}\cos(V_i+j\,g+j'\,g')$$

$$\left[\begin{array}{l} i=-\infty\,.\;.\;-1,0,1,2,\;.\;.\;+\infty\\ n,\,n'=0,\;1,\;2,\;3,\;.\;.\;.\;.\;\;\infty\\ j=n,\;n-2,\;n-4,\;\;.\;.\;.\;-n\\ j'=n',\;n'-2,\;n'-4,\;\;.\;.\;-n'\end{array}\right]$$

Expanding, and so merging like terms that $j+j'$ shall never be negative:

$$[\Delta^{-1}]_0=\sum_{i=-\infty}^{i=+\infty}$$

		Argument in terms of w, w', g, g'.	
Terms of Class 0.		$-i\,w$	$+i\,w'$
$[j+j'=0]$		$+g$	$+g'$
		$\times$	$\times$
$\frac{1}{2}\{P^0_0+e^2 P^2_0+e'^2 P^{0,2}_{0,0}+\;.\;.\;.\;.\;.\;.\;.$	$\}\cos V_i$	$-i$	i
$+\;\{e\,e'\,P^{1,1}_{-1,1}+e^3 e'\,P^{3,1}_{-1,1}+e\,e'^3\,P^{1,3}_{-1,1}+\;.\;.$	$\}\cos(V_i-g+g')$	$-i-1$	$i+1$
$+\;\{e^2 e'^2\,P^{2,2}_{-2,2}+e^4 e'^2\,P^{4,2}_{-2,2}+e^2 e'^4\,P^{2,4}_{-2,2}+\;.$	$\}\cos(V_i-2\,g+2\,g')$	$-i-2$	$i+2$
$+\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.$		.	.

		Argument in terms of w, w', g, g'.	
		$-i\,w$	$+i\,w'$
		$+g$	$+g'$
$[j+j'=1]$		$\times$	$\times$
. . . .	. . . . .	.	.
$+\ \{e^2 e' P^{2,1}_{2,-1}+e^4 e' P^{4,1}_{2,-1}+e^2 e'^3 P^{2,3}_{2,-1}+\ldots$	$\}\cos(V_i+2g-g')$	$-i+2$	$i-1$
$+\ \{e P^1_1+e^3 P^3_1+e e'^2 P^{1,2}_{1,0}+\ldots$	$\}\cos(V_i+g)$	$-i+1$	i
$+\ \{e' P^{0,1}_{0,1}+e^2 e' P^{2,1}_{0,1}+e'^3 P^{0,3}_{0,1}+\ldots$	$\}\cos(V_i+g')$	$-i$	$i+1$
$+\ \{e e'^2 P^{1,2}_{-1,2}+e^3 e'^2 P^{3,2}_{-1,2}+e e'^4 P^{1,4}_{-1,2}+\ldots$	$\}\cos(V_i-g+2g')$	$-i-1$	$i+2$
$+$	. . . . .	.	.
$[j+j'=2]$			
. . . .	. . . . .	.	.
$+\ \{e^3 e' P^{3,1}_{3,-1}+e^5 e' P^{5,1}_{3,-1}+e^3 e'^3 P^{3,3}_{3,-1}+\ldots$	$\}\cos(V_i+3g-g')$	$-i+3$	$i-1$
$+\ \{e^2 P^2_2+e^4 P^4_2+e^2 e'^2 P^{2,2}_{2,0}+\ldots$	$\}\cos(V_i+2g)$	$-i+2$	i
$+\ \{e e' P^{1,1}_{1,1}+e^3 e' P^{3,1}_{1,1}+e e'^3 P^{1,3}_{1,1}+\ldots$	$\}\cos(V_i+g+g')$	$-i+1$	$i+1$
$+\ \{e'^2 P^{0,2}_{0,2}+e^2 e'^2 P^{2,2}_{0,2}+e'^4 P^{0,4}_{0,2}+\ldots$	$\}\cos(V_i+2g')$	$-i$	$i+2$
$+\ \{e e'^3 P^{1,3}_{-1,3}+e^3 e'^3 P^{3,3}_{-1,3}+e e'^5 P^{1,5}_{-1,3}+\ldots$	$\}\cos(V_i-g+3g')$	$-i-1$	$i+3$
$+$	. . . . .	.	.
$[j+j'=3]$			
. . . .	. . . . .	.	.
$+\ \{e^4 e' P^{4,1}_{4,-1}+e^6 e' P^{6,1}_{4,-1}+e^4 e'^3 P^{4,3}_{4,-1}+\ldots$	$\}\cos(V_i+4g-g')$	$-i+4$	$i-1$
$+\ \{e^3 P^3_3+e^5 P^5_3+e^3 e'^2 P^{3,2}_{3,0}+\ldots$	$\}\cos(V_i+3g)$	$-i+3$	i
$+\ \{e^2 e' P^{2,1}_{2,1}+e^4 e' P^{4,1}_{2,1}+e^2 e'^3 P^{2,3}_{2,1}+\ldots$	$\}\cos(V_i+2g+g')$	$-i+2$	$i+1$
$+\ \{e e'^2 P^{1,2}_{1,2}+e^3 e'^2 P^{3,2}_{1,2}+e e'^4 P^{1,4}_{1,2}+\ldots$	$\}\cos(V_i+g+2g')$	$-i+1$	$i+2$
$+\ \{e'^3 P^{0,3}_{0,3}+e^2 e'^3 P^{2,3}_{0,3}+e'^5 P^{0,5}_{0,3}+\ldots$	$\}\cos(V_i+3g')$	$-i$	$i+3$
$+$	. . . . .	.	. .
$[j+j'=4]$			
. . . .	. . . . .	.	.
$+\ \{e^4 P^4_4+e^6 P^6_4+e^4 e'^2 P^{4,2}_{4,0}+\ldots$	$\}\cos(V_i+4g)$	$-i+4$	i
$+\ \{e^3 e' P^{3,1}_{3,1}+e^5 e' P^{5,1}_{3,1}+e^3 e'^3 P^{3,3}_{3,1}+\ldots$	$\}\cos(V_i+3g+g')$	$-i+3$	$i+1$
$+\ \{e^2 e'^2 P^{2,2}_{2,2}+e^4 e'^2 P^{4,2}_{2,2}+e^2 e'^4 P^{2,4}_{2,2}+\ldots$	$\}\cos(V_i+2g+2g')$	$-i+2$	$i+2$
$+\ \{e e'^3 P^{1,3}_{1,3}+e^3 e'^3 P^{3,3}_{1,3}+e e'^5 P^{1,5}_{1,3}+\ldots$	$\}\cos(V_i+g+3g')$	$-i+1$	$i+3$
$+\ \{e'^4 P^{0,4}_{0,4}+e^2 e'^4 P^{2,4}_{0,4}+e'^6 P^{0,6}_{0,4}+\ldots$	$\}\cos(V_i+4g')$	$-i$	$i+4$
$+$	. . . . .	.	.

$$
\begin{array}{llll}
 & & \multicolumn{2}{l}{\textit{Argument in terms}} \\
 & & \multicolumn{2}{l}{\textit{of } w, w', g, g'} \\
 & & -i\,w & +i\,w' \\
 & & +g & +g' \\
 & [j+j'=5] & \times & \times \\
 & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \cdot & \cdot \\
+ & \{e^5\,\mathrm{P}^5_5+e^7\,\mathrm{P}^7_5+e^5\,e'^2\,\mathrm{P}^{5,2}_{5,0}+\ \ldots\ldots\}\cos(\mathrm{V}_i+5\,g) & -i+5 & i \\
+ & \{e^4\,e'\,\mathrm{P}^{4,1}_{4,1}+e^6\,e'\,\mathrm{P}^{6,1}_{4,1}+e^4\,e'^3\,\mathrm{P}^{4,3}_{4,1}+\ \ldots\}\cos(\mathrm{V}_i+4\,g+g') & -i+4 & i+1 \\
+ & \{e^3\,e'^2\,\mathrm{P}^{3,2}_{3,2}+e^5\,e'^2\,\mathrm{P}^{5,2}_{3,2}+e^3\,e'^4\,\mathrm{P}^{3,4}_{3,2}+\ \ldots\}\cos(\mathrm{V}_i+3\,g+2\,g') & -i+3 & i+2 \\
+ & \{e^2\,e'^3\,\mathrm{P}^{2,3}_{2,3}+e^4\,e'^3\,\mathrm{P}^{4,3}_{2,3}+e^2\,e'^5\,\mathrm{P}^{2,5}_{2,3}+\ \ldots\}\cos(\mathrm{V}_i+2\,g+3\,g') & -i+2 & i+3 \\
+ & \{e\,e'^4\,\mathrm{P}^{1,4}_{1,4}+e^3\,e'^4\,\mathrm{P}^{3,4}_{1,4}+e\,e'^6\,\mathrm{P}^{1,6}_{1,4}+\ \ldots\}\cos(\mathrm{V}_i+g+4\,g') & -i+1 & i+4 \\
+ & \{e'^5\,\mathrm{P}^{0,5}_{0,6}+e^2\,e'^5\,\mathrm{P}^{2,5}_{0,5}+e'^7\,\mathrm{P}^{0,7}_{0,5}+\ \ldots\}\cos(\mathrm{V}_i+5\,g') & -i & i+5 \\
+ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \cdot & \cdot
\end{array}
$$

The expansion may be continued indefinitely by the following rules:

1. *To find the coefficient of the cosine of any argument*

$$\mathrm{V}_i+j\,g+j'\,g'.$$

The leading term is $\frac{1}{2}\,e^n\,e'^{n'}\,\mathrm{P}^{n,n'}_{j,j'}$, n and n' being the values of j and j' taken positively. The remaining terms are formed by giving n and n', separately and independently, the increments 2, 4, 6, etc., j and j' remaining unchanged. Every pair of terms whose arguments transform into each other by changing the signs of j, j' and i may then be merged into one.

2. *To find all the terms containing a given power or product of powers of e and e', say*

$$e^n\,e'^{n'}$$

Write the complete series of terms of the form

$$\frac{1}{2}\,e^n\,e'^{n'}\,\mathrm{P}^{n,n}_{j,j'}$$

which may be formed by giving j and j' separately each alternate integral value from $+n$ to $-n$ and from $+n'$ to $-n'$, respectively. The sum of the products formed by multiplying each term into the cosine of its corresponding argument will be that required. Pairs of terms which will be equal when summed as to i may then be combined.

Terms of Class 1.

$$[\varDelta^{-1}]_1=\Sigma\,e^n\,e'^{n'}\,\mathrm{P}'^{n,n'}_{j,j'}\cos(\mathrm{V}'_i+j\,g+j'\,g')$$

[i, j, j', n, n' going through the same range of values as in class 0]

Expanding and arranging groups of terms according to the values of $j+j'$:

$$
\begin{array}{lll}
[\Delta^{-1}]_1 = \sum\limits_{i=-\infty}^{i=+\infty} & & \begin{array}{c}\textit{Argument in terms of}\\ w,\,w',\,g,\,g'\end{array} \\
 & & (1-i)w+(i+1)w' \\
\qquad [j+j'=0] & & +g \quad +g' \\
 & & \times \quad \times \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & & \cdot \qquad \cdot \\
+\{e^2e'^2P'^{2,\,2}_{2,\,-2}+e^4e'^2P'^{4,\,2}_{2,\,-2}+e^2e'^4P^{2,\,4}_{2,\,-2}+ & .\ \}\cos(V'_i+2g-2g') & -i+3 \quad i-1 \\
+\{ee'P'^{1,\,1}_{1,\,-1}+e^3e'P'^{3,\,1}_{1,\,-1}+ee'^3P'^{1,\,3}_{1,\,-1}+ & .\ .\ \}\cos(V'_i+g-g') & -i+2 \quad i \\
+\{P'^{0,0}_{0,0}+e^2P'^{2,0}_{0,0}+e'^2P'^{0,2}_{0,0}+ & .\ .\ .\ .\ .\ \}\cos V'_i & -i+1 \quad i+1 \\
+\{ee'P'^{1,1}_{-1,1}+e^3e'P'^{3,1}_{-1,1}+ee'^3P'^{1,3}_{-1,1}+ & .\ .\ \}\cos(V'_i-g+g') & -i \qquad i+2 \\
+\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & & \cdot \qquad \cdot \\
\qquad [j+j'=-1] & & \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & & \cdot \qquad \cdot \\
+\{e^2e'P'^{2,1}_{-2,1}+e^4e'P'^{4,1}_{-2,1}+e^2e'^3P'^{2,3}_{-2,1}+ & .\ \}\cos(V'_i-2g+g') & -i-1 \quad i+2 \\
+\{eP'^{1,0}_{-1,0}+e^3P'^{3,0}_{-1,0}+ee'^2P'^{1,2}_{-1,0}+ & .\ .\ .\ \}\cos(V'_i-g) & -i \qquad i+1 \\
+\{e'P'^{0,\,1}_{0,\,-1}+e^2e'P'^{2,\,1}_{0,\,-1}+e'^3P'^{0,\,3}_{0,\,-1}+ & .\ .\ .\ \}\cos(V'_i-g') & -i+1 \quad i \\
+\{ee'^2P'^{1,\,2}_{1,\,-2}+e^3e'^2P'^{3,\,2}_{1,\,-2}+ee'^4P'^{1,\,4}_{1,\,-2}+ & .\ \}\cos(V'_i+g-2g') & -i+2 \quad i-1 \\
+\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & & \cdot \qquad \cdot \\
\qquad [j+j'=+1] & & \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & & \cdot \qquad \cdot \\
+\{e^2e'P'^{2,\,1}_{2,\,-1}+e^4e'P'^{4,\,1}_{2,\,-1}+e^2e'^3P'^{2,\,3}_{2,\,-1}+ & .\ \}\cos(V'_i+2g-g') & -i+3 \quad i \\
+\{eP'^{1,0}_{1,0}+e^3P'^{3,0}_{1,0}+ee'^2P'^{1,2}_{1,0}+ & .\ .\ .\ .\ \}\cos(V'_i+g) & -i+2 \quad i+1 \\
+\{e'P'^{0,1}_{0,1}+e^2e'P'^{2,1}_{0,1}+e'^3P'^{0,3}_{0,1}+ & .\ .\ .\ .\ \}\cos(V'_i+g') & -i+1 \quad i+2 \\
+\{ee'^2P'^{1,2}_{-1,2}+e^3e'^2P'^{3,2}_{-1,2}+ee'^4P'^{1,4}_{-1,2}+ & .\ .\ \}\cos(V'_i-g+2g') & -i \qquad i+3 \\
+\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & & \cdot \qquad \cdot
\end{array}
$$

The expansion may be continued indefinitely by the rules for class 0, except that the coefficient $\frac{1}{2}$ does not appear and there is no merging of equal terms.

Terms of class 2, 3 . . . . . *k*.

The expansions are formed from those of class 1 by writing k accents over the symbols P and V_i. The arguments in terms of w, w', g, and g' are of the general form

$$(-i+k)\,w+(i+k)\,w'+(-i+k+j)\,g+(i+k+j')\,g'$$

Exhibit IV.

Values of the coefficients P.

$$P^{n,n'}_{j,j'} = \Pi^{n,n'}_{j,j'}\, a'\, A_i$$
$$P'^{n,n'}_{j,j'} = \sigma^2\, \Pi'^{n,n'}_{j,j'}\, a'\, B_i$$
$$P''^{n,n'}_{j,j'} = \sigma^4\, \Pi''^{n,n'}_{j,j'}\, a'\, C_i$$
$$P'''^{n,n'}_{j,j'} = \sigma^6\, \Pi'''^{n,n'}_{j,j'}\, a'\, D_i$$
$$\overset{\text{IV}}{P}{}^{n,n'}_{j,j'} = \sigma^8\, \overset{\text{IV}}{\Pi}{}^{n,n'}_{j,j'}\, a'\, E_i$$

where

$$a'\,A_i = b_1^i - \frac{1}{2}\sigma^2\alpha\, b_3^{(i-1)} - \frac{1}{2}\sigma^2\alpha\, b_3^{(i+1)} + \frac{3}{8}\sigma^4(\alpha^2 b_5^{(i-2)} + 4\,\alpha^2 b_5^{(i)} + \alpha^2 b_5^{(i+2)})$$
$$- \frac{5}{16}\sigma^6(\alpha^3 b_7^{(i-3)} + 9\,\alpha^3 b_7^{(i-1)} + 9\,\alpha^3 b_7^{(i+1)} + \alpha^3 b_7^{(i+3)})$$
$$+ \frac{35}{128}\sigma^8(\alpha^4 b_9^{(i-4)} + 16\,\alpha^4 b_9^{(i-2)} + 36\,\alpha^4 b_9^{(i)} + 16\,\alpha^4 b_9^{(i+2)} + \alpha^4 b_9^{(i+4)}) + \ldots$$

$$a'\,B_i = \frac{1}{2}\alpha\, b_3^{(i)} - \frac{3}{4}\sigma^2(\alpha^2 b_5^{(i-1)} + \alpha^2 b_5^{(i+1)}) + \frac{15}{16}\sigma^4(\alpha^3 b_7^{(i-2)} + 3\,\alpha^3 b_7^{(i)} + \alpha^3 b_7^{(i+2)})$$
$$- \frac{35}{32}\sigma^6(\alpha^4 b_9^{(i-3)} + 6\,\alpha^4 b_9^{(i-1)} + 6\,\alpha^4 b_9^{(i+1)} + \alpha^4 b_9^{(i+3)}) + \ldots$$

$$a'\,C_i = \frac{3}{8}\alpha^2 b_5^{(i)} - \frac{15}{16}\sigma^2(\alpha^3 b_7^{(i-1)} + \alpha^3 b_7^{(i+1)}) + \frac{35}{64}\sigma^4(3\,\alpha^4 b_9^{(i-2)} + 8\,\alpha^4 b_9^{(i)} + 3\,\alpha^4 b_9^{(i+2)}) + \ldots$$

$$a'\,D_i = \frac{5}{16}\alpha^3 b_7^{(i)} - \frac{35}{32}\sigma^2(\alpha^4 b_9^{(i-1)} + \alpha^4 b_9^{(i+1)}) + \ldots$$

$$a'\,E_i = \frac{35}{128}\alpha^4 b_9^{(i)} + \ldots$$

while the operating symbols Π are shown as in Exhibit V.

Exhibit V.

Values of the operating symbols $\Pi^{n,n'}_{j,j'}$.

Class 0; operating on A_i.

I.—*Operators relating only to the Inner Planet.*

$$\Pi_0^0 = 1$$

$$2\,\Pi_1^1 = -2i - D$$

$$4\,\Pi_0^2 = -4i^2 + D + D^2$$

$$8\,\Pi_2^2 = +4i^2 - 5i + (4i - 3)D + D^2$$

$$16\,\Pi_1^3 = \ldots 8i^3 - 10i^2 + 2i + (+4i^2 - 5i + 3)D + (-2i + 1)D^2 - D^3$$

$$48\,\Pi_3^3 = -8i^3 + 30i^2 - 26i + (-12i^2 + 33i - 17)D + (-6i + 9)D^2 - D^3$$

$$64\,\Pi_0^4 = +16i^4 - 9i^2 + 2D + (-8i^2 - 1)D^2 - 2D^3 + D^4$$

$$96\,\Pi_2^4 = -16i^4 + 60i^3 - 64i^2 + 22i + (-16i^3 + 48i^2 - 47i + 22)D + (0i^2 - 3i - 1)D^2 + (+4i - 6)D^3 + D^4$$

$$384\,\Pi_4^4 = +16i^4 - 120i^3 + 283i^2 - 206i + (+32i^3 - 192i^2 + 330i - 142)D + (+24\ \ -102i + 95)D^2$$
$$+ (+8i - 18)D^3 + D^4$$

$$384\,\Pi_1^5 = -32i^5 + 80i^4 - 26i^3 - 12i^2 - 10i + (-16i^4 + 24i^3 - 9i^2 + 10i + 3)D + (+16i^3 - 36i^2 + 12i - 8)D^2$$
$$+ (+8i^2 - 2i - 5)D^3 + (-2i + 6)D^4 - D^5$$

$$768\,\Pi_3^5 = +32i^5 - 240i^4 + 614i^3 - 648i^2 + 258i + (+48i^4 - 296i^3 + 617i^2 - 572i + 231)D$$
$$+ (+16i^3 - 60i^2 + 76i - 68)D^2 + (-8i^2 + 42i - 41)D^3 + (-6i + 14)D^4 - D^5$$

$$3\,840\,\Pi_5^5 = -32i^5 + 400i^4 - 1\,790i^3 + 3\,360i^2 - 2\,194i + (-80i^4 + 840i^3 - 2\,995i^2 + 4\,080i - 1\,569)D$$
$$+ (-80i^3 + 660i^2 - 1\,660i + 1\,220)D^2 + (-40i^2 + 230i - 305)D^3 + (-10i + 30)D^4 - D^5$$

$$2\,304\,\Pi_0^6 = -64i^6 + 196i^4 - 172i^2 + (-48i^4 + 39i^2 + 24)D + (+48i^4 - 93i^2 - 26)D^2 + (48i^2 - 15)D^3$$
$$+ (-12i^2 + 25)D^4 - 9D^5 + D^6$$

$$3\,072\,\Pi_2^6 = +64i^6 - 400i^5 + 788i^4 - 593i^3 + 314i^2 - 136i + (+64i^5 - 304i^4 + 452i^3 - 399i^2 + 344i - 72)D$$
$$+ (-16i^4 + 136i^3 - 255i^2 + 115i - 58)D^2 + (-32i^3 + 112i^2 - 66i + 5)D^3$$
$$+ (-4i^2 - 17i + 41)D^4 + (+4i - 13)D^5 + D^6$$

$$7\,680\,\Pi_4^6 = -64i^6 + 800i^5 - 3\,740i^4 + 8\,200i^3 - 8\,588i^2 + 3\,608i$$
$$+ (-128i^5 + 1\,360i^4 - 5\,280i^3 + 9\,535i^2 - 8\,454i + 3\,096)D$$
$$+ (-80i^4 + 640i^3 - 1\,765i^2 + 2\,280i - 1\,466)D^2 + (0i^3 - 80i^2 + 340i - 255)D^3$$
$$+ (+20i^2 - 130i + 185)D^4 + (+8i - 25)D^5 + D^6$$

$$46\,080\,\Pi_6^6 = +64i^6 - 1\,200i^5 + 8\,660i^4 - 29\,835i^3 + 48\,538i^2 - 29\,352i$$
$$+ (+192i^5 - 3\,120i^4 + 18\,860i^3 - 51\,615i^2 + 60\,752i - 21\,576)D$$
$$+ (+240i^4 - 3\,240i^3 + 15\,345i^2 - 29\,535i + 18\,694)D^2 + (+160i^3 - 1\,680i^2 + 5\,530i - 5\,595)D^3$$
$$+ (+60i^2 - 435i + 745)D^4 + (+12i - 45)D^5 + D^6$$

$$18\,432\,\Pi_1^7 = +128i^7 - 480i^6 + 40i^5 + 874i^4 + 494i^3 - 842i^2 - 214i$$
$$+ (+64i^6 - 48i^5 - 364i^4 + 89i^3 + 481i^2 - 109i + 127)D$$
$$+ (-96i^5 + 384i^4 - 90i^3 - 441i^2 - 26i - 159)D^2 + (-48i^4 - 24i^3 + 351i^2 - 57i - 46)D^3$$
$$+ (+24i^3 - 126i^2 + 10i + 147)D^4 + (+12i^2 + 9i - 76)D^5 + (-2i + 15)D^6 - D^7$$

$$30\,720\,\Pi_3^7 = -128i^7 + 1\,440i^6 - 5\,960i^5 + 11\,610i^4 - 12\,246i^3 + 8\,262i^2 - 2\,850i$$
$$+ (-192i^6 + 1\,776i^5 - 5\,900i^4 + 9\,695i^3 - 10\,755i^2 + 8\,045i - 2\,229)D$$
$$+ (-32i^5 + 0i^4 + 810i^3 - 1\,485i^2 + 182i - 55)D^2 + (+80i^4 - 680i^3 + 1\,675i^2 - 1\,295i + 446)D^3$$
$$+ (+40i^3 - 150i^2 - 50i + 275)D^4 + (-4i^2 + 75i - 160)D^5 + (-6i + 23)D^6 - D^7$$

$$92\,160\,\Pi_5^7 = +128i^7 - 2\,400i^6 + 17\,800i^5 - 66\,790i^4 + 134\,246i^3 - 138\,970i^2 + 59\,570i$$
$$+ (+320i^6 - 5\,232i^5 + 33\,140i^4 - 104\,095i^3 + 172\,821i^2 - 147\,149i + 50\,195)D$$
$$+ (+288i^5 - 3\,840i^4 + 19\,030i^3 - 45\,045i^2 + 55\,026i - 30\,679)D^2$$
$$+ (+80i^4 - 600i^3 + 995i^2 + 175i + 326)D^3 + (-40i^3 + 570i^2 - 2\,270i + 2\,515)D^4$$
$$+ (-36i^2 + 285i - 520)D^5 + (-10i + 39)D^6 - D^7$$

$$645\,120\,\Pi_7^7 = -128i^7 + 3\,360i^6 - 35\,560i^5 + 193\,130i^4 - 563\,486i^3 + 828\,758i^2 - 472\,730i$$
$$+ (-448i^6 + 10\,416i^5 - 95\,340i^4 + 432\,775i^3 - 1\,000\,727i^2 + 1\,062\,285i - 355\,081)D$$
$$+ (-672i^5 + 13\,440i^4 - 101\,990i^3 + 361\,935i^2 - 587\,230i + 334\,369)D^2$$
$$+ (-560i^4 + 9\,240i^3 - 54\,425i^2 + 133\,945i - 113\,974)D^3$$
$$+ (-280i^3 + 3\,570i^2 - 14\,490i + 18\,515)D^4 + (-84i^2 + 735i - 1\,540)D^5 + (-14i + 63)D^6 - D^7$$

$147\,456\ \Pi_0^8 =$

$$+256 i^8 - 2\,272 i^6 + 7\,801 i^4 - 7\,140 i^2 + (+512 i^6 - 2\,720 i^4 + 2\,012 i^2 + 720)\,\mathrm{D}$$
$$+(-256 i^6 + 2\,128 i^4 - 3\,674 i^2 - 1\,044)\,\mathrm{D}^2 + (-576 i^4 + 2\,268 i^2 - 140)\,\mathrm{D}^3$$
$$+(+96 i^4 - 958 i^2 + 889)\,\mathrm{D}^4 + (192 i^2 - 560)\,\mathrm{D}^5 + (-16 i^2 + 154)\,\mathrm{D}^6 - 20\,\mathrm{D}^7 + \mathrm{D}^8$$

$184\,320\ \Pi_2^8 =$

$$-256 i^8 + 2\,240 i^7 - 6\,032 i^6 + 4\,380 i^5 - 1\,352 i^4 + 9\,586 i^3 - 6\,164 i^2 - 688 i$$
$$+(-256 i^7 + 1\,536 i^6 - 1\,856 i^5 - 880 i^4 - 3\,919 i^3 + 8\,768 i^2 - 2\,032 i + 1\,296)\,\mathrm{D}$$
$$+(+128 i^6 - 1\,488 i^5 + 4\,480 i^4 - 2\,695 i^3 - 1\,671 i^2 - 642 i - 1\,084)\,\mathrm{D}^2$$
$$+(+192 i^5 - 960 i^4 + 180 i^3 + 2\,610 i^2 - 275 i - 168)\,\mathrm{D}^3 + (0 i^4 + 420 i^3 - 1\,535 i^2 + 385 i + 1\,009)\,\mathrm{D}^4$$
$$+(-48 i^3 + 240 i^2 + 79 i - 720)\,\mathrm{D}^5 + (-8 i^2 - 47 i + 202)\,\mathrm{D}^6 + (+4 i - 24)\,\mathrm{D}^7 + \mathrm{D}^8$$

$368\,640\ \Pi_4^8 =$

$$+256 i^8 - 4\,480 i^7 + 30\,944 i^6 - 109\,320 i^5 + 218\,177 i^4 - 262\,622 i^3 + 192\,608 i^2 - 65\,968 i$$
$$+(+512 i^7 - 7\,680 i^6 + 44\,544 i^5 - 131\,680 i^4 + 230\,926 i^3 - 269\,444 i^2 + 191\,916 i - 55\,728)\,\mathrm{D}$$
$$+(+256 i^6 - 2\,592 i^5 + 8\,400 i^4 - 11\,530 i^3 + 18\,306 i^2 - 31\,906 i + 15\,052)\,\mathrm{D}^2$$
$$+(-128 i^5 + 2\,240 i^4 - 12\,360 i^3 + 26\,580 i^2 - 22\,074 i + 8\,612)\,\mathrm{D}^3$$
$$+(-160 i^4 + 1\,560 i^3 - 4\,290 i^2 + 2\,560 i + 889)\,\mathrm{D}^4$$
$$+(-32 i^3 - 0 i^2 + 1\,062 i - 2\,000)\,\mathrm{D}^5 + (+16 i^2 - 190 i + 442)\,\mathrm{D}^6 + (+8 i - 36)\,\mathrm{D}^7 + \mathrm{D}^8$$

$1\,290\,240\ \Pi_6^8 =$

$$-256 i^8 + 6\,720 i^7 - 72\,464 i^6 + 415\,380 i^5 - 1\,366\,792 i^4 + 2\,590\,518 i^3 - 2\,643\,828 i^2 + 1\,139\,472 i$$
$$+(-768 i^7 + 17\,920 i^6 - 168\,896 i^5 + 830\,480 i^4 - 2\,298\,653 i^3 + 3\,594\,304 i^2 - 2\,950\,800 i + 952\,848)\,\mathrm{D}$$
$$+(-896 i^6 + 17\,808 i^5 - 138\,880 i^4 + 546\,875 i^3 - 1\,171\,023 i^2 + 1\,351\,154 i - 687\,996)\,\mathrm{D}^2$$
$$+(-448 i^5 + 6\,720 i^4 - 35\,260 i^3 + 89\,250 i^2 - 114\,793 i + 84\,280)\,\mathrm{D}^3$$
$$+(\quad 0 i^4 - 980 i^3 + 10\,745 i^2 - 35\,525 i + 33\,649)\,\mathrm{D}^4 + (+112 i^3 - 1\,680 i^2 + 7\,693 i - 10\,640)\,\mathrm{D}^5$$
$$+(+56 i^2 - 525 i + 1\,162)\,\mathrm{D}^6 + (+12 i - 56)\,\mathrm{D}^7 + \mathrm{D}^8$$

$10\,321\,920\ \Pi_8^8 =$

$$+256 i^8 - 8\,960 i^7 + 130\,592 i^6 - 1\,023\,120 i^5 + 4\,628\,057 i^4 - 12\,000\,604 i^3 + 16\,352\,684 i^2 - 8\,902\,448 i$$
$$+(+1\,024 i^7 - 32\,256 i^6 + 415\,744 i^5 - 2\,812\,320 i^4 + 10\,613\,820 i^3 - 21\,771\,428 i^2 + 21\,370\,136 i - 6\,805\,296)\,\mathrm{D}$$
$$+(+1\,792 i^6 - 49\,728 i^5 + 550\,480 i^4 - 3\,082\,100 i^3 + 9\,077\,782 i^2 - 13\,040\,916 i + 6\,852\,460)\,\mathrm{D}^2$$
$$+(+1\,792 i^5 - 42\,560 i^4 + 388\,080 i^3 - 1\,683\,780 i^2 + 3\,433\,388 i - 2\,581\,964)\,\mathrm{D}^3$$
$$+(+1\,120 i^4 - 21\,840 i^3 + 153\,650 i^2 - 458\,640 i + 484\,729)\,\mathrm{D}^4 + (+448 i^3 - 6\,720 i^2 + 32\,396 i - 49\,840)\,\mathrm{D}^5$$
$$+(+112 i^2 - 1\,148 i + 2\,842)\,\mathrm{D}^6 + (+16 i - 84)\,\mathrm{D}^7 + \mathrm{D}^8$$

II.—*Operators relating only to Outer Planet.*

$$\Pi^{0.\,0}_{0.\,0} = 1$$

$$2\,\Pi^{0.\,1}_{0.\,1} = +2i + 1 + D$$

$$4\,\Pi^{0.\,2}_{0.\,0} = -4i^2 + D + D^2$$

$$8\,\Pi^{0.\,2}_{0.\,2} =$$
$$4i^2 + 9i + 4 + (+4i + 5)\,D + D^2$$

$$16\,\Pi^{0.\,3}_{0.\,1} =$$
$$-8i^3 - 14i^2 - 5i - 1 + (-4i^2 - i + 2)\,D + (+2i + 4)\,D^2 + D^3$$

$$48\,\Pi^{0.\,3}_{0.\,3} =$$
$$+8i^3 + 42i^2 + 65i + 27 + (+12i^2 + 45i + 38)\,D + (+6i + 12)\,D^2 + D^3$$

$$64\,\Pi^{0.\,4}_{0.\,0} =$$
$$+16i^4 - 17i^2 + (-16i^2 + 6)\,D + (-8i^2 + 11)\,D^2 + 6\,D^3 + D^4$$

$$96\,\Pi^{0.\,4}_{0.\,2} =$$
$$-16i^4 - 76i^3 - 112i^2 - 62i - 16 + (-16i^3 - 48i^2 - 29i - 2)\,D + (0i^2 + 15i + 23)\,D^2 + (+4i + 10)\,D^3 + D^4$$

$$384\,\Pi^{0.\,4}_{0.\,4} =$$
$$16i^4 + 152i^3 + 499i^2 + 646i + 256 + (+32i^3 + 240i^2 + 558i + 390)\,D + (+24i^2 + 126i + 155)\,D^2$$
$$+ (+8i + 22)\,D^3 + D^4$$

$$384\,\Pi^{0.\,5}_{0.\,1} =$$
$$+32i^5 + 96i^4 + 34i^3 - 47i^2 + 8i + 1 + (+16i^4 - 8i^3 - 87i^2 - 12i + 25)\,D + (-16i^3 - 60i^2 - 6i + 53)\,D^2$$
$$+ (-8i^2 + 6i + 39)\,D^3 + (+2i + 11)\,D^4 + D^5$$

$$768\,\Pi^{0.\,5}_{0.\,3} =$$
$$-32i^5 - 288i^4 - 926i^3 - 1\,317i^2 - 858i - 243 + (-48i^4 - 328i^3 - 713i^2 - 574i - 183)\,D$$
$$+ (-16i^3 - 36i^2 + 86i + 149)\,D^2 + (+8i^2 + 66i + 107)\,D^3 + (+6i + 19)\,D^4 + D^5$$

$$3\,840\,\Pi^{0.\,5}_{0.\,5} =$$
$$32i^5 + 480i^4 + 2\,710i^3 + 7\,055i^2 + 8\,174i + 3\,125 + (+80i^4 + 1\,000i^3 + 4\,435i^2 + 8\,130i + 5\,049)\,D$$
$$+ (+80i^3 + 780i^2 + 2\,410i + 2\,325)\,D^2 + (+40i^2 + 270i + 435)\,D^3 + (+10i + 35)\,D^4 + D^5$$

$$2\,304\,\Pi^{0.\,6}_{0.\,0} =$$
$$-64i^6 + 292i^4 - 364i^2 + (+144i^4 - 417i^2 + 120)\,D + (+48i^4 - 309i^2 + 274)\,D^2 + (-96i^2 + 225)\,D^3$$
$$+ (-12i^2 + 85)\,D^4 + 15\,D^5 + D^6$$

$$3\,072\,\Pi^{0.\,6}_{0.\,2} =$$
$$+64i^6 + 464i^5 + 1\,076i^4 + 877i^3 + 342i^2 + 320i + 64 + (+64i^5 + 272i^4 + 84i^3 - 463i^2 + 4i + 176)\,D$$
$$+ (-16i^4 - 232i^3 - 615i^2 - 171i + 318)\,D^2 + (-32i^3 - 128i^2 + 42i + 309)\,D^3$$
$$+ (-4i^2 + 37i + 121)\,D^4 + (+4i + 19)\,D^5 + D^6$$

$$7\,680\,\Pi^{0.\,6}_{0.\,4} =$$
$$-64i^6 - 928i^5 - 5\,180i^4 - 14\,120i^3 - 19\,788i^2 - 13\,864i - 4\,096$$
$$+ (-128i^5 - 1\,520i^4 - 6\,560i^3 - 12\,745i^2 - 11\,434i - 4\,392)\,D + (-80i^4 - 640i^3 - 1\,405i^2 - 400i + 674)\,D^2$$
$$+ (\;\;0i^3 + 160i^2 + 940i + 1\,265)\,D^3 + (+20i^2 + 170i + 325)\,D^4 + (+8i + 31)\,D^5 + D^6$$

$46\,080\,\Pi^{0.\,6}_{0.\,6} =$

$$+64\,i^6 + 1\,392\,i^5 + 12\,020\,i^4 + 52\,095\,i^3 + 117\,238\,i^2 + 125\,616\,i + 46\,656$$
$$+(+192\,i^5 + 3\,600\,i^4 + 25\,820\,i^3 + 87\,585\,i^2 + 138\,212\,i + 78\,960)\,\mathrm{D}$$
$$+(+240\,i^4 + 3\,720\,i^3 + 20\,745\,i^2 + 48\,855\,i + 40\,414)\,\mathrm{D}^2 + (+160\,i^3 + 1\,920\,i^2 + 7\,390\,i + 9\,045)\,\mathrm{D}^3$$
$$+(+60\,i^2 + 495\,i + 985)\,\mathrm{D}^4 + (12\,i + 51)\,\mathrm{D}^5 + \mathrm{D}^6$$

$18\,432\,\Pi^{0.\,7}_{0.\,1} =$

$$-128\,i^7 - 544\,i^6 + 8\,i^5 + 1\,670\,i^4 - 363\,i^3 - 2\,253\,i^2 + 75\,i - 1$$
$$+(-64\,i^6 + 144\,i^5 + 1\,276\,i^4 + 101\,i^3 - 2\,980\,i^2 - 211\,i + 758)\,\mathrm{D}$$
$$+(+96\,i^5 + 528\,i^4 - 126\,i^3 - 2\,370\,i^2 - 85\,i + 1\,867)\,\mathrm{D}^2 + (+48\,i^4 - 120\,i^3 - 975\,i^2 + 33\,i + 1\,729)\,\mathrm{D}^3$$
$$+(-24\,i^3 - 186\,i^2 + 65\,i + 787)\,\mathrm{D}^4 + (-12\,i^2 + 21\,i + 187)\,\mathrm{D}^5 + (+2\,i + 22)\,\mathrm{D}^6 + \mathrm{D}^7$$

$30\,720\,\Pi^{0.\,7}_{0.\,3} =$

$$+128\,i^7 + 1\,632\,i^6 + 7\,768\,i^5 + 17\,430\,i^4 + 20\,411\,i^3 + 15\,711\,i^2 + 9\,549\,i + 2\,187$$
$$+(+192\,i^6 + 1\,840\,i^5 + 5\,660\,i^4 + 5\,875\,i^3 + 2\,180\,i^2 + 4\,407\,i + 2\,826)\,\mathrm{D}$$
$$+(+32\,i^5 - 240\,i^4 - 3\,090\,i^3 - 7\,370\,i^2 - 2\,927\,i + 2\,223)\,\mathrm{D}^2 + (-80\,i^4 - 840\,i^3 - 2\,235\,i^2 - 225\,i + 2\,749)\,\mathrm{D}^3$$
$$+(-40\,i^3 - 130\,i^2 + 515\,i + 1\,455)\,\mathrm{D}^4 + (+4\,i^2 + 111\,i + 319)\,\mathrm{D}^5 + (+6\,i + 30)\,\mathrm{D}^6 + \mathrm{D}^7$$

$92\,160\,\Pi^{0.\,7}_{0.\,5} =$

$$-128\,i^7 - 2\,720\,i^6 - 23\,320\,i^5 - 103\,850\,i^4 - 257\,931\,i^3 - 357\,225\,i^2 - 259\,005\,i - 78\,125$$
$$+(-320\,i^6 - 5\,808\,i^5 - 41\,060\,i^4 - 143\,795\,i^3 - 263\,436\,i^2 - 246\,111\,i - 99\,630)\,\mathrm{D}$$
$$+(-288\,i^5 - 4\,080\,i^4 - 20\,590\,i^3 - 44\,250\,i^2 - 37\,881\,i - 10\,761)\,\mathrm{D}^2$$
$$+(-80\,i^4 - 440\,i^3 + 1\,645\,i^2 + 12\,305\,i + 15\,749)\,\mathrm{D}^3 + (+40\,i^3 + 750\,i^2 + 3\,845\,i + 5\,735)\,\mathrm{D}^4$$
$$+(+36\,i^2 + 345\,i + 775)\,\mathrm{D}^5 + (+10\,i + 46)\,\mathrm{D}^6 + \mathrm{D}^7$$

$645\,120\,\Pi^{0.\,7}_{0.\,7} =$

$$+128\,i^7 + 3\,808\,i^6 + 46\,648\,i^5 + 302\,470\,i^4 + 1\,107\,771\,i^3 + 2\,249\,499\,i^2 + 2\,271\,429\,i + 823\,543$$
$$+(+448\,i^6 + 11\,760\,i^5 + 123\,900\,i^4 + 665\,595\,i^3 + 1\,902\,572\,i^2 + 2\,700\,299\,i + 1\,447\,886)\,\mathrm{D}$$
$$+(+672\,i^5 + 15\,120\,i^4 + 131\,390\,i^3 + 547\,470\,i^2 + 1\,083\,565\,i + 803\,747)\,\mathrm{D}^2$$
$$+(+560\,i^4 + 10\,360\,i^3 + 69\,545\,i^2 + 199\,535\,i + 204\,729)\,\mathrm{D}^3 + (+280\,i^3 + 3\,990\,i^2 + 18\,375\,i + 27\,195)\,\mathrm{D}^4$$
$$+(+84\,i^2 + 819\,i + 1\,939)\,\mathrm{D}^5 + (+14\,i + 70)\,\mathrm{D}^6 + \mathrm{D}^7$$

$147\,456\,\Pi^{0.\,8}_{0\;\;0} =$

$$256\,i^8 - 3\,040\,i^6 + 13\,321\,i^4 - 16\,260\,i^2 + (-1\,024\,i^6 + 9\,088\,i^4 - 21\,052\,i^2 + 5\,048)\,\mathrm{D}$$
$$+(-256\,i^6 + 4\,432\,i^4 - 18\,386\,i^2 + 13\,068)\,\mathrm{D}^2 + (+960\,i^4 - 8\,340\,i^2 + 13\,132)\,\mathrm{D}^3 + (+96\,i^4 - 2\,158\,i^2 + 6\,769)\,\mathrm{D}^4$$
$$+(-288\,i^2 + 1\,960)\,\mathrm{D}^5 + (-16\,i^2 + 322)\,\mathrm{D}^6 + 28\,\mathrm{D}^7 + \mathrm{D}^8$$

$184\,320\,\Pi^{0.\,8}_{0.\,2} =$

$$-256\,i^8 + 2\,496\,i^7 - 7\,440\,i^6 + 4\,556\,i^5 + 4\,968\,i^4 + 11\,098\,i^3 - 20\,996\,i^2 - 2\,832\,i - 256$$
$$+(+256\,i^7 - 1\,280\,i^6 - 1\,696\,i^5 + 12\,720\,i^4 - 91\,i^3 - 27\,328\,i^2 - 1\,656\,i + 8\,656)\,\mathrm{D}$$
$$+(+128\,i^6 - 2\,064\,i^5 + 7\,360\,i^4 - 235\,i^3 - 21\,231\,i^2 + 914\,i + 16\,236)\,\mathrm{D}^2$$
$$+(-192\,i^5 + 960\,i^4 + 1\,980\,i^3 - 11\,310\,i^2 - 55\,i + 16\,340)\,\mathrm{D}^3 + (0\,i^4 + 660\,i^3 - 2\,855\,i^2 - 855\,i + 8\,549)\,\mathrm{D}^4$$
$$+(+48\,i^3 - 288\,i^2 - 445\,i + 2\,492)\,\mathrm{D}^5 + (-8\,i^2 - 75\,i + 398)\,\mathrm{D}^6 + (-4\,i + 32)\,\mathrm{D}^7 + \mathrm{D}^8$$

$368\,640\,\Pi^{0.8}_{0.4} =$

$$256\,i^8 - 4\,992\,i^7 + 38\,880\,i^6 - 156\,328\,i^5 + 355\,857\,i^4 - 491\,126\,i^3 + 449\,504\,i^2 - 266\,416\,i + 65\,536$$
$$+(-512\,i^7 + 8\,192\,i^6 - 49\,344\,i^5 + 141\,120\,i^4 - 210\,506\,i^3 + 209\,252\,i^2 + 198\,060\,i - 34\,992)\,\mathrm{D}$$
$$+(+256\,i^6 - 2\,208\,i^5 + 720\,i^4 + 35\,230\,i^3 - 86\,934\,i^2 + 36\,038\,i + 21\,664)\,\mathrm{D}^2$$
$$+(+128\,i^5 - 2\,880\,i^4 + 18\,920\,i^3 - 43\,420\,i^2 + 17\,614\,i + 25\,100)\,\mathrm{D}^3$$
$$+(-160\,i^4 + 1\,720\,i^3 - 4\,050\,i^2 - 5\,880\,i + 18\,849)\,\mathrm{D}^4 + (+32\,i^3 + 96\,i^2 - 2\,370\,i + 5\,464)\,\mathrm{D}^5$$
$$+(+16\,i^2 - 246\,i + 722)\,\mathrm{D}^6 + (+8\,i + 44)\,\mathrm{D}^7 + \mathrm{D}^8$$

$1\,290\,240\ \Pi^{0.8}_{0.6} =$

$$-256\,i^8 + 7\,488\,i^7 - 91\,280\,i^6 + 602\,532\,i^5 - 2\,342\,872\,i^4 + 5\,471\,214\,i^3 - 7\,485\,924\,i^2 + 5\,512\,464\,i - 1\,679\,616$$
$$+ (+768\,i^7 - 19\,712\,i^6 + 205\,856\,i^5 - 1\,128\,400\,i^4 + 3\,496\,703\,i^3 - 6\,152\,384\,i^2 + 5\,813\,688\,i - 2\,386\,512)\,\mathrm{D}$$
$$+ (-896\,i^6 + 19\,152\,i^5 - 159\,040\,i^4 + 648\,655\,i^3 - 1\,356\,663\,i^2 + 1\,397\,326\,i - 613\,908)\,\mathrm{D}^2$$
$$+ (+448\,i^5 - 6\,720\,i^4 + 31\,220\,i^3 - 28\,350\,i^2 - 115\,157\,i + 181\,972)\,\mathrm{D}^3$$
$$+ (\ \ 0\,i^4 - 1\,540\,i^3 + 19\,985\,i^2 - 82\,285\,i + 106\,309)\,\mathrm{D}^4 + (-112\,i^3 + 2\,016\,i^2 - 11\,095\,i + 18\,844)\,\mathrm{D}^5$$
$$+ (+56\,i^2 - 609\,i + 1\,582)\,\mathrm{D}^6 + (-12\,i + 64)\,\mathrm{D}^7 + \mathrm{D}^8$$

$10\,321\,920\ \Pi^{0.8}_{0.8} =$

$$+256\,i^8 - 9\,984\,i^7 + 164\,640\,i^6 - 1\,490\,384\,i^5 + 8\,034\,537\,i^4 - 26\,106\,892\,i^3 + 49\,046\,156\,i^2 + 47\,239\,088\,i + 16\,777\,216$$
$$+ (-1\,024\,i^7 + 35\,840\,i^6 - 520\,576\,i^5 + 4\,045\,440\,i^4 - 18\,031\,860\,i^3 + 45\,627\,204\,i^2 - 59\,755\,672\,i + 30\,461\,872)\,\mathrm{D}$$
$$+ (+1\,792\,i^6 - 55\,104\,i^5 + 684\,880\,i^4 - 4\,381\,860\,i^3 + 15\,119\,902\,i^2 - 26\,434\,436\,i + 18\,049\,548)\,\mathrm{D}^2$$
$$+ (-1\,792\,i^5 + 47\,130\,i^4 - 479\,920\,i^3 + 2\,367\,820\,i^2 - 5\,615\,428\,i + 5\,079\,116)\,\mathrm{D}^3$$
$$+ (+1\,120\,i^4 - 24\,080\,i^3 + 188\,930\,i^2 - 638\,400\,i + 779\,569)\,\mathrm{D}^4 + (-448\,i^3 + 7\,392\,i^2 - 39\,620\,i + 68\,712)\,\mathrm{D}^5$$
$$+ (+112\,i^2 - 1\,260\,i + 3\,458)\,\mathrm{D}^6 + (-16\,i + 92)\,\mathrm{D}^7 + \mathrm{D}^8$$

III.—*Operators relating to both Planets.*

$$4\ \Pi^{1.1}_{-1.1} = +4\,i^2 + 2\,i - \mathrm{D} - \mathrm{D}^2$$

$$4\ \Pi^{1.1}_{1.1} = -4\,i^2 - 2\,i + (-4\,i - 1)\,\mathrm{D} - \mathrm{D}^2$$

$$8\ \Pi^{1.2}_{1.0} = +8\,i^3 + (4\,i^2 - 2\,i)\,\mathrm{D} + (-2\,i - 1)\,\mathrm{D}^2 - \mathrm{D}^3$$

$$16\ \Pi^{1.2}_{-1.2} = +8\,i^3 + 18\,i^2 + 8\,i + (+4\,i^2 + i - 4)\,\mathrm{D} + (-2\,i - 5)\,\mathrm{D}^2 - \mathrm{D}^3$$

$$16\ \Pi^{1.2}_{1.2} = -8\,i^3 - 18\,i^2 - 8\,i + (-12\,i^2 - 19\,i - 4)\,\mathrm{D} + (-6\,i - 5)\,\mathrm{D}^2 - \mathrm{D}^3$$

$$16\ \Pi^{2.1}_{-2.1} = +8\,i^3 + 14\,i^2 + 5\,i + (-4\,i^2 - 5\,i - 3)\,\mathrm{D} + (-2\,i - 2)\,\mathrm{D}^2 + \mathrm{D}^3$$

$$8\ \Pi^{2.1}_{0.1} = -8\,i^3 - 4\,i^2 + (-4\,i^2 + 2\,i + 1)\,\mathrm{D} + (2\,i + 2)\,\mathrm{D}^2 + \mathrm{D}^3$$

$$16\ \Pi^{2.1}_{2.1} = +8\,i^3 - 6\,i^2 - 5\,i + (+12\,i^2 - 7\,i - 3)\,\mathrm{D} + (+6\,i - 2)\,\mathrm{D}^2 + \mathrm{D}^3$$

$$32\ \Pi^{1.3}_{-1.1} = -16\,i^4 - 28\,i^3 - 10\,i^2 - 2\,i + (0\,i^3 + 12\,i^2 + 9\,i + 1)\,\mathrm{D} + (+8\,i^2 + 9\,i - 2)\,\mathrm{D}^2 + (0\,i - 4)\,\mathrm{D}^3 - \mathrm{D}^4$$

$$32\ \Pi^{1.3}_{1.1} = +16\,i^4 + 28\,i^3 + 10\,i^2 + 2\,i + (+15\,i^3 + 16\,i^2 + 1\,i + 1)\,\mathrm{D} + (0\,i^2 - 7\,i - 2)\,\mathrm{D}^2 + (-4\,i - 4)\,\mathrm{D}^3 - \mathrm{D}^4$$

$$96\ \Pi^{1.3}_{-1.3} = +16\,i^4 + 84\,i^3 + 130\,i^2 + 54\,i + (+16\,i^3 + 48\,i^2 + 11\,i - 27)\,\mathrm{D} + (+0\,i^2 - 21\,i - 38)\,\mathrm{D}^2$$
$$+ (-4\,i - 12)\,\mathrm{D}^3 - \mathrm{D}^4$$

$$96\ \Pi^{1.3}_{1.3} = -16\,i^4 - 84\,i^3 - 130\,i^2 - 54\,i + (-32\,i^3 - 132\,i^2 - 141\,i - 27)\,\mathrm{D} + (-24\,i^2 - 69\,i - 38)\,\mathrm{D}^2$$
$$+ (-8\,i - 12)\,\mathrm{D}^3 - \mathrm{D}^4$$

$$16\ \Pi^{2.2}_{0.0} = +16\,i^4 - 8\,i^2\,\mathrm{D} + (-8\,i^2 + 1)\,\mathrm{D}^2 + 2\,\mathrm{D}^3 + \mathrm{D}^4$$

$$32\ \Pi^{2.2}_{2.0} = -16\,i^4 + 20\,i^3 + (-16\,i^3 + 16\,i^2 - 5\,i + 0)\,\mathrm{D} + (0\,i^2 - i - 3)\,\mathrm{D}^2 + (4\,i - 2)\,\mathrm{D}^3 + \mathrm{D}^4$$

$$64\ \Pi^{2.2}_{-2.2} = +16\,i^4 + 56\,i^3 + 61\,i^2 + 20\,i + (0\,i^3 - 8\,i^2 - 18\,i - 12)\,\mathrm{D} + (-8\,i^2 - 18\,i - 11)\,\mathrm{D}^2 + 2\,\mathrm{D}^3 + \mathrm{D}^4$$

$$32\ \Pi^{2.2}_{0.2} = -16\,i^4 - 36\,i^3 - 16\,i^2 + (-16\,i^3 - 16\,i^2 + 9\,i + 4)\,\mathrm{D} + (0\,i^2 + 13\,i + 9)\,\mathrm{D}^2 + (4\,i + 6)\,\mathrm{D}^3 + \mathrm{D}^4$$

$$64\ \Pi^{2.2}_{2.2} = +16\,i^4 + 16\,i^3 - 29\,i^2 - 20\,i + (+32\,i^3 + 24\,i^2 - 36\,i - 12)\,\mathrm{D} + (+24\,i^2 + 12\,i - 11)\,\mathrm{D}^2 + (+8\,i + 2)\,\mathrm{D}^3 + \mathrm{D}^4$$

$$96\ \Pi^{3.1}_{-3.1} = +16\,i^4 + 68\,i^3 + 82\,i^2 + 26\,i + (-16\,i^3 - 48\,i^2 - 41\,i - 17)\,\mathrm{D} + (0\,i^2 - 9\,i - 8)\,\mathrm{D}^2 + (+4\,i + 8)\,\mathrm{D}^3 - \mathrm{D}^4$$

$$32\,\Pi^{3.1}_{-1.1} = -16i^4 - 28i^3 - 14i^2 - 2i + (0i^3 + 4i^2 + 9i + 3)\,D + (+8i^2 + 9i + 4)\,D^2 + (0i + 0)\,D^3 - D^4$$

$$32\,\Pi^{3.1}_{1.1} = +16i^4 - 12i^3 - 6i^2 + 2i + (16i^3 - 16i^2 + 3i + 3)\,D + (0i^2 - 5i + 4)\,D^2 + (-4i + 0)\,D^3 - D^4$$

$$96\,\Pi^{3.1}_{3.1} = -16i^4 + 52i^3 - 22i^2 - 26i + (-32i^3 + 84i^2 - 27i - 17)\,D + (-24i^2 + 45i - 8)\,D^2 + (-8i + 8)\,D^3 - D^4$$

$$128\,\Pi^{1.4}_{1.0} = -32i^5 + 34i^3 + (-16i^4 + 32i^3 + 17i^2 - 12i)\,D + (+16i^3 + 16i^2 - 22i - 6)\,D^2 + (+8i^2 - 12i - 11)\,D^3$$
$$+ (-2i - 6)\,D^4 - D^5$$

$$192\,\Pi^{1.4}_{-1.2} = -32i^5 - 152i^4 - 224i^3 - 124i^2 - 32i + (-16i^4 - 20i^3 + 54i^2 + 58i + 16)\,D$$
$$+ (+16i^3 + 78i^2 + 75i + 2)\,D^2 + (+8i^2 + 5i - 23)\,D^3 + (-2i - 10)\,D^4 - D^6$$

$$192\,\Pi^{1.4}_{1.2} = +32i^5 + 152i^4 + 224i^3 + 124i^2 + 32i + (+48i^4 + 172i^3 + 170i^2 + 66i + 16)\,D$$
$$+ (+16i^3 + 18i^2 - 17i + 2)\,D^2 + (-8i^2 - 35i - 23)\,D^3 + (-6i - 10)\,D^4 - D^5$$

$$768\,\Pi^{1.4}_{-1.4} = +32i^5 + 304i^4 + 998i^3 + 1\,292i^2 + 512i + (+48i^4 + 328i^3 + 617i^2 + 134i - 256)\,D$$
$$+ (+16i^3 + 12i^2 - 248i - 390)\,D^2 + (-8i^2 - 82i - 155)\,D^3 + (-6i - 22)\,D^4 - D^5$$

$$768\,\Pi^{1.4}_{1.4} = -32i^5 - 304i^4 - 998i^3 - 1\,292i^2 - 512i + (-80i^4 - 632i^3 - 1\,615i^2 - 1\,426i - 256)\,D$$
$$+ (-80i^3 - 492i^2 - 868i - 390)\,D^2 + (-40i^2 - 170i - 155)\,D^3 + (-10i - 22)\,D^4 - D^6$$

$$128\,\Pi^{2.3}_{-2.1} = -32i^5 - 96i^4 - 90i^3 - 29i^2 - 5i + (+16i^4 + 56i^3 + 65i^2 + 29i + 3)\,D + (+16i^3 + 28i^2 + 10i - 7)\,D^2$$
$$+ (-8i^2 - 18i - 10)\,D^3 + (-2i + 1)\,D^4 + D^5$$

$$64\,\Pi^{2.3}_{0.1} = +32i^5 + 56i^4 + 20i^3 + 4i^2 + (+16i^4 - 4i^3 - 22i^2 - 5i - 1)\,D + (-16i^3 - 34i^2 - 6i + 1)\,D^2$$
$$+ (-8i^2 + i + 6)\,D^3 + (+2i + 5)\,D^4 + D^6$$

$$128\,\Pi^{2.3}_{2.1} = -32i^5 - 16i^4 + 50i^3 + 21i^2 + 5i + (-48i^4 - 16i^3 + 35i^2 + i + 3)\,D + (-16i^3 + 0i^2 - 14i - 7)\,D^2$$
$$+ (+8i^2 + 4i - 10)\,D^3 + (+6i + 1)\,D^4 + D^5$$

$$384\,\Pi^{2.3}_{-2.3} = +32i^5 + 208i^4 + 470i^3 + 433i^2 + 135i + (+16i^4 + 48i^3 - 9i^2 - 113i - 81)\,D$$
$$+ (-16i^3 - 96i^2 - 162i - 87)\,D^2 + (-8i^2 - 16i + 2)\,D^3 + (+2i + 9)\,D^4 + D^6$$

$$192\,\Pi^{2.3}_{0.3} = -32i^5 - 168i^4 - 260i^3 - 108i^2 + (-48i^4 - 172i^3 - 110i^2 + 65i + 27)\,D + (-16i^3 + 6i^2 + 110i + 65)\,D^2$$
$$+ (+8i^2 + 51i + 50)\,D^3 + (+6i + 13)\,D^4 + D^6$$

$$384_8\,\Pi^{2.3}_{2.3} = +32i^5 + 128i^4 + 50i^3 - 217i^2 - 135i + (+80i^4 + 264i^3 + 61i^2 - 277i - 81)\,D$$
$$+ (+80i^3 + 204i^2 + 22i - 87)\,D^2 + (+40i^2 + 70i + 2)\,D^3 + (+10i + 9)\,D^4 + D^5$$

$$64\,\Pi^{3.2}_{1.0} = -32i^5 + 40i^4 - 8i^3 + (-16i^4 + 28i^3 - 22i^2 + 2i + 0)\,D + (+16i^3 - 10i^2 - 3i + 3)\,D^2$$
$$+ (+8i^2 - 7i + 4)\,D^3 - 2i\,D^4 - D^5$$

$$192\,\Pi^{3.2}_{3.0} = +32i^5 - 120i^4 + 104i^3 + (+48i^4 - 140i^3 + 98i^2 - 26i + 0)\,D + (+16i^3 - 18i^2 + 7i - 17)\,D^2$$
$$+ (-8i^2 + 27i - 8)\,D^3 + (-6i + 8)\,D^4 - D^5$$

$$384\,\Pi^{3.2}_{-3.2} = +32i^5 + 192i^4 + 406i^3 + 354i^2 + 104i + (-16i^4 - 80i^3 - 159i^2 - 155i - 68)\,D$$
$$+ (-16i^3 - 72i^2 - 102i - 49)\,D^2 + (+8i^2 + 24i + 24)\,D^3 + (+2i + 4)\,D^4 - D^5$$

$$128\,\Pi^{3.2}_{-1.2} = -32i^5 - 112i^4 - 130i^3 - 58i^2 - 8i + (-16i^4 - 24i^3 + 15i^2 + 37i + 12)\,D + (+16i^3 + 52i^2 + 52i + 19)\,D^2$$
$$+ (+8i^2 + 10i + 4)\,D^3 + (-2i - 4)\,D^4 - D^5$$

$$128\,\Pi^{3.2}_{1.2} = +32i^5 + 32i^4 - 50i^3 - 22i^2 + 8i + (+48i^4 + 16i^3 - 59i^2 + 17i + 12)\,D + (+16i^3 - 24i^2 - 10i + 19)\,D^2$$
$$+ (-8i^2 - 20i + 4)\,D^3 + (-6i - 4)\,D^4 - D^5$$

$$384\,\Pi^{3.2}_{3.2} = -32i^5 + 48i^4 + 134i^3 - 114i^2 - 104i + (-80i^4 + 104i^3 + 227i^2 - 151i - 68)\,D$$
$$+ (-80i^3 + 84i^2 + 128i - 49)\,D^2 + (-40i^2 + 30i + 24)\,D^3 + (-10i + 4)\,D^4 - D^5$$

$$768\,\Pi^{4.\,1}_{-4.\,1} = +32i^5+256i^4+686i^3+695i^2+206i+(-48i^4-296i^3-569i^2-408i-142)\,\mathrm{D}$$
$$+(+16i^3+36i^2-38i-47)\,\mathrm{D}^2+(+8i^2+58i+77)\,\mathrm{D}^3+(-6i-17)\,\mathrm{D}^4+\mathrm{D}^5$$

$$192\,\Pi^{4.\,1}_{-2.\,1} = -32i^5-136i^4-188i^3-108i^2-22i+(+16i^4+52i^3+78i^2+69i+22)\,\mathrm{D}$$
$$+(+16i^3+54i^2+48i+21)\,\mathrm{D}^2+(-8i^2-13i-7)\,\mathrm{D}^3+(-2i-5)\,\mathrm{D}^4+\mathrm{D}^5$$

$$128\,\Pi^{4.\,1}_{0.\,1} = +32i^5+16i^4-18i^3-9i^2+(+16i^4+0i^3-9i^2+4i+2)\,\mathrm{D}+(-16i^3-8i^2-2i+1)\,\mathrm{D}^2$$
$$+(-8i^2-4i-3)\,\mathrm{D}^3+(+2i-1)\,\mathrm{D}^4+\mathrm{D}^5$$

$$192\,\Pi^{4.\,1}_{2.\,1} = -32i^5+104i^4-68i^3-20i^2+22i+(-48i^4+140i^3-110i^2+19i+22)\,\mathrm{D}$$
$$+(-16i^3+42i^2-52i+21)\,\mathrm{D}^2+(+8i^2-11i-7)\,\mathrm{D}^3+(+6i-5)\,\mathrm{D}^4+\mathrm{D}^5$$

$$768\,\Pi^{4.\,1}_{4.\,1} = +32i^5-224i^4+446i^3-129i^2-206i+(+80i^4-472i^3+751i^2-160i-142)\,\mathrm{D}$$
$$+(+80i^3-372i^2+418i-47)\,\mathrm{D}^2+(+40i^2-130i+77)\,\mathrm{D}^3+(+10i-17)\,\mathrm{D}^4+\mathrm{D}^5$$

$$768\,\Pi^{1.\,5}_{-1.\,1} = +64i^6+192i^5+68i^4-94i^3+16i^2+2i+(0i^5-112i^4-208i^3+23i^2+42i-1)\,\mathrm{D}$$
$$+(-48i^4-112i^3+75i^2+118i-25)\,\mathrm{D}^2+(0i^3+72i^2+84i-53)\,\mathrm{D}^3+(+12i^2+16i-39)\,\mathrm{D}^4$$
$$+(0i-11)\,\mathrm{D}^5-\mathrm{D}^6$$

$$768\,\Pi^{1.\,5}_{1.\,1} = -64i^6-192i^5-68i^4+94i^3-16i^2-2i+(-64i^5-80i^4+140i^3+71i^2-58i-1)\,\mathrm{D}$$
$$+(+16i^4+128i^3+99i^2-94i-25)\,\mathrm{D}^2+(+32i^3+48i^2-72i-53)\,\mathrm{D}^3+(+4i^2-28i-39)\,\mathrm{D}^4$$
$$+(-4i-11)\,\mathrm{D}^5-\mathrm{D}^6$$

$$1\,536\,\Pi^{1.\,5}_{-1.\,3} = -64i^6-576i^5-1\,852i^4-2\,634i^3-1\,716i^2-486i+(-64i^5-368i^4-500i^3+169i^2+492i+243)\,\mathrm{D}$$
$$+(+16i^4+256i^3+885i^2+872i+183)\,\mathrm{D}^2+(+32i^3+168i^2+128i-149)\,\mathrm{D}^3+(+4i^2-28i-107)\,\mathrm{D}^4$$
$$+(-4i-19)\,\mathrm{D}^5-\mathrm{D}^6$$

$$1\,536\,\Pi^{1.\,5}_{1.\,3} = +64i^6+576i^5+1\,852i^4+2\,634i^3+1\,716i^2+486i$$
$$+(+128i^5+944i^4+2\,352i^3+2\,465i^2+1\,224i+243)\,\mathrm{D}+(+80i^4+400i^3+541i^2+276i+183)\,\mathrm{D}^2$$
$$+(0i^3-96i^2-300i-149)\,\mathrm{D}^3+(-20i^2-104i-107)\,\mathrm{D}^4+(-8i-19)\,\mathrm{D}^5-\mathrm{D}^6$$

$$7\,680\,\Pi^{1.\,5}_{-1.\,5} = +64i^6+960i^5+5\,420i^4+14\,110i^3+16\,348i^2+6\,250i$$
$$+(+128i^5+1\,520i^4+6\,160i^3+9\,205i^2+1\,924i-3\,125)\,\mathrm{D}$$
$$+(+80i^4+560i^3+385i^2-3\,480i-5\,049)\,\mathrm{D}^2+(0i^3-240i^2-1\,540i-2\,325)\,\mathrm{D}^3$$
$$+(-20i^2-200i-435)\,\mathrm{D}^4+(-8i-35)\,\mathrm{D}^5-\mathrm{D}^6$$

$$7\,680\,\Pi^{1.\,5}_{1.\,5} = -64i^6-960i^5-5\,420i^4-14\,110i^3-16\,348i^2-6\,250i$$
$$+(-192i^5-2\,480i^4-11\,580i^3-23\,315i^2-18\,272i-3\,125)\,\mathrm{D}$$
$$+(-240i^4-2\,560i^3-9\,255i^2-12\,780i-5\,049)\,\mathrm{D}^2+(-160i^3-1\,320i^2-3\,280i-2\,325)\,\mathrm{D}^3$$
$$+(-60i^2-340i-435)\,\mathrm{D}^4+(-12i-35)\,\mathrm{D}^5-\mathrm{D}^6$$

$$256\,\Pi^{2.\,4}_{0.\,0} = -64i^6+68i^4+(+80i^4-41i^2)\,\mathrm{D}+(+48i^4-77i^2+6)\,\mathrm{D}^2+(-48i^2+17)\,\mathrm{D}^3+(-12i^2+17)\,\mathrm{D}^4$$
$$-7\,\mathrm{D}^5+\mathrm{D}^6$$

$$512\,\Pi^{2.\,4}_{2.\,0} = +64i^6-80i^5-68i^4+85i^3+0i^2+0i+(+64i^5-112i^4+12i^3+75i^2-30i+0)\,\mathrm{D}$$
$$+(-16i^4-24i^3+75i^2-31i-18)\,\mathrm{D}^2+(-32i^3+32i^2+14i-27)\,\mathrm{D}^3+(-4i^2+19i-7)\,\mathrm{D}^4$$
$$+(+4i+3)\,\mathrm{D}^5+\mathrm{D}^6$$

$$768\,\Pi^{2.\,4}_{-2.\,2} = -64i^6-384i^5-828i^4-808i^3-374i^2-80i+(0i^5+80i^4+320i^3+431i^2+240i+48)\,\mathrm{D}$$
$$+(+48i^4+224i^3+315i^2+148i-10)\,\mathrm{D}^2+(0i^3-48i^2-116i-71)\,\mathrm{D}^3+(-12i^2-32i-7)\,\mathrm{D}^4$$
$$+(0i+7)\,\mathrm{D}^5+\mathrm{D}^6$$

$$\begin{aligned}384\,\Pi^{2.\,4}_{0.\,2} = & +64 i^6 + 304 i^5 + 448 i^4 + 248 i^3 + 64 i^2 + 0 i + (+64 i^5 + 176 i^4 + 40 i^3 - 104 i^2 - 62 i - 16)\,\mathrm{D} \\ & + (-16 i^4 - 152 i^3 - 252 i^2 - 91 i - 18)\,\mathrm{D}^2 + (-32 i^3 - 88 i^2 - 14 i + 21)\,\mathrm{D}^3 + (-4 i^2 + 19 i + 33)\,\mathrm{D}^4 \\ & + (+4 i + 11)\,\mathrm{D}^5 + \mathrm{D}^6\end{aligned}$$

$$\begin{aligned}768\,\Pi^{2.\,4}_{2.\,2} = & -64 i^6 - 224 i^5 - 68 i^4 + 312 i^3 + 246 i^2 + 80 i + (-128 i^5 - 368 i^4 - 96 i^3 + 225 i^2 + 132 i + 48)\,\mathrm{D} \\ & + (-80 i^4 - 160 i^3 - 67 i^2 - 98 i - 10)\,\mathrm{D}^2 + (0 i^3 + 32 i^2 - 32 i - 71)\,\mathrm{D}^3 + (+20 i^2 + 38 i - 7)\,\mathrm{D}^4 \\ & + (+8 i + 7)\,\mathrm{D}^5 + \mathrm{D}^6\end{aligned}$$

$$\begin{aligned}3\,072\,\Pi^{2.\,4}_{-2.\,4} = & +64 i^6 + 688 i^5 + 2\,756 i^4 + 5\,079 i^3 + 4\,254 i^2 + 1\,280 i + (+64 i^5 + 464 i^4 + 980 i^3 + 269 i^2 - 1\,012 i - 768)\,\mathrm{D} \\ & + (-16 i^4 - 280 i^3 - 1\,203 i^2 - 1\,813 i - 914)\,\mathrm{D}^2 + (-32 i^3 - 208 i^2 - 330 i - 75)\,\mathrm{D}^3 + (-4 i^2 + 19 i + 89)\,\mathrm{D} \\ & + (4 i + 19)\,\mathrm{D}^5 + \mathrm{D}^6\end{aligned}$$

$$\begin{aligned}1\,536\,\Pi^{2.\,4}_{0\;\,4} = & -64 i^6 - 608 i^5 - 1\,996 i^4 - 2\,584 i^3 - 1\,024 i^2 + 0 i + (-128 i^5 - 944 i^4 - 2\,080 i^3 - 1\,061 i^2 + 646 i + 256)\,\mathrm{D} \\ & + (-80 i^4 - 320 i^3 + 119 i^2 + 1\,204 i + 646)\,\mathrm{D}^2 + (0 i^3 + 176 i^2 + 684 i + 545)\,\mathrm{D}^3 + (+20 i^2 + 134 i + 177)\,\mathrm{D}^4 \\ & + (+8 i + 23)\,\mathrm{D}^5 + \mathrm{D}^6\end{aligned}$$

$$\begin{aligned}3\,072\,\Pi^{2.\,4}_{2.\,4} = & +64 i^6 + 528 i^5 + 1\,236 i^4 + 89 i^3 - 2\,206 i^2 - 1\,280 i \\ & + (+192 i^5 + 1\,360 i^4 + 2\,572 i^3 - 143 i^2 - 2\,864 i - 768)\,\mathrm{D} + (+240 i^4 + 1\,400 i^3 + 2\,001 i^2 - 243 i - 914)\,\mathrm{D}^2 \\ & + (+160 i^3 + 720 i^2 + 690 i - 75)\,\mathrm{D}^3 + (+60 i^2 + 185 i + 89)\,\mathrm{D}^4 + (+12 i + 19)\,\mathrm{D}^5 + \mathrm{D}^6\end{aligned}$$

$$\begin{aligned}768\,\Pi^{3.\,3}_{-3.\,1} = & -64 i^6 - 352 i^5 - 668 i^4 - 522 i^3 - 160 i^2 - 26 i + (+64 i^5 + 304 i^4 + 540 i^3 + 449 i^2 + 170 i + 17)\,\mathrm{D} \\ & + (+16 i^4 + 80 i^3 + 93 i^2 + 4 i - 43)\,\mathrm{D}^2 + (-32 i^3 - 112 i^2 - 132 i - 49)\,\mathrm{D}^3 + (+4 i^2 + 10 i + 17)\,\mathrm{D}^4 \\ & + (+4 i + 5)\,\mathrm{D}^5 - \mathrm{D}^6\end{aligned}$$

$$\begin{aligned}256\,\Pi^{3.\,3}_{-1.\,1} = & +64 i^6 + 192 i^5 + 196 i^4 + 86 i^3 + 20 i^2 + 2 i + (0 i^5 - 48 i^4 - 112 i^3 - 89 i^2 - 24 i - 3)\,\mathrm{D} \\ & + (-48 i^4 - 112 i^3 - 77 i^2 - 8 i + 5)\,\mathrm{D}^2 + (0 i^3 + 24 i^2 + 32 i + 15)\,\mathrm{D}^3 + (+12 i^2 + 16 i + 5)\,\mathrm{D}^4 - 3\,\mathrm{D}^5 - \mathrm{D}^6\end{aligned}$$

$$\begin{aligned}256\,\Pi^{3.\,3}_{1.\,1} = & -64 i^6 - 32 i^5 + 84 i^4 + 14 i^3 + 0 i^2 - 2 i + (-64 i^5 + 16 i^4 + 44 i^3 - 43 i^2 - 6 i - 3)\,\mathrm{D} \\ & + (+16 i^4 + 48 i^3 - 39 i^2 - 8 i + 5)\,\mathrm{D}^2 + (+32 i^3 + 8 i^2 - 12 i + 15)\,\mathrm{D}^3 + (+4 i^2 - 10 i + 5)\,\mathrm{D}^4 \\ & + (-4 i - 3)\,\mathrm{D}^5 - \mathrm{D}^6\end{aligned}$$

$$\begin{aligned}768\,\Pi^{3.\,3}_{3.\,1} = & +64 i^6 - 128 i^5 - 172 i^4 + 222 i^3 + 100 i^2 + 26 i + (+128 i^5 - 208 i^4 - 208 i^3 + 171 i^2 + 0 i + 17)\,\mathrm{D} \\ & + (+80 i^4 - 80 i^3 - 17 i^2 - 60 i - 43)\,\mathrm{D}^2 + (+0 i^3 + 32 i^2 + 56 i - 49)\,\mathrm{D}^3 + (-20 i^2 + 28 i + 17)\,\mathrm{D}^4 \\ & + (-8 i + 5)\,\mathrm{D}^5 - \mathrm{D}^6\end{aligned}$$

$$\begin{aligned}2\,304\,\Pi^{3.\,3}_{-3.\,3} = & +64 i^6 + 576 i^5 + 1\,988 i^4 + 3\,258 i^3 + 2\,500 i^2 + 702 i + (0 i^5 - 48 i^4 - 336 i^3 - 873 i^2 - 1\,008 i - 459)\,\mathrm{D} \\ & + (-48 i^4 - 336 i^3 - 861 i^2 - 960 i - 403)\,\mathrm{D}^2 + (0 i^3 + 24 i^2 + 96 i + 111)\,\mathrm{D}^3 \\ & + (+12 i^2 + 48 i + 53)\,\mathrm{D}^4 - 3\,\mathrm{D}^5 - \mathrm{D}^6\end{aligned}$$

$$\begin{aligned}768\,\Pi^{3.\,3}_{-1.\,3} = & -64 i^6 - 416 i^5 - 956 i^4 - 950 i^3 - 400 i^2 - 54 i + (-64 i^5 - 272 i^4 - 284 i^3 + 89 i^2 + 254 i + 81)\,\mathrm{D} \\ & + (+16 i^4 + 176 i^3 + 453 i^2 + 420 i + 141)\,\mathrm{D}^2 + (+32 i^3 + 128 i^2 + 132 i + 47)\,\mathrm{D}^3 + (+4 i^2 - 10 i - 23)\,\mathrm{D}^4 \\ & + (-4 i - 11)\,\mathrm{D}^5 - \mathrm{D}^6\end{aligned}$$

$$\begin{aligned}768\,\Pi^{3.\,3}_{1.\,3} = & +64 i^6 + 256 i^5 + 116 i^4 - 350 i^3 - 140 i^2 + 54 i + (+128 i^5 + 368 i^4 - 48 i^3 - 381 i^2 + 136 i + 81)\,\mathrm{D} \\ & + (+80 i^4 + 80 i^3 - 233 i^2 - 20 i + 141)\,\mathrm{D}^2 + (0 i^3 - 112 i^2 - 136 i + 47)\,\mathrm{D}^3 + (-20 i^2 - 68 i - 23)\,\mathrm{D}^4 \\ & + (-8 i - 11)\,\mathrm{D}^5 - \mathrm{D}^6\end{aligned}$$

$$\begin{aligned}2\,304\,\Pi^{3.\,3}_{3.\,3} = & -64 i^6 - 96 i^5 + 532 i^4 + 642 i^3 - 880 i^2 - 702 i + (-192 i^5 - 240 i^4 + 1\,204 i^3 + 1\,077 i^2 - 1\,202 i - 459)\,\mathrm{D} \\ & + (-240 i^4 - 240 i^3 + 1\,017 i^2 + 600 i - 403)\,\mathrm{D}^2 + (-160 i^3 - 120 i^2 + 380 i + 111)\,\mathrm{D}^3 \\ & + (-60 i^2 - 30 i + 53)\,\mathrm{D}^4 + (-12 i - 3)\,\mathrm{D}^5 - \mathrm{D}^6\end{aligned}$$

$$256\,\Pi^{4.\,2}_{0.\,0} = -64 i^6 + 36 i^4 + (+16 i^4 - 17 i^2)\,\mathrm{D} + (+48 i^4 - 5 i^2 + 2)\,\mathrm{D}^2 + \mathrm{D}^3 + (-12 i^2 - 3)\,\mathrm{D}^4 - \mathrm{D}^5 + \mathrm{D}^6$$

$$\begin{aligned}384\,\Pi^{4.2}_{2.0}=&+64\,i^6-240\,i^5+256\,i^4-88\,i^3+0\,i^2+0\,i+(+64\,i^5-208\,i^4+248\,i^3-152\,i^2+22\,i+0)\,\mathrm{D}\\&+(-16\,i^4+56\,i^3-12\,i^2-25\,i+22)\,\mathrm{D}^2+(-32\,i^3+72\,i^2-50\,i+21)\,\mathrm{D}^3+(-4\,i^2+i-7)\,\mathrm{D}^4\\&+(4\,i-5)\,\mathrm{D}^5+\mathrm{D}^6\end{aligned}$$

$$\begin{aligned}1\,536\,\Pi^{4.2}_{4.0}=&-64\,i^6+480\,i^5-1\,132\,i^4+824\,i^3+0\,i^2+0\,i+(-128\,i^5+784\,i^4-1\,440\,i^3+851\,i^2-206\,i+0)\,\mathrm{D}\\&+(-80\,i^4+320\,i^3-289\,i^2+124\,i-142)\,\mathrm{D}^2+(-0\,i^3-96\,i^2+228\,i-47)\,\mathrm{D}^3+(+20\,i^2-94\,i+77)\,\mathrm{D}^4\\&+(+8\,i-17)\,\mathrm{D}^5+\mathrm{D}^6\end{aligned}$$

$$\begin{aligned}3\,072\,\Pi^{4.2}_{-4.2}=&+64\,i^6+624\,i^5+2\,276\,i^4+3\,851\,i^3+2\,986\,i^2+824\,i\\&+(-64\,i^5-496\,i^4-1\,444\,i^3-2\,067\,i^2-1\,568\,i-568)\,\mathrm{D}+(-16\,i^4-184\,i^3-603\,i^2-749\,i-330)\,\mathrm{D}^2\\&+(+32\,i^3+192\,i^2+366\,i+261)\,\mathrm{D}^3+(-4\,i^2-i+9)\,\mathrm{D}^4+(-4\,i-13)\,\mathrm{D}^5+\mathrm{D}^6\end{aligned}$$

$$\begin{aligned}768\,\Pi^{4.2}_{-2.2}=&-64\,i^6-384\,i^5-860\,i^4-904\,i^3-454\,i^2-88\,i+(0\,i^5+16\,i^4+128\,i^3+295\,i^2+276\,i+88)\,\mathrm{D}\\&+(+48\,i^4+224\,i^3+387\,i^2+304\,i+106)\,\mathrm{D}^2+(0\,i^3+0\,i^2-12\,i-7)\,\mathrm{D}^3+(-12\,i^2-32\,i-27)\,\mathrm{D}^4-\mathrm{D}^5+\mathrm{D}^6\end{aligned}$$

$$\begin{aligned}512\,\Pi^{4.2}_{0.2}=&+64\,i^6+144\,i^5+28\,i^4-81\,i^3-36\,i^2+0\,i+(+64\,i^5+80\,i^4-36\,i^3-37\,i^2+18\,i+8)\,\mathrm{D}\\&+(-16\,i^4-72\,i^3-45\,i^2-i+6)\,\mathrm{D}^2+(-32\,i^3-48\,i^2-22\,i-11)\,\mathrm{D}^3+(-4\,i^2+i-7)\,\mathrm{D}^4\\&+(+4\,i+3)\,\mathrm{D}^5+\mathrm{D}^6\end{aligned}$$

$$\begin{aligned}768\,\Pi^{4.2}_{2.2}=&-64\,i^6+96\,i^5+220\,i^4-248\,i^3-58\,i^2+88\,i+(-128\,i^5+208\,i^4+224\,i^3-375\,i^2+120\,i+88)\,\mathrm{D}\\&+(-80\,i^4+160\,i^3-43\,i^2-146\,i+106)\,\mathrm{D}^2+(0\,i^3+48\,i^2-104\,i-7)\,\mathrm{D}^3+(+20\,i^2+2\,i-27)\,\mathrm{D}^4\\&+(+8\,i-1)\,\mathrm{D}^5+\mathrm{D}^6\end{aligned}$$

$$\begin{aligned}3\,072\,\Pi^{4.2}_{4.2}=&+64\,i^6-336\,i^5+116\,i^4+1\,243\,i^3-722\,i^2-824\,i+(+192\,i^5-880\,i^4+252\,i^3+2\,225\,i^2-988\,i-568)\,\mathrm{D}\\&+(+240\,i^4-920\,i^3+201\,i^2+1\,323\,i-330)\,\mathrm{D}^2+(+160\,i^3-480\,i^2+70\,i+261)\,\mathrm{D}^3\\&+(+60\,i^2-125\,i+9)\,\mathrm{D}^4+(12\,i-13)\,\mathrm{D}^5+\mathrm{D}^6\end{aligned}$$

$$\begin{aligned}7\,680\,\Pi^{5.1}_{-5.1}=&+64\,i^6+832\,i^5+3\,980\,i^4+8\,510\,i^3+7\,748\,i^2+2\,194\,i\\&+(-128\,i^5-1\,360\,i^4-5\,040\,i^3-7\,795\,i^2-5\,024\,i-1\,569)\,\mathrm{D}+(+80\,i^4+560\,i^3+985\,i^2+20\,i-349)\,\mathrm{D}^2\\&+(0\,i^3+160\,i^2+820\,i+915)\,\mathrm{D}^3+(-20\,i^2-160\,i-275)\,\mathrm{D}^4+(+8\,i+29)\,\mathrm{D}^5-\mathrm{D}^6\end{aligned}$$

$$\begin{aligned}1\,536\,\Pi^{5.1}_{-3.1}=&-64\,i^6-512\,i^5-1\,468\,i^4-1\,910\,i^3-1\,146\,i^2-258\,i+(+64\,i^5+400\,i^4+916\,i^3+1\,113\,i^2+776\,i+231)\,\mathrm{D}\\&+(+16\,i^4+160\,i^3+405\,i^2+360\,i+163)\,\mathrm{D}^2+(-32\,i^3-152\,i^2-200\,i-109)\,\mathrm{D}^3+(+4\,i^2-8\,i-27)\,\mathrm{D}^4\\&+(+4\,i+13)\,\mathrm{D}^5-\mathrm{D}^6\end{aligned}$$

$$\begin{aligned}768\,\Pi^{5.1}_{-1.1}=&+64\,i^6+192\,i^5+132\,i^4+2\,i^3+8\,i^2+10\,i+(0\,i^5+16\,i^4-16\,i^3-41\,i^2+6\,i+3)\,\mathrm{D}\\&+(-48\,i^4-112\,i^3-69\,i^2-38\,i-5)\,\mathrm{D}^2+(0\,i^3-24\,i^2-20\,i-13)\,\mathrm{D}^3+(+12\,i^2+16\,i+1)\,\mathrm{D}^4+5\,\mathrm{D}^5-\mathrm{D}^6\end{aligned}$$

$$\begin{aligned}768\,\Pi^{5.1}_{1.1}=&-64\,i^6+128\,i^5+28\,i^4-50\,i^3-32\,i^2-10\,i+(-64\,i^5+112\,i^4-20\,i^3-i^2+6\,i+3)\,\mathrm{D}\\&+(+16\,i^4-32\,i^3-21\,i^2+6\,i-5)\,\mathrm{D}^2+(+32\,i^3-32\,i^2+0\,i-13)\,\mathrm{D}^3+(+4\,i^2+8\,i+1)\,\mathrm{D}^4\\&+(-4\,i+5)\,\mathrm{D}^5-\mathrm{D}^6\end{aligned}$$

$$\begin{aligned}1\,536\,\Pi^{5.1}_{3.1}=&+64\,i^6-448\,i^5+988\,i^4-682\,i^3-132\,i^2+258\,i+(+128\,i^5-784\,i^4+1\,552^{3}-1\,175\,i^2+148\,i+231)\,\mathrm{D}\\&+(+80\,i^4-400\,i^3+709\,i^2-632\,i+163)\,\mathrm{D}^2+(0\,i^3+16\,i^2+36\,i-109)\,\mathrm{D}^3+(-20\,i^2+64\,i-27)\,\mathrm{D}^4\\&+(-8\,i+13)\,\mathrm{D}^5-\mathrm{D}^6\end{aligned}$$

$$\begin{aligned}7\,680\,\Pi^{5.1}_{5.1}=&-64\,i^6+768\,i^5-3\,180\,i^4+4\,930\,i^3-1\,028\,i^2-2\,194\,i\\&+(-192\,i^5+2\,000\,i^4-6\,940\,i^3+8\,525\,i^2-1\,252\,i-1\,569)\,\mathrm{D}\\&+(-240\,i^4+2\,080\,i^3-5\,655\,i^2+4\,860\,i-349)\,\mathrm{D}^2+(-160\,i^3+1\,080\,i^2-2\,040\,i+915)\,\mathrm{D}^3\\&+(-60\,i^2+280\,i-275)\,\mathrm{D}^4+(-12\,i+29)\,\mathrm{D}^5-\mathrm{D}^6\end{aligned}$$

$$\begin{aligned}4\,608\,\Pi^{1.\,6}_{1.\,0} = {} & +128\,i^7-584\,i^5+728\,i^3+(+64\,i^6-288\,i^5-292\,i^4+834\,i^3+364\,i^2-240\,i)\,\mathrm{D}\\ & +(-96\,i^5-144\,i^4+618\,i^3+417\,i^2-548\,i-120)\,\mathrm{D}^2+(-48\,i^4+192\,i^3+309\,i^2-450\,i-274)\,\mathrm{D}^3\\ & +(+24\,i^3+96\,i^2-170\,i-225)\,\mathrm{D}^4+(+12\,i^2-30\,i-85)\,\mathrm{D}^5+(-2\,i-15)\,\mathrm{D}^6-\mathrm{D}^7\end{aligned}$$

$$\begin{aligned}6\,144\,\Pi^{1.\,6}_{-1.\,2} = {} & +128\,i^7+928\,i^6+2\,152\,i^5+1\,754\,i^4+684\,i^3+640\,i^2+128\,i\\ & +(+64\,i^6+80\,i^5-908\,i^4-1\,803\,i^3-334\,i^2+32\,i-64)\,\mathrm{D}\\ & +(-96\,i^5-736\,i^4-1\,314\,i^3+121\,i^2+632\,i-176)\,\mathrm{D}^2+(-48\,i^4-24\,i^3+699\,i^2+789\,i-318)\,\mathrm{D}^3\\ & +(+24\,i^3+202\,i^2+200\,i-309)\,\mathrm{D}^4+(+12\,i^2+i-121)\,\mathrm{D}^5+(-2\,i-19)\,\mathrm{D}^6-\mathrm{D}^7\end{aligned}$$

$$\begin{aligned}6\,144\,\Pi^{1.\,6}_{1.\,2} = {} & -128\,i^7-928\,i^6-2\,152\,i^5-1\,754\,i^4-684\,i^3-640\,i^2-128\,i\\ & +(-192\,i^6-1\,008\,i^5-1\,244\,i^4+49\,i^3-350\,i^2-672\,i-64)\,\mathrm{D}\\ & +(-32\,i^5+192\,i^4+1\,146\,i^3+805\,i^2-640\,i-176)\,\mathrm{D}^2+(+80\,i^4+488\,i^3+531\,i^2-447\,i-318)\,\mathrm{D}^3\\ & +(+40\,i^3+54\,i^2-284\,i-309)\,\mathrm{D}^4+(-4\,i^2-75\,i-121)\,\mathrm{D}^5+(-6\,i-19)\,\mathrm{D}^6-\mathrm{D}^7\end{aligned}$$

$$\begin{aligned}15\,360\,\Pi^{1.\,6}_{-1.\,4} = {} & -128\,i^7-1\,856\,i^6-10\,360\,i^5-28\,240\,i^4-39\,576\,i^3-27\,728\,i^2-8\,192\,i\\ & +(-192\,i^6-2\,112\,i^5-7\,940\,i^4-11\,370\,i^3-3\,080\,i^2+5\,080\,i+4\,096)\,\mathrm{D}\\ & +(-32\,i^5+240\,i^4+3\,750\,i^3+11\,945\,i^2+12\,782\,i+4\,392)\,\mathrm{D}^2\\ & +(+80\,i^4+960\,i^3+3\,285\,i^2+2\,930\,i-674)\,\mathrm{D}^3+(+40\,i^3+180\,i^2-290\,i-1\,265)\,\mathrm{D}^4\\ & +(-4\,i^2-108\,i-325)\,\mathrm{D}^5+(-6\,i-31)\,\mathrm{D}^6-\mathrm{D}^7\end{aligned}$$

$$\begin{aligned}15\,360\,\Pi^{1.\,6}_{1.\,4} = {} & +128\,i^7+1\,856\,i^6+10\,360\,i^5+28\,240\,i^4+39\,576\,i^3+27\,728\,i^2+8\,192\,i\\ & +(+320\,i^6+3\,968\,i^5+18\,300\,i^4+39\,610\,i^3+42\,656\,i^2+22\,648\,i+4\,096)\,\mathrm{D}\\ & +(+288\,i^5+2\,800\,i^4+9\,370\,i^3+13\,545\,i^2+10\,086\,i+4\,392)\,\mathrm{D}^2\\ & +(+80\,i^4+320\,i^3-475\,i^2-2\,130\,i-674)\,\mathrm{D}^3+(-40\,i^3-500\,i^2-1\,590\,i-1\,265)\,\mathrm{D}^4\\ & +(-36\,i^2-232\,i-325)\,\mathrm{D}^5+(-10\,i-31)\,\mathrm{D}^6-\mathrm{D}^7\end{aligned}$$

$$\begin{aligned}92\,160\,\Pi^{1.\,6}_{-1.\,6} = {} & +128\,i^7+2\,784\,i^6+24\,040\,i^5+104\,190\,i^4+234\,476\,i^3+251\,232\,i^2+93\,312\,i\\ & +(+320\,i^6+5\,808\,i^5+39\,620\,i^4+123\,075\,i^3+159\,186\,i^2+32\,304\,i-46\,656)\,\mathrm{D}\\ & +(+288\,i^5+3\,840\,i^4+15\,670\,i^3+10\,125\,i^2-57\,384\,i-78\,960)\,\mathrm{D}^2\\ & +(+80\,i^4+120\,i^3-5\,965\,i^2-30\,765\,i-40\,414)\,\mathrm{D}^3+(-40\,i^3-930\,i^2-5\,420\,i-9\,045)\,\mathrm{D}^4\\ & +(-36\,i^2-393\,i-985)\,\mathrm{D}^5+(-10\,i-51)\,\mathrm{D}^6-\mathrm{D}^7\end{aligned}$$

$$\begin{aligned}92\,160\,\Pi^{1.\,6}_{1.\,6} = {} & -128\,i^7-2\,784\,i^6-24\,040\,i^5-104\,190\,i^4-234\,476\,i^3-251\,232\,i^2-93\,312\,i\\ & +(-448\,i^6-8\,592\,i^5-63\,660\,i^4-227\,265\,i^3-393\,662\,i^2-283\,536\,i-46\,656)\,\mathrm{D}\\ & +(-672\,i^5-11\,040\,i^4-67\,310\,i^3-185\,295\,i^2-219\,040\,i-78\,960)\,\mathrm{D}^2\\ & +(-560\,i^4-7\,560\,i^3-35\,525\,i^2-66\,945\,i-40\,414)\,\mathrm{D}^3+(-280\,i^3-2\,910\,i^2-9\,360\,i-9\,045)\,\mathrm{D}^4\\ & +(-84\,i^2-597\,i-985)\,\mathrm{D}^5+(-14\,i-51)\,\mathrm{D}^6-\mathrm{D}^7\end{aligned}$$

$$\begin{aligned}3\,072\,\Pi^{2.\,5}_{-2.\,1} = {} & +128\,i^7+544\,i^6+616\,i^5-18\,i^4-203\,i^3+44\,i^2+5\,i\\ & +(-64\,i^6-432\,i^5-812\,i^4-397\,i^3+149\,i^2+97\,i-3)\,\mathrm{D}\\ & +(-96\,i^5-240\,i^4+82\,i^3+444\,i^2+209\,i-74)\,\mathrm{D}^2+(+48\,i^4+264\,i^3+303\,i^2-11\,i-134)\,\mathrm{D}^3\\ & +(+24\,i^3-6\,i^2-125\,i-64)\,\mathrm{D}^4+(-12\,i^2-39\,i+6)\,\mathrm{D}^5+(-2\,i+8)\,\mathrm{D}^6+\mathrm{D}^7\end{aligned}$$

$$\begin{aligned}1\,536\,\Pi^{2.\,5}_{0.\,1} = {} & -128\,i^7-384\,i^6-136\,i^5+188\,i^4-32\,i^3-4\,i^2+0\,i+(-64\,i^6+64\,i^5+444\,i^4+82\,i^3-147\,i^2+8\,i+1)\,\mathrm{D}\\ & +(+96\,i^5+352\,i^4+50\,i^3-346\,i^2-4\,i+26)\,\mathrm{D}^2+(+48\,i^4-48\,i^3-303\,i^2-18\,i+78)\,\mathrm{D}^3\\ & +(-24\,i^3-112\,i^2+0\,i+92)\,\mathrm{D}^4+(-12\,i^2+8\,i+50)\,\mathrm{D}^5+(+2\,i+12)\,\mathrm{D}^6+\mathrm{D}^7\end{aligned}$$

$$\begin{aligned}3\,072\,\Pi^{2.\,5}_{2.\,1} = {} & +128\,i^7+224\,i^6-344\,i^5-358\,i^4+267\,i^3-36\,i^2-5\,i\\ & +(+192\,i^6+176\,i^5-460\,i^4+97\,i^3+333\,i^2-145\,i-3)\,\mathrm{D}\\ & +(+32\,i^5-144\,i^4-14\,i^3+408\,i^2-121\,i-74)\,\mathrm{D}^2+(-80\,i^4-136\,i^3+195\,i^2+23\,i-134)\,\mathrm{D}^3\\ & +(-40\,i^3+22\,i^2+77\,i-64)\,\mathrm{D}^4+(+4\,i^2+39\,i+6)\,\mathrm{D}^5+(+6\,i+8)\,\mathrm{D}^6+\mathrm{D}^7\end{aligned}$$

$$\begin{aligned}6\,144\,\Pi^{2.5}_{-2.3} = & -128\,i^7 - 1\,312\,i^6 - 5\,144\,i^5 - 9\,898\,i^4 - 10\,017\,i^3 - 5\,262\,i^2 - 1\,215\,i \\ & + (-64\,i^6 - 304\,i^5 + 76\,i^4 + 2\,185\,i^3 + 3\,781\,i^2 + 2\,631\,i + 729)\,D \\ & + (+96\,i^5 + 944\,i^4 + 3\,074\,i^3 + 4\,144\,i^2 + 2\,341\,i + 306)\,D^2 + (+48\,i^4 + 168\,i^3 - 191\,i^2 - 893\,i - 630)\,D^3 \\ & + (-24\,i^3 - 218\,i^2 - 445\,i - 172)\,D^4 + (-12\,i^2 - 23\,i + 50)\,D^5 + (+2\,i + 16)\,D^6 + D^7\end{aligned}$$

$$\begin{aligned}3\,072\,\Pi^{2.5}_{0.3} = & +128\,i^7 + 1\,152\,i^6 + 3\,704\,i^5 + 5\,268\,i^4 + 3\,432\,i^3 + 972\,i^2 + 0\,i \\ & + (+192\,i^6 + 1\,280\,i^5 + 2\,564\,i^4 + 1\,370\,i^3 - 585\,i^2 - 858\,i - 243)\,D \\ & + (+32\,i^5 - 192\,i^4 - 1\,598\,i^3 - 2\,626\,i^2 - 1\,432\,i - 426)\,D^2 + (-80\,i^4 - 608\,i^3 - 1\,177\,i^2 - 488\,i - 34)\,D^3 \\ & + (-40\,i^3 - 104\,i^2 + 152\,i + 256)\,D^4 + (+4\,i^2 + 72\,i + 126)\,D^5 + (+6\,i + 20)\,D^6 + D^7\end{aligned}$$

$$\begin{aligned}6\,144\,\Pi^{2.5}_{2.3} = & -128\,i^7 - 992\,i^6 - 2\,264\,i^5 - 638\,i^4 + 3\,153\,i^3 + 3\,318\,i^2 + 1\,215\,i \\ & + (-320\,i^6 - 2\,128\,i^5 - 4\,052\,i^4 - 1\,221\,i^3 + 2\,657\,i^2 + 2\,517\,i + 729)\,D \\ & + (-288\,i^5 - 1\,520\,i^4 - 2\,270\,i^3 - 1\,308\,i^2 - 613\,i + 306)\,D^2 + (-80\,i^4 - 200\,i^3 - 163\,i^2 - 771\,i - 630)\,D^3 \\ & + (+40\,i^3 + 250\,i^2 + 221\,i - 172)\,D^4 + (+36\,i^2 + 119\,i + 50)\,D^5 + (+10\,i + 16)\,D^6 + D^7\end{aligned}$$

$$\begin{aligned}30\,720\,\Pi^{2.5}_{-2.5} = & +128\,i^7 + 2\,080\,i^6 + 13\,240\,i^5 + 41\,770\,i^4 + 67\,971\,i^3 + 53\,370\,i^2 + 15\,625\,i \\ & + (+192\,i^6 + 2\,384\,i^5 + 10\,460\,i^4 + 18\,345\,i^3 + 6\,985\,i^2 - 11\,777\,i - 9\,375)\,D \\ & + (+32\,i^5 - 240\,i^4 - 4\,490\,i^3 - 17\,420\,i^2 - 24\,787\,i - 12\,022)\,D^2 \\ & + (-80\,i^4 - 1\,080\,i^3 - 4\,455\,i^2 - 6\,225\,i - 1\,926)\,D^3 + (-40\,i^3 - 230\,i^2 + 35\,i + 1\,020)\,D^4 \\ & + (+4\,i^2 + 105\,i + 330)\,D^5 + (+6\,i + 32)\,D^6 + D^7\end{aligned}$$

$$\begin{aligned}15\,360\,\Pi^{2.5}_{0.5} = & -128\,i^7 - 1\,920\,i^6 - 10\,840\,i^5 - 28\,220\,i^4 - 32\,696\,i^3 - 12\,500\,i^2 + 0\,i \\ & + (-320\,i^6 - 3\,968\,i^5 - 17\,260\,i^4 - 29\,810\,i^3 - 13\,141\,i^2 + 8\,174\,i + 3\,125)\,D \\ & + (-288\,i^5 - 2\,560\,i^4 - 5\,930\,i^3 + 2\,190\,i^2 + 16\,304\,i + 8\,174)\,D^2 \\ & + (-80\,i^4 + 0\,i^3 + 3\,475\,i^2 + 10\,540\,i + 7\,374)\,D^3 + (+40\,i^3 + 680\,i^2 + 2\,680\,i + 2\,760)\,D^4 \\ & + (+36\,i^2 + 280\,i + 470)\,D^5 + (+10\,i + 36)\,D^6 + D^7\end{aligned}$$

$$\begin{aligned}30\,720\,\Pi^{2.5}_{2.5} = & +128\,i^7 + 1\,760\,i^6 + 8\,440\,i^5 + 14\,670\,i^4 - 2\,579\,i^3 - 28\,370\,i^2 - 15\,625\,i \\ & + (+448\,i^6 + 5\,424\,i^5 + 22\,140\,i^4 + 30\,435\,i^3 - 8\,923\,i^2 - 37\,267\,i - 9\,375)\,D \\ & + (+672\,i^5 + 6\,960\,i^4 + 23\,190\,i^3 + 23\,520\,i^2 - 7\,645\,i - 12\,022)\,D^2 \\ & + (+560\,i^4 + 4\,760\,i^3 + 12\,125\,i^2 + 8\,025\,i - 1\,926)\,D^3 \\ & + (+280\,i^3 + 1\,830\,i^2 + 3\,165\,i + 1\,020)\,D^4 + (+84\,i^2 + 375\,i + 330)\,D^5 + (+14\,i + 32)\,D^6 + D^7\end{aligned}$$

$$\begin{aligned}1\,024\,\Pi^{3.4}_{1.0} = & +128\,i^7 - 160\,i^6 - 104\,i^5 + 170\,i^4 - 34\,i^3 + 0\,i^2 + 0\,i + (+64\,i^6 - 208\,i^5 + 140\,i^4 + 101\,i^3 - 111\,i^2 + 12\,i)\,D \\ & + (-96\,i^5 + 32\,i^4 + 186\,i^3 - 151\,i^2 - 8\,i + 18)\,D^2 + (-48\,i^4 + 120\,i^3 - 39\,i^2 - 55\,i + 39)\,D^3 \\ & + (+24\,i^3 + 22\,i^2 - 50\,i + 23)\,D^4 + (+12\,i^2 - 17\,i - 2)\,D^5 + (-2\,i - 5)\,D^6 - D^7\end{aligned}$$

$$\begin{aligned}3\,072\,\Pi^{3.4}_{3.0} = & -128\,i^7 + 480\,i^6 - 280\,i^5 - 510\,i^4 + 442\,i^3 + 0\,i^2 + 0\,i \\ & + (-192\,i^6 + 656\,i^5 - 548\,i^4 - 193\,i^3 + 469\,i^2 - 156\,i + 0)\,D \\ & + (-32\,i^5 + 96\,i^4 - 306\,i^3 + 377\,i^2 - 88\,i - 102)\,D^2 + (+80\,i^4 - 216\,i^3 + 57\,i^2 + 171\,i - 133)\,D^3 \\ & + (+40\,i^3 - 98\,i^2 + 106\,i - 9)\,D^4 + (-4\,i^2 - 3\,i + 26)\,D^5 + (-6\,i + 3)\,D^6 - D^7\end{aligned}$$

$$\begin{aligned}4\,608\,\Pi^{3.4}_{-3.2} = & -128\,i^7 - 1\,088\,i^6 - 3\,592\,i^5 - 5\,832\,i^4 - 4\,900\,i^3 - 2\,092\,i^2 - 416\,i \\ & + (+64\,i^6 + 576\,i^5 + 2\,036\,i^4 + 3\,598\,i^3 + 3\,328\,i^2 + 1\,530\,i + 272)\,D \\ & + (+96\,i^5 + 624\,i^4 + 1\,482\,i^3 + 1\,497\,i^2 + 503\,i - 110)\,D^2 + (-48\,i^4 - 336\,i^3 - 861\,i^2 - 965\,i - 393)\,D^3 \\ & + (-24\,i^3 - 84\,i^2 - 70\,i + 39)\,D^4 + (-12\,i^2 + 48\,i + 50)\,D^5 + (-2\,i - 1)\,D^6 - D^7\end{aligned}$$

$$1\,536\,\Pi_{-1,\,2}^{3,\,4} = +128\,i^7 + 768\,i^6 + 1\,688\,i^5 + 1\,768\,i^4 + 972\,i^3 + 284\,i^2 + 32\,i$$
$$+ (+64\,i^6 + 160\,i^5 - 132\,i^4 - 634\,i^3 - 632\,i^2 - 262\,i - 48)\,\mathrm{D}$$
$$+ (-96\,i^5 - 560\,i^4 - 1\,038\,i^3 - 793\,i^2 - 237\,i - 22)\,\mathrm{D}^2 + (-48\,i^4 - 96\,i^3 + 65\,i^2 + 169\,i + 83)\,\mathrm{D}^3$$
$$+ (+24\,i^3 + 128\,i^2 + 150\,i + 55)\,\mathrm{D}^4 + (+12\,i^2 + 14\,i - 10)\,\mathrm{D}^5 + (-2\,i - 9)\,\mathrm{D}^6 - \mathrm{D}^7$$

$$1\,536\,\Pi_{1,\,2}^{3,\,4} = -128\,i^7 - 448\,i^6 - 168\,i^5 + 472\,i^4 + 268\,i^3 + 36\,i^2 - 32\,i$$
$$+ (-192\,i^6 - 448\,i^5 + 100\,i^4 + 262\,i^3 - 128\,i^2 - 110\,i - 48)\,\mathrm{D}$$
$$+ (-32\,i^5 + 144\,i^4 + 258\,i^3 - 195\,i^2 - 61\,i - 22)\,\mathrm{D}^2 + (+80\,i^4 + 256\,i^3 + 47\,i^2 - 13\,i + 83)\,\mathrm{D}^3$$
$$+ (+40\,i^3 + 28\,i^2 - 38\,i + 55)\,\mathrm{D}^4 + (-4\,i^2 - 36\,i - 10)\,\mathrm{D}^5 + (-6\,i - 9)\,\mathrm{D}^6 - \mathrm{D}^7$$

$$4\,608\,\Pi_{3,\,2}^{3,\,4} = +128\,i^7 + 128\,i^6 - 968\,i^5 - 888\,i^4 + 1\,180\,i^3 + 1\,132\,i^2 + 416\,i$$
$$+ (+320\,i^6 + 288\,i^5 - 1\,684\,i^4 - 1\,266\,i^3 + 744\,i^2 + 578\,i + 272)\,\mathrm{D}$$
$$+ (+288\,i^5 + 240\,i^4 - 710\,i^3 - 453\,i^2 - 633\,i - 110)\,\mathrm{D}^2 + (+80\,i^4 + 80\,i^3 + 269\,i^2 + 57\,i - 393)\,\mathrm{D}^3$$
$$+ (-40\,i^3 + 0\,i^2 + 262\,i + 39)\,\mathrm{D}^4 + (-36\,i^2 - 6\,i + 50)\,\mathrm{D}^5 + (-10\,i - 1)\,\mathrm{D}^6 - \mathrm{D}^7$$

$$18\,432\,\Pi_{-3,\,4}^{3,\,4} = +128\,i^7 + 1\,696\,i^6 + 8\,968\,i^5 + 24\,090\,i^4 + 34\,402\,i^3 + 24\,476\,i^2 + 6\,656\,i$$
$$+ (+64\,i^6 + 528\,i^5 + 1\,220\,i^4 - 703\,i^3 - 6\,665\,i^2 - 9\,290\,i - 4\,352)\,\mathrm{D}$$
$$+ (-96\,i^5 - 1\,152\,i^4 - 5\,154\,i^3 - 10\,881\,i^2 - 10\,976\,i - 4\,326)\,\mathrm{D}^2 + (-48\,i^4 - 312\,i^3 - 549\,i^2 + 31\,i + 619)\,\mathrm{D}^3$$
$$+ (+24\,i^3 + 234\,i^2 + 670\,i + 631)\,\mathrm{D}^4 + (+12\,i^2 + 45\,i + 26)\,\mathrm{D}^5 + (-2\,i - 13)\,\mathrm{D}^6 - \mathrm{D}^7$$

$$6\,144\,\Pi_{-1,\,4}^{3,\,4} = -128\,i^7 - 1\,376\,i^6 - 5\,544\,i^5 - 10\,462\,i^4 - 9\,506\,i^3 - 3\,852\,i^2 - 512\,i$$
$$+ (-192\,i^6 - 1\,552\,i^5 - 4\,124\,i^4 - 3\,645\,i^3 + 735\,i^2 + 2\,438\,i + 768)\,\mathrm{D}$$
$$+ (-32\,i^5 + 192\,i^4 + 2\,130\,i^3 + 5\,059\,i^2 + 4\,472\,i + 1\,426)\,\mathrm{D}^2 + (+80\,i^4 + 728\,i^3 + 1\,943\,i^2 + 1\,801\,i + 599)\,\mathrm{D}^3$$
$$+ (+40\,i^3 + 154\,i^2 + 10\,i - 169)\,\mathrm{D}^4 + (-4\,i^2 - 69\,i - 130)\,\mathrm{D}^5 + (-6\,i - 21)\,\mathrm{D}^6 - \mathrm{D}^7$$

$$6\,144\,\Pi_{1,\,4}^{3,\,4} = +128\,i^7 + 1\,056\,i^6 + 2\,504\,i^5 + 482\,i^4 - 3\,414\,i^3 - 1\,268\,i^2 + 512\,i$$
$$+ (+320\,i^6 + 2\,128\,i^5 + 3\,412\,i^4 - 1\,435\,i^3 - 3\,493\,i^2 + 1\,438\,i + 768)\,\mathrm{D}$$
$$+ (+288\,i^5 + 1\,280\,i^4 + 310\,i^3 - 2\,601\,i^2 + 168\,i + 1\,426)\,\mathrm{D}^2 + (+80\,i^4 - 120\,i^3 - 1\,517\,i^2 - 1\,221\,i + 599)\,\mathrm{D}^3$$
$$+ (-40\,i^3 - 430\,i^2 - 826\,i - 169)\,\mathrm{D}^4 + (-36\,i^2 - 167\,i - 130)\,\mathrm{D}^5 + (-10\,i - 21)\,\mathrm{D}^6 - \mathrm{D}^7$$

$$18\,432\,\Pi_{3,\,4}^{3,\,4} = -128\,i^7 - 736\,i^6 + 152\,i^5 + 5\,850\,i^4 + 4\,358\,i^3 - 9\,116\,i^2 - 6\,656\,i$$
$$+ (-448\,i^6 - 2\,256\,i^5 + 660\,i^4 + 13\,511\,i^3 + 6\,955\,i^2 - 12\,674\,i - 4\,352)\,\mathrm{D}$$
$$+ (-672\,i^5 - 2\,880\,i^4 + 970\,i^3 + 11\,643\,i^2 + 3\,632\,i - 4\,326)\,\mathrm{D}^2$$
$$+ (-560\,i^4 - 1\,960\,i^3 + 655\,i^2 + 4\,437\,i + 619)\,\mathrm{D}^3 + (-280\,i^3 - 750\,i^2 + 210\,i + 631)\,\mathrm{D}^4$$
$$+ (-84\,i^2 - 153\,i + 26)\,\mathrm{D}^5 + (-14\,i - 13)\,\mathrm{D}^6 - \mathrm{D}^7$$

$$6\,144\,\Pi_{-4,\,1}^{4,\,3} = -128\,i^7 - 1\,184\,i^6 - 4\,024\,i^5 - 6\,226\,i^4 - 4\,419\,i^3 - 1\,313\,i^2 - 206\,i$$
$$+ (+192\,i^6 + 1\,488\,i^5 + 4\,268\,i^4 + 5\,881\,i^3 + 4\,190\,i^2 + 1\,452\,i + 142)\,\mathrm{D}$$
$$+ (-32\,i^5 - 48\,i^4 + 186\,i^3 + 194\,i^2 - 271\,i - 379)\,\mathrm{D}^2 + (-80\,i^4 - 568\,i^3 - 1\,287\,i^2 - 1\,191\,i - 360)\,\mathrm{D}^3$$
$$+ (+40\,i^3 + 174\,i^2 + 265\,i + 201)\,\mathrm{D}^4 + (+4\,i^2 + 33\,i + 25)\,\mathrm{D}^5 + (-6\,i - 14)\,\mathrm{D}^6 + \mathrm{D}^7$$

$$1\,536\,\Pi_{-2,\,1}^{4,\,3} = +128\,i^7 + 704\,i^6 + 1\,432\,i^5 + 1\,388\,i^4 + 688\,i^3 + 174\,i^2 + 22\,i$$
$$+ (-64\,i^6 - 352\,i^5 - 844\,i^4 - 1\,058\,i^3 - 697\,i^2 - 201\,i - 22)\,\mathrm{D}$$
$$+ (-96\,i^5 - 416\,i^4 - 606\,i^3 - 340\,i^2 - 14\,i + 45)\,\mathrm{D}^2 + (+48\,i^4 + 192\,i^3 + 327\,i^2 + 251\,i + 92)\,\mathrm{D}$$
$$+ (+24\,i^3 + 68\,i^2 + 50\,i + 5)\,\mathrm{D}^4 + (-12\,i^2 - 26\,i - 23)\,\mathrm{D}^5 + (-2\,i - 2)\,\mathrm{D}^6 + \mathrm{D}^7$$

$$1\,024\,\Pi_{0,\,1}^{4,\,3} = -128\,i^7 - 224\,i^6 - 8\,i^5 + 110\,i^4 + 45\,i^3 + 9\,i^2 + (-64\,i^6 - 16\,i^5 + 68\,i^4 - 7\,i^3 - 46\,i^2 - 10\,i - 2)\,\mathrm{D}$$
$$+ (+96\,i^5 + 176\,i^4 + 30\,i^3 - 22\,i^2 + 3\,i + 5)\,\mathrm{D}^2 + (+48\,i^4 + 24\,i^3 + 7\,i^2 + 15\,i + 8)\,\mathrm{D}^3$$
$$+ (-24\,i^3 - 38\,i^2 - 5\,i - 7)\,\mathrm{D}^4 + (-12\,i^2 - 5\,i - 7)\,\mathrm{D}^5 + (+2\,i + 2)\,\mathrm{D}^6 + \mathrm{D}^7$$

$$
\begin{aligned}
1\,536\,\Pi^{4,3}_{2,1} = {} & +128\,i^7-256\,i^6-248\,i^5+436\,i^4-48\,i^3-46\,i^2-22\,i \\
& +(+192\,i^6-384\,i^5-52\,i^4+354\,i^3-271\,i^2-19\,i-22)\,D \\
& +(+32\,i^5-96\,i^4+270\,i^3-128\,i^2-20\,i+45)\,D^2+(-80\,i^4+96\,i^3+105\,i^2-101\,i+92)\,D^3 \\
& +(-40\,i^3+48\,i^2-52\,i+5)\,D^4+(+4\,i^2+0\,i-23)\,D^5+(+6\,i-2)\,D^6+D^7
\end{aligned}
$$

$$
\begin{aligned}
6\,144\,\Pi^{4,3}_{4,1} = {} & -128\,i^7+736\,i^6-664\,i^5-1\,730\,i^4+1\,589\,i^3+747\,i^2+206\,i \\
& +(-320\,i^6+1\,552\,i^5-1\,092\,i^4-2\,255\,i^3+1\,302\,i^2-32\,i+142)\,D \\
& +(-288\,i^5+1\,040\,i^4-430\,i^3-270\,i^2-395\,i-379)\,D^2+(-80\,i^4+40\,i^3+157\,i^2+613\,i-360)\,D^3 \\
& +(+40\,i^3-250\,i^2+141\,i+201)\,D^4+(+36\,i^2-107\,i+25)\,D^5+(+10\,i-14)\,D^6+D^7
\end{aligned}
$$

$$
\begin{aligned}
18\,432\,\Pi^{4,3}_{-4,3} = {} & +128\,i^7+1\,632\,i^6+8\,344\,i^5+21\,766\,i^4+30\,287\,i^3+21\,031\,i^2+5\,562\,i \\
& +(-64\,i^6-720\,i^5-3\,380\,i^4-8\,573\,i^3-12\,574\,i^2-10\,312\,i-3\,834)\,D \\
& +(-96\,i^5-1\,008\,i^4-4\,074\,i^3-7\,950\,i^2-7\,529\,i-2\,831)\,D^2 \\
& +(+48\,i^4+408\,i^3+1\,365\,i^2+2\,159\,i+1\,420)\,D^3 \\
& +(+24\,i^3+174\,i^2+415\,i+341)\,D^4+(-12\,i^2-57\,i-83)\,D^5+(-2\,i-6)\,D^6+D^7
\end{aligned}
$$

$$
\begin{aligned}
4\,608\,\Pi^{4,3}_{-2,3} = {} & -128\,i^7-1\,152\,i^6-4\,072\,i^5-7\,196\,i^4-6\,704\,i^3-3\,158\,i^2-594\,i \\
& +(-64\,i^6-384\,i^5-644\,i^4+278\,i^3+1\,853\,i^2+1\,863\,i+594)\,D \\
& +(+96\,i^5+768\,i^4+2\,346\,i^3+3\,456\,i^2+2\,528\,i+809)\,D^2+(+48\,i^4+240\,i^3+405\,i^2+245\,i+64)\,D^3 \\
& +(-24\,i^3-144\,i^2-280\,i-191)\,D^4+(-12\,i^2-36\,i-35)\,D^5+(+2\,i+6)\,D^6+D^7
\end{aligned}
$$

$$
\begin{aligned}
3\,072\,\Pi^{4,3}_{0,3} = {} & +128\,i^7+672\,i^6+968\,i^5+54\,i^4-585\,i^3-243\,i^2+0\,i \\
& +(+192\,i^6+720\,i^5+500\,i^4-389\,i^3-258\,i^2+130\,i+54)\,D \\
& +(+32\,i^5-144\,i^4-582\,i^3-342\,i^2+25\,i+49)\,D^2+(-80\,i^4-376\,i^3-409\,i^2-163\,i-68)\,D^3 \\
& +(-40\,i^3-78\,i^2-31\,i-59)\,D^4+(+4\,i^2+33\,i+13)\,D^5+(+6\,i+10)\,D^6+D^7
\end{aligned}
$$

$$
\begin{aligned}
4\,608\,\Pi^{4,3}_{2,3} = {} & -128\,i^7-192\,i^6+968\,i^5+956\,i^4-1\,616\,i^3-298\,i^2+594\,i \\
& +(-320\,i^6-288\,i^5+1\,924\,i^4+554\,i^3-2\,277\,i^2+997\,i+594)\,D \\
& +(-288\,i^5+0\,i^4+1\,190\,i^3-900\,i^2-678\,i+809)\,D^2+(-80\,i^4-240\,i^3+91\,i^2-851\,i+64)\,D^3 \\
& +(+40\,i^3+180\,i^2-142\,i-191\,D^4+(+36\,i^2+54\,i-35)\,D^5+(+10\,i+6)\,D^6+D^7
\end{aligned}
$$

$$
\begin{aligned}
18\,432\,\Pi^{4,3}_{4,3} = {} & +128\,i^7-288\,i^6-1\,736\,i^5+2\,870\,i^4+6\,503\,i^3-5\,749\,i^2-5\,562\,i \\
& +(+448\,i^6-912\,i^5-4\,740\,i^4+6\,811\,i^3+11\,786\,i^2-8\,148\,i-3\,834)\,D \\
& +(+672\,i^5-1\,200\,i^4-5\,170\,i^3+6\,018\,i^2+7\,099\,i-2\,831)\,D^2 \\
& +(+560\,i^4-840\,i^3-2\,815\,i^2+2\,347\,i+1\,420)\,D^3+(-280\,i^3-330\,i^2-765\,i+341)\,D^4 \\
& +(+84\,i^2-69\,i-83)\,D^5+(+14\,i-6)\,D^6+D^7
\end{aligned}
$$

$$
\begin{aligned}
1\,536\,\Pi^{5,2}_{1,0} = {} & +128\,i^7-320\,i^6+104\,i^5+48\,i^4+40\,i^3+0\,i^2+0\,i+(+64\,i^6-128\,i^5+116\,i^4-66\,i^3-24\,i^2-10\,i)\,D \\
& +(-96\,i^5+208\,i^4-50\,i^3+11\,i^2+0\,i+3)\,D^2+(-48\,i^4+48\,i^3-25\,i^2+22\,i-5)\,D^3 \\
& +(+24\,i^3-52\,i^2+10\,i-13)\,D^4+(+12\,i^2-4\,i+1)\,D^5+(-2\,i+5)\,D^6-D^7
\end{aligned}
$$

$$
\begin{aligned}
3\,072\,\Pi^{5,2}_{3,0} = {} & -128\,i^7+960\,i^6-2\,456\,i^5+2\,592\,i^4-1\,032\,i^3+0\,i^2+0\,i \\
& +(-192\,i^6+1\,216\,i^5-2\,708\,i^4+2\,902\,i^3-1\,572\,i^2+258\,i+0)\,D \\
& +(-32\,i^5+48\,i^4+14\,i^3+241\,i^2-314\,i+231)\,D^2+(+80\,i^4-448\,i^3+721\,i^2-496\,i+163)\,D^3 \\
& +(+40\,i^3-124\,i^2+118\,i-109)\,D^4+(-4\,i^2+36\,i-27)\,D^5+(-6\,i+13)\,D^6-D^7
\end{aligned}
$$

$$
\begin{aligned}
15\,360\,\Pi^{5,2}_{5,0} = {} & +128\,i^7-1\,600\,i^6+7\,160\,i^5-13\,440\,i^4+8\,776\,i^3+0\,i^2+0\,i \\
& +(+320\,i^6-3\,392\,i^5+12\,380\,i^4-18\,110\,i^3+9\,636\,i^2-2\,194\,i+0)\,D \\
& +(+288\,i^5-2\,320\,i^4+5\,690\,i^3-4\,515\,i^2+1\,886\,i-1\,569)\,D^2 \\
& +(+80\,i^4-160\,i^3-1\,115\,i^2+2\,420\,i-349)\,D^3 \\
& +(-40\,i^3+500\,i^2-1\,430\,i+915)\,D^4+(-36\,i^2+220\,i-275)\,D^5+(-10\,i+29)\,D^6-D^7
\end{aligned}
$$

$$30\,720\,\Pi^{5,2}_{-5,2} = +128i^7+1\,888i^6+10\,888i^5+31\,150i^4+46\,176i^3+33\,186i^2+8\,776i$$
$$+(-192i^6-2\,320i^5-10\,700i^4-24\,245i^3-29\,400i^2-19\,471i-6\,276)D$$
$$+(+32i^5+0i^4-1\,490i^3-5\,475i^2-6\,862i-2\,965)D^2+(80i^4+920i^3+3\,495i^2+5\,435i+3\,311)D^3$$
$$+(-40i^3-250i^2-400i-185)D^4+(-4i^2-69i-159)D^5+(6i+25)D^6-D^7$$

$$6\,144\,\Pi^{5,2}_{-3,2} = -128i^7-1\,248i^6-4\,744i^5-9\,078i^4-9\,320i^3-4\,914i^2-1\,032i$$
$$+(+64i^6+496i^5+1\,668i^4+3\,363i^3+4\,268i^2+3\,077i+924)D$$
$$+(+96i^5+800i^4+2\,426i^3+3\,529i^2+2\,610i+883)D^2+(-48i^4-264i^3-561i^2-617i-273)D^3$$
$$+(-24i^3-158i^2-300i-217)D^4+(+12i^2+35i+25)D^5+(+2i+9)D^6-D^7$$

$$3\,072\,\Pi^{5,2}_{-1,2} = +128i^7+608i^6+952i^5+506i^4+36i^3+42i^2+40i$$
$$+(+64i^6+240i^5+188i^4-135i^3-134i^2+37i+12)D$$
$$+(-96i^5-384i^4-566i^3-381i^2-148i-17)D^2+(-48i^4-168i^3-207i^2-139i-57)D^3$$
$$+(+24i^3+54i^2+40i-9)D^4+(+12i^2+27i+21)D^5+(-2i+1)D^6-D^7$$

$$3\,072\,\Pi^{5,2}_{1,2} = -128i^7+32i^6+488i^5+38i^4-252i^3-138i^2-40i$$
$$+(-192i^6+112i^5+412i^4-123i^3-34i^2+17i+12)D+(-32i^5+96i^4-154i^3-85i^2+28i-17)D^2$$
$$+(+80i^4+24i^3-147i^2-15i-57)D^3+(+40i^3+2i^2+28i-9)D^4$$
$$+(-4i^2+3i+21)D^5+(-6i+1)D^6-D^7$$

$$6\,144\,\Pi^{5,2}_{3,2} = +128i^7-672i^6+424i^5+1\,974i^4-2\,344i^3-270i^2+1\,032i$$
$$+(+320i^6-1\,552i^5+1\,252i^4+2\,559i^3-3\,964i^2+1\,081i+924)D$$
$$+(+288i^5-1\,280i^4+1\,430i^3+321i^2-1\,986i+883)D^2+(+80i^4-360i^3+803i^2-665i-273)D^3$$
$$+(-40i^3+70i^2+224i-217)D^4+(-36i^2+59i+25)D^5+(-10i+9)D^6-D^7$$

$$30\,720\,\Pi^{5,2}_{5,2} = -128i^7+1\,312i^6-3\,688i^5-1\,070i^4+14\,304i^3-6\,306i^2-8\,776i$$
$$+(-448i^6+4\,080i^5-9\,900i^4-2\,785i^3+26\,488i^2-8\,771i-6\,276)D$$
$$+(-672i^5+5\,280i^4-10\,590i^3-2\,715i^2+16\,270i-2\,965)D^2$$
$$+(-560i^4+3\,640i^3-5\,645i^2-1\,165i+3\,311)D^3+(-280i^3+1\,410i^2-1\,500i-185)D^4$$
$$+(-84i^2+291i-159)D^5+(-14i+25)D^6-D^7$$

$$92\,160\,\Pi^{6,1}_{-6,1} = +128i^7+2\,464i^6+18\,520i^5+68\,330i^4+126\,911i^3+107\,242i^2+29\,352i$$
$$+(-320i^6-5\,232i^5-32\,180i^4-92\,255i^3-124\,581i^2-74\,552i-21\,576)D$$
$$+(+288i^5+3\,600i^4+15\,070i^3+22\,800i^2+6\,171i-2\,882)D^2$$
$$+(-80i^4-280i^3+2\,605i^2+12\,815i+13\,099)D^3$$
$$+(-40i^3-750i^2-3\,605i-4\,850)D^4+(+36i^2+333i+700)D^5+(-10i-44)D^6+D^7$$

$$15\,360\,\Pi^{6,1}_{-4,1} = -128i^7-1\,664i^6-8\,280i^5-20\,140i^4-25\,376i^3-15\,804i^2-3\,608i$$
$$+(+192i^6+2\,048i^5+8\,180i^4+16\,150i^3+17\,855i^2+11\,038i+3\,096)D$$
$$+(-32i^5+0i^4+1\,110i^3+3\,210i^2+3\,242i+1\,630)D^2+(-80i^4-800i^3-2\,525i^2-3\,130i-1\,721)D^3$$
$$+(+40i^3+200i^2+160i-70)D^4+(+4i^2+72i+160)D^5+(-6i-24)D^6+D^7$$

$$6\,144\,\Pi^{6,1}_{-2,1} = +128i^7+864i^6+1\,976i^5+1\,974i^4+1\,221i^3+586i^2+136i$$
$$+(-64i^6-272i^5-420i^4-657i^3-773i^2-352i-72)D$$
$$+(-96i^5-592i^4-1\,098i^3-884i^2-575i-130)D^2+(+48i^4+120i^3-11i^2-39i-53)D^3$$
$$+(+24i^3+142i^2+165i+46)D^4+(-12i^2-13i+28)D^5+(-2i-12)D^6+D^7$$

$$4\,608\,\Pi^{6,1}_{0,1} = -128i^7-64i^6+392i^5+196i^4-344i^3-172i^2$$
$$+(-64i^6-96i^5+148i^4+78i^3-133i^2+48i+24)D+(+96i^5+0i^4-186i^3-54i^2-52i-2)D^2$$
$$+(+48i^4+96i^3-45i^2-30i-41)D^3+(-24i^3+36i^2+50i+10)D^4$$
$$+(-12i^2-18i+16)D^5+(+2i-8)D^6+D^7$$

$$6\,144\,\Pi^{6,1}_{2,1} = +128i^7 - 736i^6 + 1\,176i^5 - 398i^4 + 35i^3 + 42i^2 - 136i$$
$$+(+192i^6 - 944i^5 + 1\,388i^4 - 939i^3 + 603i^2 + 64i - 72)\,D$$
$$+(+32i^5 - 48i^4 + 78i^3 - 424i^2 + 343i - 130)\,D^2 + (-80i^4 + 328i^3 - 275i^2 + 59i - 53)\,D^3$$
$$+(-40i^3 + 74i^2 - i + 46)\,D^4 + (+4i^2 - 39i + 28)\,D^5 + (+6i - 12)\,D^6 + D^7$$

$$15\,360\,\Pi^{6,1}_{4,1} = -128i^7 + 1\,536i^6 - 6\,680i^5 + 12\,660i^4 - 8\,976i^3 - 1\,372i^2 + 3\,608i$$
$$+(-320i^6 + 3\,392i^5 - 12\,940i^4 + 21\,990i^3 - 15\,961i^2 + 1\,346i + 3\,096)\,D$$
$$+(-288i^5 + 2\,560i^4 - 8\,170i^3 + 12\,330i^2 - 9\,106i + 1\,630)\,D^2$$
$$+(-80i^4 + 480i^3 - 1\,165i^2 + 2\,110i - 1\,721)\,D^3 + (+40i^3 - 320i^2 + 580i - 70)\,D^4$$
$$+(+36i^2 - 172i + 160)\,D^5 + (+10i - 24)\,D^6 + D^7$$

$$92\,160\,\Pi^{6,1}_{6,1} = +128i^7 - 2\,336i^6 + 16\,120i^5 - 51\,010i^4 + 67\,241i^3 - 10\,166i^2 - 29\,352i$$
$$+(+448i^6 - 7\,248i^5 + 43\,260i^4 - 114\,205i^3 + 118\,427i^2 - 11\,752i - 21\,576)\,D$$
$$+(+672i^5 - 9\,360i^4 + 46\,310i^3 - 95\,340i^2 + 68\,605i - 2\,882)\,D^2$$
$$+(+560i^4 - 6\,440i^3 + 24\,725i^2 - 35\,195i + 13\,099)\,D^3$$
$$+(+280i^3 - 2\,490i^2 + 6\,585i - 4\,850)\,D^4 + (+84i^2 - 513i + 700)\,D^5 + (+14i - 44)\,D^6 + D^7$$

CLASS 1.—OPERATORS ON B_i.

I.—*Operators relating only to Inner Planet.*

$$\Pi'^{0}_{0} = 1$$

$$2\,\Pi'^{1}_{-1} = +2i - 2 - D$$

$$2\,\Pi'^{1}_{1} = -2i + 2 - D$$

$$8\,\Pi'^{2}_{-2} = +4i^2 - 3i - 1 + (-4i + 1)\,D + D^2$$

$$4\,\Pi'^{2}_{0} = -4i^2 + 8i - 4 + D + D^2$$

$$8\,\Pi'^{2}_{2} = +4i^2 - 13i + 9 + (+4i - 7)\,D + D^2$$

$$48\,\Pi'^{3}_{-3} = +8i^3 + 6i^2 - 10i - 4 + (-12i^2 - 9i + 4)\,D + (+6i + 3)\,D^2 - D^3$$

$$16\,\Pi'^{3}_{-1} = -8i^3 + 14i^2 - 6i + (+4i^2 - 3i + 2)\,D + (+2i - 1)\,D^2 - D^3$$

$$16\,\Pi'^{3}_{1} = +8i^3 - 34i^2 + 46i - 20 + (+4i^2 - 13i + 12)\,D + (-2i + 3)\,D^2 - D^3$$

$$48\,\Pi'^{3}_{3} = -8i^3 + 54i^2 - 110i + 64 + (-12i^2 + 57i - 62)\,D + (-6i + 15)\,D^2 - D^3$$

$$384\,\Pi'^{4}_{-4} = +16i^4 + 56i^3 + 19i^2 - 64i - 27 + (-32i^3 - 96i^2 - 42i + 28)\,D$$
$$+(+24i^2 + 54i + 17)\,D^2 + (-8i - 10)\,D^3 + D^4$$

$$96\,\Pi'^{4}_{-2} = -16i^4 + 4i^3 + 20i^2 - 10i + 2 + (+16i^3 + 0i^2 - 1i + 7)\,D + (0i^2 + 3i - 4)\,D^2 + (-4i - 2)\,D^3 + D^4$$

$$64\,\Pi'^{4}_{0} = +16i^4 - 64i^3 + 87i^2 - 46i + 7 + (0i^3 + 0i^2 + 0i + 2)\,D + (-8i^2 + 16i - 9)\,D^2 + (+0i - 2)\,D^3 + D^4$$

$$96\,\Pi'^{4}_{2} = -16i^4 + 124i^3 - 340i^2 + 394i - 162 + (-16i^3 + 96i^2 - 191i + 133)\,D$$
$$+(0i^2 - 3i + 2)\,D^2 + (+4i - 10)\,D^3 + D^4$$

$$384\,\Pi'^{4}_{4} = +16i^4 - 184i^3 + 739i^2 - 1\,196i + 625 + (+32i^3 - 288i^2 + 810i - 696)\,D$$
$$+(+24i^2 - 150i + 221)\,D^2 + (+8i - 26)\,D^3 + D^4$$

$$3\,840\,\Pi'^{5}_{-5} = +32i^5 + 240i^4 + 510i^3 + 70i^2 - 596i - 256 + (-80i^4 - 520i^3 - 955i^2 - 290i + 276)\,D$$
$$+(+80i^3 + 420i^2 + 580i + 140)\,D^2 + (-40i^2 - 150i - 115)\,D^3 + (+10i + 20)\,D^4 - D^5$$

$$768\ \Pi'^{\,5}_{-3} = -32i^5 - 80i^4 + 26i^3 + 74i^2 - 4i + 16 + (+48i^4 + 104i^3 + 17i^2 + 34i + 28)\,\mathrm{D}$$
$$+ (-16i^3 - 12i^3 - 4i - 36)\,\mathrm{D}^2 + (-8i^2 - 26i - 7)\,\mathrm{D}^3 + (+6i + 8)\,\mathrm{D}^4 - \mathrm{D}^5$$

$$384\ \Pi'^{\,5}_{-1} = +32i^5 - 80i^4 + 26i^3 + 70i^2 - 48i + 0 + (-16i^4 + 40i^3 - 33i^2 + 0i + 12)\,\mathrm{D}$$
$$+ (-16i^3 + 12i^2 + 12i - 16)\,\mathrm{D}^2 + (+8i^2 - 14i + 1)\,\mathrm{D}^3 + (+2i + 4)\,\mathrm{D}^4 - \mathrm{D}^5$$

$$384\ \Pi'^{\,5}_{\,1} = -32i^5 + 240i^4 - 666i^3 + 866i^2 - 544i + 136 + (-16i^4 + 88i^3 - 177i^2 + 164i - 56)\,\mathrm{D}$$
$$+ (+16i^3 - 84i^2 + 132i - 72)\,\mathrm{D}^2 + (+8i^2 - 18i + 5)\,\mathrm{D}^3 + (-2i + 8)\,\mathrm{D}^4 - \mathrm{D}^5$$

$$768\ \Pi'^{\,5}_{\,3} = +32i^5 - 400i^4 + 1\,894i^3 - 4\,250i^2 + 4\,516i - 1\,792 + (+48i^4 - 488i^3 + 1\,793i^2 - 2\,886i + 1\,764)\,\mathrm{D}$$
$$+ (+16i^3 - 108i^2 + 244i - 220)\,\mathrm{D}^2 + (-8i^2 + 58i - 91)\,\mathrm{D}^3 + (-6i + 20)\,\mathrm{D}^4 - \mathrm{D}^5$$

$$3\,840\ \Pi'^{\,5}_{\,5} = -32i^5 + 560i^4 - 3\,710i^3 + 11\,450i^2 - 16\,044i + 7\,776$$
$$+ (-80i^4 + 1\,160i^3 - 5\,995i^2 + 12\,910i - 9\,564)\,\mathrm{D} + (-80i^3 + 900i^2 - 3\,220i + 3\,620)\,\mathrm{D}^2$$
$$+ (-40i^2 + 310i - 575)\,\mathrm{D}^3 + (-10i + 40)\,\mathrm{D}^4 - \mathrm{D}^5$$

II.—*Operators relating only to Outer Planet.*

$$\Pi'^{0.\,0}_{0.\,0} = 1$$

$$2\ \Pi'^{0.\,1}_{0-1} = -2i - 1 + \mathrm{D}$$

$$2\ \Pi'^{0.\,1}_{0.\,1} = +2i + 3 + \mathrm{D}$$

$$8\ \Pi'^{0.\,2}_{0-2} = +4i^2 - i - 1 + (-4i + 1)\,\mathrm{D} + \mathrm{D}^2$$

$$4\ \Pi'^{0.\,2}_{0.\,0} = -4i^2 - 8i - 4 + \mathrm{D} + \mathrm{D}^2$$

$$8\ \Pi'^{0.\,2}_{0.\,2} = +4i^2 + 17i + 17 + (4i + 9)\,\mathrm{D} + \mathrm{D}^2$$

$$48\ \Pi'^{0.\,3}_{0-3} = -8i^3 + 18i^2 - 5i - 4 + (+12i^2 - 21i + 5)\,\mathrm{D} + (-6i + 6)\,\mathrm{D}^2 + \mathrm{D}^3$$

$$16\ \Pi'^{0.\,3}_{0-1} = +8i^3 + 10i^2 + i - 2 + (-4i^2 - 7i - 1)\,\mathrm{D} + (-2i + 2)\,\mathrm{D}^2 + \mathrm{D}^3$$

$$16\ \Pi'^{0.\,3}_{0.\,1} = -8i^3 - 38i^2 - 57i - 28 + (-4i^2 - 9i - 3)\,\mathrm{D} + (+2i + 6)\,\mathrm{D}^2 + \mathrm{D}^3$$

$$48\ \Pi'^{0.\,3}_{0.\,3} = +8i^3 + 66i^2 + 173i + 142 + (+12i^2 + 69i + 95)\,\mathrm{D} + (+6i + 18)\,\mathrm{D}^2 + \mathrm{D}^3$$

$$384\ \Pi'^{0.\,4}_{0-4} = +16i^4 - 88i^3 + 139i^2 - 40i - 27 + (-32i^3 + 144i^2 - 174i + 40)\,\mathrm{D} + (+24i^2 - 78i + 53)\,\mathrm{D}^2$$
$$+ (-8i + 14)\,\mathrm{D}^3 + \mathrm{D}^4$$

$$96\ \Pi'^{0.\,4}_{0-2} = -16i^4 + 12i^3 + 20i^2 + 2i - 6 + (+16i^3 + 0i^2 - 19i - 5)\,\mathrm{D} + (0i^2 - 15i + 8)\,\mathrm{D}^2 + (-4i + 6)\,\mathrm{D}^3 + \mathrm{D}^4$$

$$64\ \Pi'^{0.\,4}_{0.\,0} = +16i^4 + 64i^3 + 79i^2 + 30i - 1 + (0i^3 - 16i^2 - 32i - 10)\,\mathrm{D} + (-8i^2 - 16i + 3)\,\mathrm{D}^2 + (0i + 6)\,\mathrm{D}^3 + \mathrm{D}^4$$

$$96\ \Pi'^{0.\,4}_{0.\,2} = -16i^4 - 140i^3 - 436i^2 - 578i - 282 + (-16i^3 - 96i^2 - 173i - 95)\,\mathrm{D} + (0i^2 + 15i + 38)\,\mathrm{D}^2$$
$$+ (+4i + 14)\,\mathrm{D}^3 + \mathrm{D}^4$$

$$384\ \Pi'^{0.\,4}_{0.\,4} = +16i^4 + 216i^3 + 1\,051i^2 + 2\,164i + 1\,569 + (+32i^3 + 336i^2 + 1\,134i + 1\,220)\,\mathrm{D}$$
$$+ (+24i^2 + 174i + 305)\,\mathrm{D}^2 + (+8i + 30)\,\mathrm{D}^3 + \mathrm{D}^4$$

$$3\,840\ \Pi'^{0.\,5}_{0-5} = -32i^5 + 320i^4 - 1\,110i^3 + 1\,485i^2 - 434i - 256 + (+80i^4 - 680i^3 + 1\,915i^2 - 1\,940i + 434)\,\mathrm{D}$$
$$+ (-80i^3 + 540i^2 - 1\,090i + 615)\,\mathrm{D}^2 + (+40i^2 - 190i + 205)\,\mathrm{D}^3 + (-10i + 25)\,\mathrm{D}^4 + \mathrm{D}^5$$

$$768\ \Pi'^{0.\,5}_{0-3} = +32i^5 - 128i^4 + 94i^3 + 53i^2 + 10i - 32 + (-48i^4 + 136i^3 - 17i^2 - 60i - 42)\,\mathrm{D}$$
$$+ (+16i^3 + 12i^2 - 110i + 43)\,\mathrm{D}^2 + (+8i^2 - 50i + 49)\,\mathrm{D}^3 + (-6i + 13)\,\mathrm{D}^4 + \mathrm{D}^5$$

$$384\ \Pi'^{0.\,5}_{0-1} = -32i^5 - 64i^4 + 30i^3 + 107i^2 + 20i - 24 + (+16i^4 + 72i^3 + 33i^2 - 74i - 26)\,\mathrm{D}$$
$$+ (+16i^3 - 12i^2 - 66i + 15)\,\mathrm{D}^2 + (-8i^2 - 22i + 25)\,\mathrm{D}^3 + (-2i + 9)\,\mathrm{D}^4 + \mathrm{D}^5$$

$$384\,\Pi'^{\,0.\;5}_{\,0.\;1} = +32i^5+256i^4+738i^3+951i^2+560i+124+(+16i^4+56i^3-15i^2-146i-66)\,\mathrm{D}$$
$$+(-16i^3-108i^2-174i-29)\,\mathrm{D}^2+(-8i^2-10i+37)\,\mathrm{D}^3+(+2i+13)\,\mathrm{D}^4+\mathrm{D}^5$$

$$768\,\Pi'^{\,0.\;5}_{\,0.\;3} = -32i^5-448i^4-2\,398i^3-6\,143i^2-7\,582i-3\,664+(-48i^4-520i^3-1\,985i^2-3\,176i-1\,846)\,\mathrm{D}$$
$$+(-16i^3-84i^2-34i+183)\,\mathrm{D}^2+(+8i^2+82i+181)\,\mathrm{D}^3+(+6i+25)\,\mathrm{D}^4+\mathrm{D}^5$$

$$3\,840\,\Pi'^{\,0.\;5}_{\,0.\;5} = +32i^5+640i^4+4\,950i^3+18\,385i^2+32\,494i+21\,576$$
$$+(+80i^4+1\,320i^3+7\,915i^2+20\,320i+18\,694)\,\mathrm{D}+(+80i^3+1\,020i^2+4\,210i+5\,595)\,\mathrm{D}^2$$
$$+(+40i^2+350i+745)\,\mathrm{D}^3+(+10i+45)\,\mathrm{D}^4+\mathrm{D}^5$$

III.—*Operators relating to both Planets.*

$$4\,\Pi'^{\,1.\;1}_{\,1.\;1} = -4i^2-2i+6+(-4i-1)\,\mathrm{D}-\mathrm{D}^2$$

$$4\,\Pi'^{\,1.\;1}_{-1.\;1} = +4i^2+2i-6+(0i-5)\,\mathrm{D}-\mathrm{D}^2$$

$$4\,\Pi'^{\,1.\;1}_{\,1-1} = +4i^2-2i-2+(0i+3)\,\mathrm{D}-\mathrm{D}^2$$

$$4\,\Pi'^{\,1.\;1}_{-1-1} = -4i^2+2i+2+(+4i-1)\,\mathrm{D}-\mathrm{D}^2$$

$$16\,\Pi'^{\,2.\;1}_{-2-1} = -8i^3+2i^2+5i+1+(+12i^2-1i-2)\,\mathrm{D}+(-6i+0)\,\mathrm{D}^2+\mathrm{D}^3$$

$$8\,\Pi'^{\,2.\;1}_{\,0-1} = +8i^3-12i^2+0i+4+(-4i^2+6i-5)\,\mathrm{D}+(-2i+0)\,\mathrm{D}^2+\mathrm{D}^3$$

$$16\,\Pi'^{\,2.\;1}_{\,2-1} = -8i^3+22i^2-5i-9+(-4i^2-3i+16)\,\mathrm{D}+(+2i-8)\,\mathrm{D}^2+\mathrm{D}^3$$

$$16\,\Pi'^{\,2.\;1}_{-2.\;1} = +8i^3+6i^2-11i-3+(-4i^2-13i+2)\,\mathrm{D}+(-2i+4)\,\mathrm{D}^2+\mathrm{D}^3\,.$$

$$8\,\Pi'^{\,2.\;1}_{\,0.\;1} = -8i^3+4i^2+16i-12+(-4i^2+10i-1)\,\mathrm{D}+(+2i+4)\,\mathrm{D}^2+\mathrm{D}^3$$

$$16\,\Pi'^{\,2.\;1}_{\,2.\;1} = +8i^3-14i^2-21i+27+(+12i^2-15i-12)\,\mathrm{D}+(+6i-4)\,\mathrm{D}^2+\mathrm{D}^3$$

$$16\,\Pi'^{\,1.\;2}_{-1-2} = +8i^3-10i^2+0i+2+(-12i^2+11i-1)\,\mathrm{D}+(+6i-3)\,\mathrm{D}^2-\mathrm{D}^3$$

$$16\,\Pi'^{\,1.\;2}_{\,1-2} = -8i^3+10i^2+0i-2+(+4i^2-9i+3)\,\mathrm{D}+(+2i+1)\,\mathrm{D}^2-\mathrm{D}^3$$

$$8\,\Pi'^{\,1.\;2}_{-1.\;0} = -8i^3-8i^2+8i+8+(+4i^2+10i+2)\,\mathrm{D}+(+2i-3)\,\mathrm{D}^2-\mathrm{D}^3$$

$$8\,\Pi'^{\,1.\;2}_{\,1.\;0} = +8i^3+8i^2-8i-8+(+4i^2+6i+6)\,\mathrm{D}+(-2i+1)\,\mathrm{D}^2-\mathrm{D}^3$$

$$16\,\Pi'^{\,1.\;2}_{-1.\;2} = +8i^3+26i^2+0i-34+(+4i^2-7i-35)\,\mathrm{D}+(-2i-11)\,\mathrm{D}^2-\mathrm{D}^3$$

$$16\,\Pi'^{\,1.\;2}_{\,1.\;2} = -8i^3-26i^2+0i+34+(-12i^2-27i+1)\,\mathrm{D}+(-6i-7)\,\mathrm{D}^2-\mathrm{D}^3$$

$$96\,\Pi'^{\,3.\;1}_{-3-1} = -16i^4-20i^3+14i^2+18i+4+(+32i^3+36i^2-9i-8)\,\mathrm{D}+(-24i^2-21i+1)\,\mathrm{D}^2$$
$$+(+8i+4)\,\mathrm{D}^3-\mathrm{D}^4$$

$$32\,\Pi'^{\,3.\;1}_{-1-1} = +16i^4-20i^3-2i^2+6i+0+(-16i^3+16i^2-7i-2)\,\mathrm{D}+(0i^2-3i+3)\,\mathrm{D}^2$$
$$+(+4i+0)\,\mathrm{D}^3-\mathrm{D}^4$$

$$32\,\Pi'^{\,3.\;1}_{\,1-1} = -16i^4+60i^3-58i^2-6i+20+(0i^3-12i^2+35i-32)\,\mathrm{D}+(+8i^2-17i+9)\,\mathrm{D}^2$$
$$+(0i+4)\,\mathrm{D}^3-\mathrm{D}^4$$

$$96\,\Pi'^{\,3.\;1}_{\,3-1} = +16i^4-100i^3+166i^2-18i-64+(+16i^3-48i^2-43i+126)\,\mathrm{D}+(0i^2+33i-77)\,\mathrm{D}^2$$
$$+(-4i+16)\,\mathrm{D}^3-\mathrm{D}^4$$

$$96\,\Pi'^{\,3.\;1}_{-3.\;1} = +16i^4+36i^3-2i^2-38i-12+(-16i^3-48i^2-29i+8)\,\mathrm{D}+(0i^2+15i+13)\,\mathrm{D}^2$$
$$+(+4i+0)\,\mathrm{D}^3-\mathrm{D}^4$$

$$32\,\Pi'^{\,3.\;1}_{-1.\;1} = -16i^4+4i^3+30i^2-18i+0+(0i^3+20i^2-11i+6)\,\mathrm{D}+(+8i^2+1i-1)\,\mathrm{D}^2$$
$$+(0i-4)\,\mathrm{D}^3-\mathrm{D}^4$$

$$32\,\Pi'{}^{3\ 1}_{1\ 1} = +16i^4-44i^3-10i^2+98i-60+(+16i^3-48i^2+31i+16)\,D+(0i^2-13i+21)\,D^2$$
$$+(-4i+0)\,D^3-D^4$$

$$96\,\Pi'{}^{3.\ 1}_{3.\ 1} = -16i^4+84i^3-58i^2-202i+192+(-32i^3+132i^2-63i-122)\,D$$
$$+(-24i^2+69i-17)\,D^2+(-8i+12)\,D^3-D^4$$

$$64\,\Pi'{}^{2.\ 2}_{-2-2} = +16i^4-16i^3-5i^2+4i+1+(-32i^3+24i^2+4i-2)\,D+(+24i^2-12i-1)\,D^2$$
$$+(-8i+2)\,D^3+D^4$$

$$32\,\Pi'{}^{2.\ 2}_{0-2} = -16i^4+36i^3-20i^2-4i+4+(+16i^3-32i^2+23i-5)\,D+(0i^2+3i-4)\,D^2$$
$$+(-4i+2)\,D^3+D^4$$

$$64\,\Pi'{}^{2.\ 2}_{2-2} = +16i^4-56i^3+45i^2+4i-9+(0i^3+24i^2-46i+16)\,D+(-8i^2+18i+1)\,D^2$$
$$+(0i-6)\,D^3+D^4$$

$$32\,\Pi'{}^{2.\ 2}_{-2.\ 0} = -16i^4-20i^3+12i^2+20i+4+(+16i^3+32i^2+5i-5)\,D+(0i^2-15i-4)\,D^2$$
$$+(-4i+2)\,D^3+D^4$$

$$16\,\Pi'{}^{2.\ 2}_{0\ 0} = +16i^4+0i^3-32i^2+0i+16+(0i^3-8i^2+0i-8)\,D+(-8i^2+0i-7)\,D^2+(0i+2)\,D^3+D^4$$

$$32\,\Pi'{}^{2.\ 2}_{2.\ 0} = -16i^4+20i^3+52i^2-20i-36+(-16i^3+0i^2+27i+37)\,D+(0i^2-17i-2)\,D^2$$
$$+(+4i-6)\,D^3+D^4$$

$$64\,\Pi'{}^{2.\ 2}_{-2.\ 2} = +16i^4+56i^3+13i^2-68i-17+(0i^3-40i^2-82i+8)\,D+(-8i^2-18i+25)\,D^2$$
$$+(0i+10)\,D^3+D^4$$

$$32\,\Pi'{}^{2.\ 2}_{0.\ 2} = -16i^4-36i^3+52i^2+68i-68+(-16i^3+0i^2+73i-19)\,D+(0i^2+29i+22)\,D^2$$
$$+(+4i+10)\,D^3+D^4$$

$$64\,\Pi'{}^{2.\ 2}_{2.\ 2} = +16i^4+16i^3-117i^2-68i+153+(+32i^3+24i^2-132i-38)\,D+(+24i^2+12i-37)\,D^2$$
$$+(+8i+2)\,D^3+D^4$$

$$96\,\Pi'{}^{1.\ 3}_{-1-3} = -16i^4+52i^3-46i^2+2i+8+(+32i^3-84i^2+57i-6)\,D+(-24i^2+45i-17)\,D^2$$
$$+(+8i-8)\,D^3-D^4$$

$$32\,\Pi'{}^{1.\ 3}_{-1-1} = +16i^4+4i^3-18i^2-6i+4+(-16i^3-16i^2+11i+4)\,D+(0i^2+15i-3)\,D^2+(+4i-4)\,D^3-D^4$$

$$32\,\Pi'{}^{1.\ 3}_{-1.\ 1} = -16i^4-60i^3-38i^2+58i+56+(0i^3+28i^2+69i+34)\,D+(+8i^2+17i-9)\,D^2$$
$$+(0i-8)\,D^3-D^4$$

$$96\,\Pi'{}^{1.\ 3}_{-1.\ 3} = +16i^4+116i^3+214i^2-62i-284+(+16i^3+48i^2-121i-332)\,D+(0i^2-45i-131)\,D^2$$
$$+(-4i-20)\,D^3-D^4$$

$$96\,\Pi'{}^{1.\ 3}_{1-3} = +16i^4-52i^3+46i^2-2i-8+(-16i^3+48i^2-47i+14)\,D+(0i^2-3i+7)\,D^2$$
$$+(+4i-4)\,D^3-D^4$$

$$32\,\Pi'{}^{1.\ 3}_{1-1} = -16i^4-4i^3+18i^2+6i-4+(0i^3-4i^2-13i+0)\,D+(+8i^2-1i+5)\,D^2+(0i+0)\,D^3-D^4$$

$$32\,\Pi'{}^{1.\ 3}_{1.\ 1} = +16i^4+60i^3+38i^2-58i-56+(+16i^3+48i^2+45i+22)\,D+(0i^2+1i+15)\,D^2$$
$$+(-4i-4)\,D^3-D^4$$

$$96\,\Pi'{}^{1.\ 3}_{1.\ 3} = -16i^4-116i^3-214i^2+62i+284+(-32i^3-180i^2-225i+48)\,D+(-24i^2-93i-59)\,D^2$$
$$+(-8i-16)\,D^3-D^4$$

$$768\,\Pi'{}^{4.\ 1}_{-4-1} = -32i^5-128i^4-94i^3+109i^2+118i+27+(+80i^4+280i^3+199i^2-78i-55)\,D$$
$$+(-80i^3-228i^2-130i+11)\,D^2+(+40i^2+82i+27)\,D^3+(-10i-11)\,D^4+D^5$$

$$768\,\Pi'^{\,4.\;1}_{-4.\;1} = +32i^5+160i^4+206i^3-71i^2-246i-81+(-48i^4-232i^3-353i^2-134i+57)\,D$$
$$+(+16i^3+84i^2+154i+79)\,D^2+(+8i^2+10i-13)\,D^3+(-6i-7)\,D^4+D^5$$

$$192\,\Pi'^{\,4.\;1}_{-2-1} = +32i^5+8i^4-44i^3+0i^2+6i-2+(-48i^4-12i^3+22i^2-23i-5)\,D$$
$$+(+16i^3-6i^2+4i+11)\,D^2+(+8i^2+11i-2)\,D^3+(-6i-3)\,D^4+D^5$$

$$192\,\Pi'^{\,4.\;1}_{-2.\;1} = -32i^5-40i^4+52i^3+40i^2-26i+6+(+16i^4+52i^3+18i^2+1i+23)\,D$$
$$+(+16i^3+6i^2+0i-5)\,D^2+(-8i^2-13i-10)\,D^3+(-2i+1)\,D^4+D^5$$

$$128\,\Pi'^{\,4.\;1}_{0-1} = -32i^5+112i^4-110i^3+5i^2+32i-7+(+16i^4-64i^3+87i^2-50i+5)\,D$$
$$+(+16i^3-24i^2+2i+11)\,D^2+(-8i^2+20i-7)\,D^3+(-2i-3)\,D^4+D^5$$

$$128\,\Pi'^{\,4.\;1}_{0.\;1} = +32i^5-80i^4-18i^3+169i^2-124i+21+(+16i^4-64i^3+87i^2-42i+13)\,D$$
$$+(-16i^3+8i^2+30i-25)\,D^2+(-8i^2+12i-15)\,D^3+(+2i+1)\,D^4+D^5$$

$$192\,\Pi'^{\,4.\;1}_{2-1} = +32i^5-232i^4+556i^3-448i^2-70i+162+(+16i^4-52i^3-54i^2+319i-295)\,D$$
$$+(-16i^3+102i^2-192i+131)\,D^2+(-8i^2+13i+12)\,D^3+(+2i-11)\,D^4+D^5$$

$$192\,\Pi'^{\,4.\;1}_{2.\;1} = -32i^5+200i^4-308i^3-232i^2+858i-486+(-48i^4+268i^3+434i^2+87i+237)\,D$$
$$+(-16i^3+90i^2-196i+139)\,D^2+(+8i^2-11i-28)\,D^3+(+6i-7)\,D^4+D^5$$

$$768\,\Pi'^{\,4.\;1}_{4-1} = -32i^5+352i^4-1\,294i^3+1\,653i^2-54i-625+(-48i^4+360i^3-593i^2-614i+1\,321)\,D$$
$$+(-16i^3-12i^2+518i-917)\,D^2+(+8i^2-106i+247)\,D^3+(+6i-27)\,D^4+D^5$$

$$768\,\Pi'^{\,4.\;1}_{4.\;1} = +32i^5-320i^4+926i^3-175i^2-2\,338i+1\,875+(+80i^4-664i^3+1\,495i^2-158i+1\,463)\,D$$
$$+(+80i^3-516i^2+802i-33)\,D^2+(+40i^2-178i+143)\,D^3+(+10i-23)\,D^4+D^5$$

$$384\,\Pi'^{\,3.\;2}_{-3-2} = +32i^6+16i^4-54i^3-12i^2+14i+4+(-80i^4-40i^3+83i^2+11i-8)\,D$$
$$+(+80i^3+36i^2-44i-3)\,D^2+(-40i^2-14i+8)\,D^3+(+10i+2)\,D^4-D^5$$

$$192\,\Pi'^{\,3.\;2}_{-3.\;0} = -32i^5-88i^4-40i^3+72i^2+72i+16+(+48i^4+140i^3+110i^2-6i-20)\,D$$
$$+(-16i^3-66i^2-67i-12)\,D^2+(-8i^2+5i+11)\,D^3+(+6i+2)\,D^4-D^5$$

$$384\,\Pi'^{\,3.\;2}_{-3.\;2} = +32i^5+160i^4+198i^3-84i^2-238i-68+(-16i^4-144i^3-327i^2-191i+32)\,D$$
$$+(-16i^3-24i^2+78i+83)\,D^2+(+8i^2+40i+14)\,D^3+(+2i-6)\,D^4-D^5$$

$$128\,\Pi'^{\,3.\;2}_{-1-2} = -32i^5+64i^4-30i^3-8i^2+6i+0+(+48i^4-80i^3+45i^2-5i-2)\,D$$
$$+(-16i^3+24i^2-18i+3)\,D^2+(-8i^2+4i+2)\,D^3+(+6i-2)\,D^4-D^5$$

$$64\,\Pi'^{\,3.\;2}_{-1.\;0} = +32i^5+8i^4-56i^3-8i^2+24i+0+(-16i^4-28i^3+14i^2-10i-8)\,D$$
$$+(-16i^3+6i^2-9i+6)\,D^2+(+8i^2+7i+5)\,D^3+(2i-2)\,D^4-D^5$$

$$128\,\Pi'^{\,3.\;2}_{-1.\;2} = -32i^5-80i^4+78i^3+136i^2-102i+0+(-16i^4+40i^3+127i^2-71i+34)\,D$$
$$+(+16i^3+68i^2-8i+1)\,D^2+(+8i^2-6i-24)\,D^3+(-2i-10)\,D^4-D^5$$

$$128\,\Pi'^{\,3.\;2}_{1-2} = +32i^5-144i^4+210i^3-92i^2-26i+20+(-16i^4+88i^3-161i^2+127i-32)\,D$$
$$+(-16i^3+36i^2-16i-11)\,D^2+(+8i^2-26i+16)\,D^3+(2i+2)\,D^4-D^5$$

$$64\,\Pi'^{\,3.\;2}_{1.\;0} = -32i^5+72i^4+56i^3-152i^2-24i+80+(-16i^4+28i^3+6i^2+2i-68)\,D$$
$$+(+16i^3-26i^2+17i-20)\,D^2+(+8i^2-7i+19)\,D^3+(-2i+2)\,D^4-D^5$$

$$128\,\Pi'^{\,3.\;2}_{1.\;2} = +32i^5+0i^4-258i^3+124i^2+442i-340+(+48i^4-48i^3-227i^2+317i+24)\,D$$
$$+(+16i^3-72i^2-6i+139)\,D^2+(-8i^2-36i+22)\,D^3+(-6i-6)\,D^4-D^5$$

$$384\,\Pi'^{\,3.\;2}_{3-2} = -32i^5+224i^4-486i^3+312i^2+46i-64+(-16i^4+16i^3+201i^2-361i+126)\,D$$
$$+(+16i^3-120i^2+186i-13)\,D^2+(+8i^2-8i-46)\,D^3+(-2i+14)\,D^4-D^5$$

$$192\,\Pi'{}^{3.\;2}_{3.\;0} = +32i^5 - 152i^4 + 40i^3 + 408i^2 - 72i - 256 + (+48i^4 - 140i^3 - 106i^2 + 158i + 312)\,\mathrm{D}$$
$$+ (+16i^3 + 30i^2 - 149i - 58)\,\mathrm{D}^2 + (-8i^2 + 59i - 43)\,\mathrm{D}^3 + (-6i + 14)\,\mathrm{D}^4 - \mathrm{D}^5$$

$$384\,\Pi'{}^{3.\;2}_{3.\;2} = -32i^5 + 80i^4 + 342i^3 - 696i^2 - 782i + 1\,088 + (-80i^4 + 168i^3 + 563i^2 - 819i - 478)\,\mathrm{D}$$
$$+ (-80i^3 + 132i^2 + 308i - 239)\,\mathrm{D}^2 + (-40i^2 + 46i + 56)\,\mathrm{D}^3 + (-10i + 6)\,\mathrm{D}^4 - \mathrm{D}^5$$

$$384\,\Pi'{}^{2.\;3}_{-2\,-3} = -32i^5 + 96i^4 - 66i^3 - 19i^2 + 17i + 4 + (+80i^4 - 200i^3 + 109i^2 + 17i - 9)\,\mathrm{D}$$
$$+ (-80i^3 + 156i^2 - 58i - 5)\,\mathrm{D}^2 + (+40i^2 - 54i + 10)\,\mathrm{D}^3 + (-10i + 7)\,\mathrm{D}^4 + \mathrm{D}^5$$

$$128\,\Pi'{}^{2.\;3}_{-2\,-1} = +32i^5 + 16i^4 - 34i^3 - 21i^2 + 5i + 2 + (-48i^4 - 48i^3 + 27i^2 + 19i - 1)\,\mathrm{D}$$
$$+ (+16i^3 + 48i^2 - 6i - 5)\,\mathrm{D}^2 + (+8i^2 - 20i + 0)\,\mathrm{D}^3 + (-6i + 3)\,\mathrm{D}^4 + \mathrm{D}^5$$

$$128\,\Pi'{}^{2.\;3}_{-2.\;1} = -32i^5 - 128i^4 - 106i^3 + 97i^2 + 141i + 28 + (+16i^4 + 120i^3 + 209i^2 + 73i - 25)\,\mathrm{D}$$
$$+ (+16i^3 + 12i^2 - 74i - 37)\,\mathrm{D}^2 + (-8i^2 - 34i + 2)\,\mathrm{D}^3 + (-2i + 7)\,\mathrm{D}^4 + \mathrm{D}^5$$

$$384\,\Pi'{}^{2.\;3}_{-2.\;3} = +32i^5 + 240i^4 + 486i^3 - 17i^2 - 599i - 142 + (+16i^4 - 16i^3 - 465i^2 - 749i + 47)\,\mathrm{D}$$
$$+ (-16i^3 - 144i^2 - 198i + 219)\,\mathrm{D}^2 + (-8i^2 + 0i + 112)\,\mathrm{D}^3 + (+2i + 19)\,\mathrm{D}^4 + \mathrm{D}^5$$

$$192\,\Pi'{}^{2.\;3}_{0\,-3} = +32i^5 - 136i^4 + 196i^3 - 96i^2 - 12i + 16 + (-48i^4 + 172i^3 - 218i^2 + 119i - 24)\,\mathrm{D}$$
$$+ (+16i^3 - 42i^2 + 46i - 23)\,\mathrm{D}^2 + (+8i^2 - 19i + 7)\,\mathrm{D}^3 + (-6i + 7)\,\mathrm{D}^4 + \mathrm{D}^5$$

$$64\,\Pi'{}^{2.\;3}_{0\,-1} = -32i^5 + 24i^4 + 44i^3 - 24i^2 - 20i + 8 + (+16i^4 + 4i^3 - 25i^2 + 21i + 2)\,\mathrm{D}$$
$$+ (+16i^3 - 18i^2 + 18i - 11)\,\mathrm{D}^2 + (-8i^2 - i - 3)\,\mathrm{D}^3 + (-2i + 3)\,\mathrm{D}^4 + \mathrm{D}^5$$

$$64\,\Pi'{}^{2.\;3}_{0.\;1} = +32i^5 + 88i^4 - 44i^3 - 192i^2 + 4i + 112 + (+16i^4 - 4i^3 - 82i^2 - 45i - 16)\,\mathrm{D}$$
$$+ (-16i^3 - 50i^2 - 26i - 55)\,\mathrm{D}^2 + (-8i^2 + 1i - 1)\,\mathrm{D}^3 + (+2i + 7)\,\mathrm{D}^4 + \mathrm{D}^5$$

$$192\,\Pi'{}^{2.\;3}_{0\;\;3} = -32i^5 - 200i^4 - 196i^3 + 552i^2 + 444i - 568 + (-48i^4 - 172i^3 + 190i^2 + 657i - 238)\,\mathrm{D}$$
$$+ (-16i^3 + 54i^2 + 362i + 165)\,\mathrm{D}^2 + (+8i^2 + 83i + 109)\,\mathrm{D}^3 + (+6i + 19)\,\mathrm{D}^4 + \mathrm{D}^5$$

$$384\,\Pi'{}^{2.\;3}_{2\,-3} = -32i^5 + 176i^4 - 326i^3 + 211i^2 + 7i - 36 + (+16i^4 - 112i^3 + 255i^2 - 235i + 73)\,\mathrm{D}$$
$$+ (+16i^3 - 48i^2 + 30i + 15)\,\mathrm{D}^2 + (-8i^2 + 32i - 28)\,\mathrm{D}^3 + (-2i - 1)\,\mathrm{D}^4 + \mathrm{D}^5$$

$$128\,\Pi'{}^{2.\;3}_{2\,-1} = +32i^5 - 64i^4 - 54i^3 + 69i^2 + 35i - 18 + (+16i^4 + 8i^3 - 15i^2 - 65i + 5)\,\mathrm{D}$$
$$+ (-16i^3 + 44i^2 + 2i + 23)\,\mathrm{D}^2 + (-8i^2 + 2i - 6)\,\mathrm{D}^3 + (+2i - 5)\,\mathrm{D}^4 + \mathrm{D}^5$$

$$128\,\Pi'{}^{2.\;3}_{2.\;1} = -32i^5 - 48i^4 + 194i^3 + 287i^2 - 149i - 252 + (-48i^4 - 80i^3 + 107i^2 + 245i + 169)\,\mathrm{D}$$
$$+ (-16i^3 - 48i^2 - 66i + 47)\,\mathrm{D}^2 + (+8i^2 - 12i - 36)\,\mathrm{D}^3 + (+6i - 1)\,\mathrm{D}^4 + \mathrm{D}^5$$

$$384\,\Pi'{}^{2.\;3}_{2.\;3} = +32i^5 + 160i^4 - 94i^3 - 1\,087i^2 - 289i + 1\,278 + (+80i^4 + 328i^3 - 179i^2 - 1\,257i - 139)\,\mathrm{D}$$
$$+ (+80i^3 + 252i^2 - 110i - 361)\,\mathrm{D}^2 + (+40i^2 + 86i - 22)\,\mathrm{D}^3 + (+10i + 11)\,\mathrm{D}^4 + \mathrm{D}^5$$

$$768\,\Pi'{}^{1.\;4}_{-1\,-4} = +32i^5 - 208i^4 + 454i^3 - 358i^2 + 26i + 54 + (-80i^4 + 440i^3 - 775i^2 + 468i - 53)\,\mathrm{D}$$
$$+ (+80i^3 - 348i^2 + 436i - 146)\,\mathrm{D}^2 + (-40i^2 + 122i - 81)\,\mathrm{D}^3 + (+10i - 16)\,\mathrm{D}^4 - \mathrm{D}^5$$

$$192\,\Pi'{}^{1.\;4}_{-1\,-2} = -32i^5 + 56i^4 + 16i^3 - 36i^2 - 16i + 12 + (+48i^4 - 44i^3 - 58i^2 + 26i + 16)\,\mathrm{D}$$
$$+ (-16i^3 - 30i^2 + 65i - 11)\,\mathrm{D}^2 + (-8i^2 + 35i - 20)\,\mathrm{D}^3 + (+6i - 8)\,\mathrm{D}^4 - \mathrm{D}^5$$

$$128\,\Pi'{}^{1.\;4}_{-1.\;0} \quad +32i^5 + 96i^4 + 30i^3 - 98i^2 - 62i + 2 + (-16i^4 - 96i^3 - 111i^2 + 14i + 21)\,\mathrm{D}$$
$$+ (-16i^3 + 0i^2 + 70i + 4)\,\mathrm{D}^2 + (8i^2 + 28i - 15)\,\mathrm{D}^3 + (2i - 8)\,\mathrm{D}^4 - \mathrm{D}^5$$

$$192\,\Pi'{}^{1.\;4}_{-1.\;2} = -32i^5 - 248i^4 - 592i^3 - 284i^2 + 592i + 564 + (-16i^4 - 20i^3 + 282i^2 + 734i + 472)\,\mathrm{D}$$
$$+ (+16i^3 + 126i^2 + 219i + 19)\,\mathrm{D}^2 + (+8i^2 + 5i - 66)\,\mathrm{D}^3 + (-2i - 16)\,\mathrm{D}^4 - \mathrm{D}^5$$

$$768\,\Pi'{}^{1.\;4}_{-1.\;4} = +32i^5 + 400i^4 + 1\,670i^3 + 2\,226i^2 - 1\,190i - 3\,138 + (+48i^4 + 392i^3 + 545i^2 - 1\,992i - 4\,009)\,\mathrm{D}$$
$$+ (+16i^3 - 36i^2 - 872i - 1\,830)\,\mathrm{D}^2 + (-8i^2 - 130i - 365)\,\mathrm{D}^3 + (-6i - 32)\,\mathrm{D}^4 - \mathrm{D}^5$$

$$768\,\Pi'^{\,1.\;4}_{\,1.\,-4} = -32\,i^5 + 208\,i^4 - 454\,i^3 + 358\,i^2 - 26\,i - 54 + (+48\,i^4 - 264\,i^3 + 497\,i^2 - 388\,i + 107)\,D$$
$$+ (-16\,i^3 + 60\,i^2 - 88\,i + 66)\,D^2 + (-8\,i^2 + 34\,i - 25)\,D^3 + (+6\,i - 12)\,D^4 - D^5$$

$$192\,\Pi'^{\,1.\;4}_{\,1.\,-2} = +32\,i^5 - 56\,i^4 - 16\,i^3 + 36\,i^2 + 16\,i - 12 + (-16\,i^4 + 20\,i^3 + 18\,i^2 - 30\,i - 4)\,D$$
$$+ (-16\,i^3 + 30\,i^2 - 27\,i + 21)\,D^2 + (+8\,i^2 - 5\,i + 4)\,D^3 + (+2\,i - 4)\,D^4 - D^5$$

$$128\,\Pi'^{\,1.\;4}_{\,1.\;0} = -32\,i^5 - 96\,i^4 - 30\,i^3 + 98\,i^2 + 62\,i - 2 + (-16\,i^4 - 32\,i^3 - 47\,i^2 - 74\,i - 19)\,D$$
$$+ (+16\,i^3 + 32\,i^2 - 6\,i + 16)\,D^2 + (+8\,i^2 + 4\,i + 9)\,D^3 + (-2\,i - 4)\,D^4 - D^5$$

$$192\,\Pi'^{\,1.\;4}_{\,1.\;2} = +32\,i^5 + 248\,i^4 + 592\,i^3 + 284\,i^2 - 592\,i - 564 + (+48\,i^4 + 300\,i^3 + 590\,i^2 + 422\,i + 92)\,D$$
$$+ (+16\,i^3 + 66\,i^2 + 127\,i + 171)\,D^2 + (-8\,i^2 - 35\,i - 10)\,D^3 + (-6\,i - 12)\,D^4 - D^5$$

$$768\,\Pi'^{\,1.\;4}_{\,1.\;4} = -32\,i^5 - 400\,i^4 - 1\,670\,i^3 - 2\,226\,i^2 + 1\,190\,i + 3\,138 + (-80\,i^4 - 824\,i^3 - 2\,647\,i^2 - 2\,336\,i + 871)\,D$$
$$+ (-80\,i^3 - 636\,i^2 - 1\,396\,i - 610)\,D^2 + (-40\,i^2 - 218\,i - 245)\,D^3 + (-10\,i - 28)\,D^4 - D^5$$

II.

INEQUALITIES OF LONG PERIOD, AND OF THE SECOND ORDER AS TO THE MASSES,

IN THE

MEAN LONGITUDES OF THE FOUR INNER PLANETS.

BY

SIMON NEWCOMB.

CONTENTS OF PART II.

INEQUALITIES OF LONG PERIOD, AND OF THE SECOND ORDER AS TO THE MASSES IN THE MEAN LONGITUDES OF THE FOUR INNER PLANETS

I.

General character of these terms.

The terms to be considered in the present paper are those which arise from close approximation to linear relations among the mean motions of three planets. From an analytic point of view they owe their origin to the fact that when the inequalities of the elements of a planet are fully developed, terms appear which contain as a factor the ratio of the mean motion n, to the linear function

$$i\, n + i'\, n' + i''\, n''$$

If for any values of the integers i, i', i'', this ratio is a large number, the inequalities in question may also become large, even when the other factors which enter into them are small. Those of the mean longitudes will be much the largest, because they and they alone contain the square of the ratio above mentioned as a factor.

In the case of the four inner planets, to which attention is confined in the present paper, there is reason to suppose that the only inequalities of the class in question which can be of importance in the present state of astronomy are those of the mean longitude. To them, therefore, attention will be confined.

In the only method at present practicable of treating these inequalities for the purpose of tabulation, the perihelia and nodes of the several planets are considered as invariable. Since this is not the case the results are not rigorous, and can hold true for only a few centuries. In the direction of obtaining more rigorous results, two steps may be taken. One would be to consider the inequalities as functions not only of the mean longitudes, but also of the longitudes of the perihelia and of the nodes, and to regard these elements as affected by their secular variations as they exist at the present time. We should then have, instead of a single pair of terms containing a given linear function of the mean longitudes as an argument, a long series of terms containing combinations of multiples of the perihelia and nodes. But the results would still not be rigorous for all time. To obtain such results we should have to adopt the system developed in the paper of the present author, published in 1874 by the Smithsonian Institution.* By this method the perihelia and nodes are replaced by linear combinations of certain invariable arguments. The result would be for each combination of the mean motions, an indefinite series of terms having different periods. In

*Smithsonian Contributions to Knowledge, Vol. XXI, No. 281. See also POINCARÉ: *Les Méthodes Nouvelles de la Mécanique Céleste,* Tome II.

the present condition of astronomy, the actual numerical development of the quantities is out of the question, and the results would not admit of application if obtained. When astronomers have to construct a single theory which shall be valid for several thousand years, this method may have to be adopted.

By considering the perihelia and nodes as constant we obtain results which are rigorous for the particular epoch chosen for computation, and deviate very slightly for several centuries from it. This remains true even if the effect of introducing the motion of the nodes and perihelia should be to materially alter the values of the coefficients for the inequalities.

Of the inequalities in question, the most important is that in the motion of the Earth around the Sun, which was, I believe, discovered by HANSEN. It arises from the modification of the action of Mars on the Earth, produced by the inequalities of the motion of these two planets by Jupiter. The argument is

$$3\,J - 8\,M + 4\,E$$

the letters being the initials of the names of the planets. The period of this inequality is about 1,850 years. In the case of the Earth, its coefficient is about 7″, and its introduction by HANSEN and LE VERRIER has caused a not unimportant change in the measurement of mean solar time. Solar time as used in astronomy is measured by the mean longitude of the Sun, which is obtained by subtracting from its actual longitude as observed all the periodic inequalities with which its motion is affected. The inequality in question attained its minimum about our time. The result might be described as a discovery that, since accurate astronomical observation commenced, the observed longitude of the Sun, corrected for all the inequalities previously known, was about 7″ too small in arc, or nearly 0.ˢ45 when referred to mean time. Thus arose a change of this amount in the sidereal time of mean noon, and thus in all determinations of mean time, through the introduction of the table of HANSEN and LE VERRIER.

It may be an open question whether this change was advisable. Its only practical object was to unify the measure of solar time as it would have been determined a thousand years ago, and as it will be determined a thousand years hence, with that determined now. Practically, however, the former determinations do not exist, and it might be questionable whether a discrepancy of a second between the determinations of time during the nineteenth and twentieth centuries and those to be determined a thousand years hence will prove troublesome. It must also be remembered that, from what has just been said, the character of the inequality, as it will be found to exist after one or two thousand years, must be very uncertain. The possible error in our own determinations of the Sun's mean motion will also affect the result. When the astronomers of a thousand years hence unify our measure of time with their own, they are as likely to fall upon BESSEL's mean longitude as the standard for our time as they are upon the corrected mean longitudes of HANSEN and LE VERRIER. Still, as the change has been made, it is probably better to adhere to it. The question of whether this shall be done is not, however, germane to the present paper.

Another inequality, depending on the argument

$$4\,M - 7\,E + 3\,V,$$

was discovered by LE VERRIER. Its period is 308 years, and as, in the case of the Sun, its coefficient is only about $0''.3$, it is hardly important to the astronomy of our time. I have, however, thought it best to follow the example of LE VERRIER and introduce it into the tables.

Another argument computed in the present paper is that depending on the combination

$$\mathrm{J} - 2\,\mathrm{M} + \mathrm{E},$$

which is sensible only in the motion of Mars.

LE VERRIER's method of computing these inequalities must have been very tedious. In accordance with the general principles on which I have constructed the planetary tables I have deemed it advisable to determine independently the numerical values of all the inequalities. The method adopted is different *ab initio* from that of LE VERRIER, for although the same development of the perturbative function is used, yet the terms enter into the inequalities in an entirely different way.

II.

General formulæ for the largest terms of the inequalities.

The terms of the mean longitude which contain the square of the ratio of the mean motion of a planet to the motion of an argument arise from the following rigorous equation for the variation of the mean distance of the planet:

$$\mathrm{D}_t \frac{\mu}{a} = -2\left(\frac{d\,\mathrm{R}}{d\,\mathrm{v}}\,\mathrm{D}_t\, v + \frac{d\,\mathrm{R}}{d\,\rho}\,\mathrm{D}_t\, \rho\right) \tag{1}$$

Let us put, for brevity,

$$s = \int_t \left(\frac{d\,\mathrm{R}}{d\,\mathrm{v}}\,\mathrm{D}_t\, v + \frac{d\,\mathrm{R}}{d\,\rho}\,\mathrm{D}_t\, \rho\right) \tag{2}$$

no arbitrary constant being added. We then have

$$\frac{\mu}{a} = \frac{\mu}{a_0} - 2\,s$$

a_0 being the constant of the mean distance, and $\frac{1}{2}\frac{\mu}{a_0}$ the arbitrary constant in (2). What we actually want for the terms in question is the value of the mean motion, n, to terms of the second order in s. From the relation

$$a^3\, n^2 = \mu$$

we have

$$\mu\, n = \mathrm{A}^{\frac{3}{2}}$$

where we put, for brevity,

$$\mathrm{A} = \frac{\mu}{a}$$

Putting n_0, A_0, etc., for the constant parts of n, A, etc., and

$$\delta n = n - n_0$$
$$\delta A = A - A_0 = -2s$$
etc., etc.,

we find, to terms of the second order,

$$\delta n = -\frac{3a_0 n_0}{\mu} s + \frac{3}{2}\frac{a_0^2 n_0}{\mu^2} s^2$$

Let us also put

$$s = s_1 + s_2$$

s_1 being the terms of the first order in s, and s_2 those of the second order. Then we have

$$n = n_0 - \frac{3a_0 n_0}{\mu} s_1 - \frac{3a_0 n_0}{\mu} s_2 + \frac{3}{2}\frac{a_0^2 n_0}{\mu^2} s_1^2$$

and for the mean longitude

$$l = \int n\,dt = n_0 t - \frac{3a_0 n_0}{\mu}\int (s_1 + s_2)\,dt + \frac{3}{2}\frac{a_0^2 n_0}{\mu^2}\int s_1^2\,dt$$

Here the terms of the second order are of two classes, those in s_2 and those in s_2^2. Let us first consider the latter. We have, to quantities of the first order,

$$\frac{3}{2}\frac{a^2 n}{\mu^2} s_1^2 = \frac{(D_t\,\delta l)^2}{6n}$$

We hereafter omit the subscript o's, putting a, n, etc., for the absolute constants a_0, n_0, etc. Thus the terms of the second order arising from s_1^2 are equivalent to

$$\delta_2 l = \int \frac{(D_t\,\delta l)^2}{6n}\,dt$$

an expression which we may utilize to determine the order of magnitude of the terms in question.

Next, for the terms in s_2, we have

$$D_t s_2 = \delta\frac{dR}{dv} D_t v + \delta\frac{dR}{d\rho} D_t \rho + \frac{dR}{dv} D_t\,\delta v + \frac{dR}{d\rho} D_t\,\delta\rho \qquad (6)$$

where

$$\delta\frac{dR}{dv} = \frac{d^2R}{dv^2}\delta v + \frac{d^2R}{dv\,d\rho}\delta\rho + \frac{d^2R}{dv\,dv'}\delta v' + \frac{d^2R}{dv\,d\rho'}\delta\rho' + \frac{d^2R}{dv\,d\gamma}\delta\gamma$$

with a corresponding expression for

$$\delta\frac{dR}{d\rho}$$

But owing to the minuteness of the terms arising from the motion of the orbital plane, we may put δv and $\delta v'$ for $\delta \mathrm{v}$ and $\delta \mathrm{v}'$ and omit $\delta \gamma$. Thus we actually use

$$\begin{aligned} \delta \frac{d\,\mathrm{R}}{d\,\mathrm{v}} &= \frac{d^2\,\mathrm{R}}{d\,\mathrm{v}^2}\,\delta\,v + \frac{d^2\,\mathrm{R}}{d\,\mathrm{v}\,d\,\rho}\,\delta\,\rho + \frac{d^2\,\mathrm{R}}{d\,\mathrm{v}\,d\,\mathrm{v}'}\,\delta\,v' + \frac{d^2\,\mathrm{R}}{d\,\mathrm{v}\,d\,\rho'}\,\delta\,\rho' \\ \delta \frac{d\,\mathrm{R}}{d\,\rho} &= \frac{d^2\,\mathrm{R}}{d\,\mathrm{v}\,d\,\rho}\,\delta\,v + \frac{d^2\,\mathrm{R}}{d\,\rho^2}\,\delta\,\rho + \frac{d^2\,\mathrm{R}}{d\,\mathrm{v}'\,d\,\rho}\,\delta\,v' + \frac{d^2\,\mathrm{R}}{d\,\rho\,d\,\rho'}\,\delta\,\rho' \end{aligned} \tag{7}$$

where $\delta\,v$ and $\delta\,v'$ are the perturbations of the longitude in orbit of the first order.

All the derivatives of R as to v, v', ρ, and ρ' are known to be equal to the corresponding derivatives of the developed function R as to λ, λ', $\log a$ and $\log a'$, where λ and λ' are the mean angular distances of the planet from the common node, provided that we first eliminate the longitudes of the perihelia by replacing them by their expressions as functions of the mean anomalies and mean longitudes. And when the eccentric is used instead of the mean anomaly, a corresponding theorem may be applied, as shown on page 18 of Volume III.

When we consider only the mutual action of two planets, the expression (6) will contain only terms having the same arguments as those of the first order, and from the smallness of the latter we infer that the terms of the second order of this class are insensible. But when we consider the changes in the mutual action of two planets due to the action of a third planet upon them, we meet with terms of triple argument, some of which are known to be appreciable, owing to their very long period and the consequent magnitude of the factor whose square enters into them. Let any term of the second derivatives of R in (7) be

$$\mathrm{D}^2\,\mathrm{R} = \frac{m'}{a_1}\,h\,\begin{matrix}\sin\\ \cos\end{matrix}\,(i'\,g' + i\,g)$$

and any term in $\delta\,v$, $\delta\,v'$, $\delta\,\rho$ or $\delta\,\rho'$ be

$$c\,\begin{matrix}\sin\\ \cos\end{matrix}\,(i''\,g'' + j'\,g')$$

Then in $2\,\delta\frac{d\,\mathrm{R}}{d\,\mathrm{v}}$ and $2\,\delta\frac{d\,\mathrm{R}}{d\,\rho}$ we shall have the terms

$$\frac{m'}{a_1}\,c\,h\,\begin{matrix}\cos\\ \sin\end{matrix}\,\big(i''\,g'' + (j' \pm i')\,g' \pm i\,g\big)$$

and in (6) we shall have terms of the same argument, as well as smaller terms in which $i\,g$ is replaced by $(i+1)\,g$, $(i-1)\,g$, etc.

Let us represent any one of these terms by

$$\frac{m'\,n}{a_1}\,\big(h_c \cos(i''\,g'' + i'\,g' + i\,g) + h_s \sin(i''\,g'' + i'\,g' + i\,g)\big)$$

Let us also put, for brevity,

$$\mathrm{N} = i''\,g'' + i'\,g' + i\,g$$

$$\mathrm{n} = i''\,n'' + i'\,n' + i\,n$$

the latter being the coefficient of t in N. We then have

$$2\,D_t s_2 = \frac{m'\,n}{a_1}\left(h_c \cos \mathrm{N} + h_s \sin \mathrm{N}\right) \tag{8}$$

$$2\,s_2 = \frac{m'\,n}{a'\,\mathrm{n}}\left(h_c \sin \mathrm{N} - h_s \cos \mathrm{N}\right)$$

Then, in (4), s_2 will give rise to the terms

$$\delta^2 l = \frac{3}{2}\,\frac{m'}{\mu}\,\frac{a}{a_1}\,\frac{n^2}{\mathrm{n}^2}\left(h_c \cos \mathrm{N} + h_s \sin \mathrm{N}\right) \tag{9}$$

In the case of the four inner planets the largest values of h_c and h_s are of the order of magitude of the perturbations in longitude, and therefore rarely exceed 10″. The ratio $\frac{m'}{\mu}$ has its largest value in the case of the Earth, for which it amounts to $1 \div 330\,000$. We may therefore regard it as certain that no term of this class can exceed 0″.1, unless

$$\nu = \frac{n}{\mathrm{n}} > 100$$

In judging of a probability that a term for which this condition is fulfilled may be sensible, we must note that h_c and h_s will always contain as factors powers of the eccentricities whose lowest exponent is $i'' + i' + i$ taken positively, so that the larger this sum the greater the value of ν to make a term become sensible.

III.

Possible arguments of the inequalities.

Let us now inquire for what triple arguments n approximates to zero. Representing by n_0, n_1, n_2, etc., the ratios of the mean motions of Mercury, Venus, Earth, Mars, and Jupiter to that of the Earth, we have

$$n_0 = 4.15209$$
$$n_1 = 1.62552$$
$$n_2 = 1.00000$$
$$n_3 = 0.53168$$
$$n_4 = 0.08430$$

which numbers may be used for the mean motions n, n', n''.

We have now to find the triplets of planets for which a linear combination of these motions may approximate to zero.

Earth—Venus—Mercury.

In order that $i\,n_0 + i'\,n_1 + i''\,n_2$ may be zero, we must have $0.15209\,i + 0.37448\,i' =$ an integer. We thus find that small values of n may be formed as follows:

$$
\begin{array}{rrrr}
15\,n_2 & +n_1 & -4\,n_0 & =+.01716 \\
6 & +4 & -3 & +.04581 \\
5 & +2 & -2 & -.05314 \\
4 & -5 & +1 & +.02449
\end{array}
$$

The last value is the only one to be considered. In the motion of Mercury disturbed by Venus it gives rise to a term in the mean longitude having a period of 40 years and a coefficient of the order of magnitude of 0″.05. It seems unnecessary to take account of this term.

Mars—Earth—Venus.

The following are the annual motions of arguments of long period:

$$
\begin{array}{rrrr}
12\,n_3 & +5\,n_2 & -7\,n_1 & =+.00152 \\
12 & -8 & +1 & +.00568 \\
8 & -1 & -2 & +.00240 \\
4 & +6 & -5 & -.00088 \\
4 & -7 & +3 & +.00328
\end{array}
$$

The last two equations alone demand consideration. The last argument was shown by LEVERRIER to lead to an inequality whose coefficient is about 0″.3 in the motion of the Earth.

Mars—Earth—Mercury.

The only close relation is

$$11\,n_3 - 9\,n_2 + n_0 = +.00057$$

A short examination of the perturbations of Mercury by the Earth shows that the term depending on this argument cannot become sensible.

In Mars—Venus—Mercury there is no term to be considered.

Jupiter—Mars—Earth.

The following are the close relations:

$$
\begin{array}{rrrr}
1\,n_4 & -\ 2\,n_3 & +1\,n_2 & =+.02094 \\
2 & -\ 6 & +3 & -.02148 \\
3 & -\ 8 & +4 & -.00054 \\
4 & -10 & +5 & +.02040 \\
7 & -\ 3 & +1 & -.00494 \\
10 & -11 & +5 & -.00548
\end{array}
$$

IV.

Numerical examination of the order of magnitude of the doubtful terms.

By comparing the equations (6) and (7) it will be seen that the principal quantities which enter into h_c and h_s are sums of products of two factors, the one a func-

tion of the derivatives of R, the other the terms of the perturbations of one or the other planet. This makes it easy to determine the order of magnitude of any term of long period in advance of an accurate computation.

It is also to be remarked that the largest term of

$$D_t v \text{ is } n$$

and of

$$D_t \rho \text{ is} - e\, n \sin g$$

the latter containing the eccentricity as a factor. The second term of (6) may therefore be omitted in determining the order of magnitude of $D_t s_2$.

It is also to be remarked that if the first derivatives of R contain terms of the form

$$\frac{d\mathrm{R}}{d\mathrm{v}} = h \begin{matrix}\sin\\ \cos\end{matrix}(i' g' + i g)$$

$$\frac{d\mathrm{R}}{d\rho} = h' \begin{matrix}\sin\\ \cos\end{matrix}(i' g + i g)$$

the order of magnitude of the corresponding terms of the second derivatives will be

$$\frac{d^2\mathrm{R}}{d\mathrm{v}^2} = i\, n\, h \begin{matrix}\sin\\ \cos\end{matrix}(i' g' + i g)$$

$$\frac{d^2\mathrm{R}}{d\mathrm{v}\, d\rho} = i\, n\, h' \begin{matrix}\sin\\ \cos\end{matrix}(i' g' + i g)$$

The rough computations of the largest coefficients follow in tabular form. P_v and P_ρ are the limiting values of the products which make up h_c and h_s.

ARGUMENT 4 EARTH — 5 VENUS + MERCURY.

The numbers for Mercury are as follows:

V., M.	$\frac{d\mathrm{R}}{d\mathrm{v}}$	$\frac{d^2\mathrm{R}}{d\mathrm{v}\, d\mathrm{v}'}$	$\frac{d^2\mathrm{R}}{d\mathrm{v}\, d\rho'}$	E., V.	$\delta v'$	$\delta \rho'$	P_v	P_ρ
					$''$	$''$	$''$	$''$
1, −1	.08	.08	.28	4, −4	0.90	0.2	0.07	0.1
2, −1	.32	.64	.80	4, −3	0.60	0.02	0.38	0.0
3, −1	.10	.30	.36	4, −2	0.01	0.00	0.00	0.0
2, −2	.46	.92	1.00	3, −3	0.60	0.02	0.60	0.0

From this it may be concluded that the order of magnitude of the largest terms in $\sqrt{h_c^2 + h_s^2}$ depending on the argument

$$4\,\mathrm{E} - 5\,\mathrm{V} + 1\,\mathrm{M}$$

is 0″.60. This coefficient is multiplied by

$$\frac{3}{2}\frac{m'}{\mu}\frac{a}{a_1}\frac{n^2}{\mathrm{n}^2} = \frac{3}{2}\frac{1}{400\,000 \times .006^2}\frac{a}{a_1}$$

a factor whose order of magnitude is $\frac{1}{15}$.

The coefficient in $\delta^2 l$ is less than 0″.02. The terms in the motion of the Earth and Venus are still smaller.

ARGUMENT 4 MARS+6 EARTH—5 VENUS.

(a) In the motion of Venus.

Earth on Venus.				Mars on Earth.				
E., V.	$\frac{dR}{dv}$	$\frac{d^2R}{dv\,dv'}$	$\frac{d^2R}{dv\,d\rho'}$	M., E.	$\delta v'$	$\delta \rho'$	P_v	P_ρ
					″	″		
6, —5	.07	.42	.48	4, 0	.001	.000	.000	.000
7, —5	.008	.056	.100	4, —1	.01	.002	.001	.000
8, —5	.001	.008	.016	4, —2	.6	.02	.005	.002
9, —5	.000	.000	.000	4, —3	.5	.10	.000	.000

The factor $\frac{m'}{\mu}\frac{a\,n^2}{a_1\,n^{2\prime}}$, in the action of the Earth on Venus, is about 4. The limiting value of the term in question in the motion of Venus is about 0″.01. Owing to the small eccentricity of Venus the contiguous term is yet smaller than 0″.01.

(b) In the motion of the Earth.

To investigate the corresponding term in the motion of the Earth we have to consider, in the action of Venus on the Earth, the modifications produced by the perturbation of the Earth by Mars, and in the action of Mars, the perturbations by Venus.

Mars on Earth.					Venus on Earth.								
M., E.	$\frac{dR}{dv}$	$\frac{dR}{d\rho}$	$\frac{d^2R}{dv^2}$	$\frac{d^2R}{dv\,d\rho}$	E., V.	δv	$\delta \rho$	$D_t\,\delta v$	$D_t\,\delta \rho$	P_v	P_ρ	P_v'	P_ρ'
						″	″	″	″	″	″	″	″
4, 0	.0001	.001	.0002	.004	6, —5	.04	.01	.09	.02	.00	.00	.00	.00
4, —1	.006	.016	.024	.06	7, —5	13	.03	.15	.03	.00	.00	.00	.00
4, —2	.08	.13	32	.5	8, —5	15	.006	.02	.001	.05	.00	.00	.00
4, —3	.40	.50	1.6	2.0	9, —5	.00	.00	.00	.00	.00	.00	.00	.co
4, —4	.46	.52	1.8	2.0	10, —5	.00	.00	.00	.00	.00	.00	.00	.00

Venus on Earth.					Mars on Earth.								
E., V.	$\frac{dR}{dv}$	$\frac{dR}{d\rho}$	$\frac{d^2R}{dv^2}$	$\frac{d^2R}{dv\,d\rho}$	M., E.	δv	$\delta \rho$	$D\,\delta v$	$D_t\,\delta \rho$	P_v	P_ρ	P_v'	P_ρ'
						″	″	″	″	″	″	″	″
6, —5	.07	.08	.4	.5	4, —0	.001	.000	.002	.000	.00	.00	.00	.00
7, —5	.01	.01	.1	.1	4, —1	.01	.002	.01	.002	.00	.00	.00	.00
8, —5	.001	.002	.01	.02	4, —2	.6	.02	.08	.003	.00	.00	.00	[illegible]
9, —5	.000	.000	.00	.00	4, —3	.5	.10	.5	.09	.00	.00	.00	.00
10, —5	.000	.000	.00	.00	4, —4	.03	.01	.06	.02	.00	.00	.00	.00

The limiting value of any of the resulting terms in the motion of the Earth is of the order of magnitude 0″.02.

(*c*) *In the motion of Mars.*

Earth on Mars.				Venus on Earth.				
M., E.	$\frac{dR}{dv'}$	$\frac{d^2R}{dv'\,dv}$	$\frac{d^2R}{dv'\,d\rho}$	E., V.	δr	$\delta\rho$	P_v	P_ρ
					″	″	″	″
4, 0	.000	.000	.004	6, −5	.04	.01	.00	.00
4, −1	.007	.028	.08	7, −5	.13	.03	.00	.00
4, −2	.09	.36	.5	8, −5	.15	.01	.05	.00
4, −3	.37	1.5	1.8	9, −5	.001	.001	.00	.00
4, −4	.46	1.8	2.0	10, −5	.001	.001	.00	.00

The largest term which can arise in the motion of Mars is less than 0″.1, and, the period being more than 1,000 years, it is without importance at our epoch.

The argument

$$4\,M-7\,E+3\,V$$

is omitted in this examination, as LEVERRIER has computed the inequalities depending upon it in all three planets.

ARGUMENT JUPITER—2 MARS + EARTH.

(*a*) *In the motion of the Earth.*

Mars on Earth.				Jupiter on Mars.				
M., E.	$\frac{dR}{dv}$	$\frac{d^2R}{dv\,dv'}$	$\frac{d^2R}{dv\,d\rho'}$	J., M.	$\delta r'$	$\delta\rho'$	P_v	P_ρ
					″	″	″	
0, −1	.02	.02	.11	1, −2	3	1	.06	0.0
1, −1	.15	.15	.6	1, −1	26	11	4.0	7.0
2, −1	.09	18	.4	1, −0	4	1	0.7	04.
2, −2	.8	1.7	2.0					

The factor $\frac{m'}{\mu}\frac{a}{a_1}\frac{n^2}{n^2}$ is less than .001. It may be inferred that the term in the motion of the Earth is less than 0″.01

(*b*) *In the motion of Mars.*

Jupiter on Mars.					Earth on Mars.				
J., M.	$\frac{dR}{dv}$	$\frac{dR}{d\rho}$	$\frac{d^2R}{dv^2}$	$\frac{d^2R}{dv\,d\rho}$	M., E.	δv	$\delta\rho$	$D_t\,\delta v$	$D_t\,\delta\rho$
						″	″	″	″
1, −2	.004	.004	.01	.01	0, −1	0.8	0.4	1.5	0.8
1, −1	.011	.031	.01	.03	1, −1	8.0	3.0	7.0	3.0
1, 0	.000	.005	.00	.01	2, −1	13.0	1.0	1.5	0.1

Earth on Mars.				
M., E.	$\frac{dR}{dv}$	$\frac{dR}{d\rho}$	$\frac{d^2R}{dv^2}$	$\frac{d^2R}{dv^2}$
1, —1	1.5	4.4	1.5	4.4
2, —1	0.1	0.7	0.2	1.4

Jupiter on Mars.				
J., M.	δv	$\delta \rho$	$D_t \delta v$	$D_t \delta \rho$
	″	″	″	″
1, —1	26.0	11.0	22.0	10.0
1, —0	4.0	1.0	0.7	0.2

The limiting value of the term is a quantity of the order of magnitude 0″.10. It seems worth while to compute the term more precisely.

We conclude that terms of the following arguments are the only ones which require an accurate computation:

(1) $4\,M-7\,E+3\,V$; period 308 years,

(2) $J-2\,M+E$; period 48 years,

(3) $3\,J-8\,M+4\,E$; period 1,850 years.

of which the second can be appreciable only in the motion of Mars.

V.

Formulæ for special cases.

In the actual computation of the terms in question, the special form of the work will vary with the case to be considered. There are three cases, according as the planet whose inequalities are required is the inner, the outer, or the middle of those whose mutual action is considered.

We shall mark the elements pertaining to the middle and outer planets by one and two accents respectively.

CASE I.—*The disturbed planet the inner one.*—In this case the principal terms to be considered are those arising from the change in the action of the inner planet produced by the action of the outer one upon it. Rigorously considered there will also be terms arising through the action of the outer on the inner planet, but we may assume that these will be too small to demand consideration at the present time, except in the combination

$$3\,J-8\,M+4\,E$$

Then in (7) the only terms required will be

$$\begin{aligned}\delta\frac{d\,R}{d\,v}&=\frac{d^2\,R}{d\,v\,d\,v'}\,\delta\,v'+\frac{d^2\,R}{d\rho\,d\rho'}\,\delta\,\rho'\\ \delta\frac{d\,R}{d\,\rho}&=\frac{d^2\,R}{d\,v'\,d\rho}\,\delta\,v'+\frac{d^2\,R}{d\,\rho\,d\rho'}\,\delta\,\rho'\end{aligned}\qquad(10)$$

Since the action of the outer on the inner planet is neglected, we shall also have

$$\mathrm{D}_t \delta v = 0;\ \mathrm{D}_t \delta \rho = 0$$

Thus (6) will become

$$\mathrm{D}_t s_2 = \delta \frac{d\,\mathrm{R}}{d\,\mathrm{v}} \mathrm{D}_t v + \delta \frac{d\,\mathrm{R}}{d\,\rho} \mathrm{D}_t \rho. \tag{11}$$

where $\mathrm{D}_t v$ and $\mathrm{D}_t \rho$ are to be taken from the terms of the undisturbed elliptic motion, which give, approximately,

$$\mathrm{D}_t v = n\left(1 + 2e\cos g + \frac{5}{2}e^2\cos 2g + \quad . \quad . \quad . \quad\right)$$

$$\mathrm{D}_t \rho = n\left(e\sin g + \frac{3}{2}e^2\sin 2g + \quad . \quad . \quad . \quad\right)$$

Taking N, as in (8), for the argument of the term of very long period, we may put

$$2\,\delta\frac{d\,\mathrm{R}}{d\,\mathrm{V}} = \frac{m'}{a_1}\begin{cases} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ +(\mathrm{V}_{-1}\cos + \mathrm{V}'_{-1}\sin)(\mathrm{N}-g) \\ +(\mathrm{V}_0\ \cos + \mathrm{V}'_0\ \sin)\ \mathrm{N} \\ +(\mathrm{V}_1\ \cos + \mathrm{V}'_1\ \sin)(\mathrm{N}+g) \\ + . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{cases} \tag{12}$$

$$2\,\delta\frac{d\,\mathrm{R}}{d\,\rho} = \frac{m'}{a_1}\begin{cases} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ +(\mathrm{R}_{-1}\cos + \mathrm{R}'_{-1}\sin)(\mathrm{N}-g) \\ +(\mathrm{R}_0\ \cos + \mathrm{R}'_0\ \sin)\mathrm{N} \\ +(\mathrm{R}_1\ \cos + \mathrm{R}'_1\ \sin)(\mathrm{N}+g) \\ + . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{cases}$$

Then, to form $2\,\mathrm{D}_t s_2$ in the equation (8) we shall have

$$\begin{aligned} h_c &= \mathrm{V}_0 + e\left(\mathrm{V}_1 + \mathrm{V}_{-1} + \frac{1}{2}\mathrm{R}'_1 - \frac{1}{2}\mathrm{R}'_{-1}\right) \\ &\quad + e^2\left(\frac{5}{4}\mathrm{V}_2 + \frac{5}{4}\mathrm{V}_{-2} + \frac{3}{4}\mathrm{R}'_2 - \frac{3}{4}\mathrm{R}'_{-2}\right) \\ &\quad + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned} \tag{13}$$

$$\begin{aligned} h_s &= \mathrm{V}'_0 + e\left(\mathrm{V}'_1 + \mathrm{V}'_{-1} - \frac{1}{2}\mathrm{R}_1 + \frac{1}{2}\mathrm{R}_{-1}\right) \\ &\quad + e^2\left(\frac{5}{4}\mathrm{V}'_2 + \frac{5}{4}\mathrm{V}'_{-2} - \frac{3}{4}\mathrm{R}_2 + \frac{3}{4}\mathrm{R}_{-2}\right) \\ &\quad + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}$$

We see that the term in N of $\delta \frac{d\,\mathrm{R}}{d\,\rho}$ drops out entirely, and that the coefficients of all the terms in both $\delta \frac{d\,\mathrm{R}}{d\,\mathrm{v}}$ and $\delta \frac{d\,\mathrm{R}}{d\,\rho}$, which belong to the argument $\mathrm{N} \pm i\,g$, are multiplied by e^i. Hence, in any case, these terms are not required to the same degree of precision as the others, and the term of $\delta \frac{d\,\mathrm{R}}{d\,\mathrm{v}}$ in N will be the controlling one, unless the adjacent terms are much larger than it is. Whether such is the case can be readily determined from the approximate values of the coefficients already given.

Having found h_c and h_s, the term of $\delta\,l$ in N is immediately obtained from (9) without modification, m' being the mass and a_1 the mean distance of the middle planet.

CASE II.—*The disturbed planet the outer one.*—In this case the terms of R and its derivatives will arise from the action of the middle on the outer planet; and the equations (10), (11), and (12), will be used without any change, except affecting v and ρ with two accents. But a and a_1 will now both be the mean distance of the outer planet, so that $\frac{a}{a_1}$ will disappear from (9), and we shall

$$\delta_2\,l'' = \frac{3}{2}\,\frac{m}{\mu}\,\frac{n'^2}{\mathrm{n}^2} \left\{ h_c \cos \mathrm{N} + h_s \sin \mathrm{N} \right\} \tag{14}$$

CASE III.—*The disturbed planet the middle one.*—We have here to consider the action of each of the extreme planets on the middle one, as modified by the perturbations of the middle one produced by the other extreme one.

We take first the value of R as arising from the action of the outer on the middle planet and combine it with the perturbations of the latter by the inner one. Thus we shall have for (7)

$$\delta \frac{d\,\mathrm{R}}{d\,\mathrm{v}'} = \frac{d\,\mathrm{R}}{d\,\mathrm{v}'^2}\,\delta\,v' + \frac{d^2\,\mathrm{R}}{d\,\mathrm{v}'\,d\,\rho'}\,\delta\,\rho'$$

$$\delta \frac{d\,\mathrm{R}}{d\,\rho'} = \frac{d^2\,\mathrm{R}}{d\,\mathrm{v}'\,d\,\rho'}\,\delta\,v' + \frac{d^2\,\mathrm{R}}{d\,\rho'^2}\,\delta\,\rho'$$

$\delta\,\mathrm{v}'$ and $\delta\,\rho'$ being due to the action of the inner planet. We have to include all four terms of (6), which now becomes

$$\mathrm{D}_t\,s_2 = \delta \frac{d\,\mathrm{R}}{d\,\mathrm{v}'}\,\mathrm{D}_t\,v' + \delta \frac{d\,\mathrm{R}}{d\,\rho'}\,\mathrm{D}_t\,\rho' + \frac{d\,\mathrm{R}}{d\,\mathrm{v}'}\,\mathrm{D}_t\,\delta\,v' + \frac{d\,\mathrm{R}}{d\,\rho'}\,\mathrm{D}_t\,\delta\,\rho'$$

The first two terms are formed as in (13); the last two by a process quite like that by which $\delta \frac{d\,\mathrm{R}}{d\,\mathrm{v}'}$ and $\delta \frac{d\,\mathrm{R}}{d\,\rho'}$ are formed. The form (9) will then become

$$\delta^2\,l' = \frac{3}{2}\,\frac{m''}{\mu}\,\frac{a'}{a''}\,\frac{n'^2}{\mathrm{n}^2} \left\{ h_c \cos \mathrm{N} + h_s \sin \mathrm{N} \right\}$$

The modification of the action of the inner planet due to the direct action of the outer one upon the middle one will commonly give inequalities of the same order of

magnitude as those just considered. Now, the terms of R will be those due to the action of the inner on the middle planet; those of $\delta v'$ and $\delta \rho'$ will be due to the action of the outer on the middle planet. The first formula (9) will become

$$\delta^2 l' = \frac{3}{2} \frac{m n'^2}{\mu n^2} \left\{ h_c \cos N + h_s \sin N \right\}$$

VI.

Accurate computation of terms depending on the argument 4 *Mars* — 7 *Earth* + 3 *Venus.*

The computation of the inequalities requires the second derivatives of R as to v and ρ, which are not computed in the determination of the inequalities of the first order. They may be computed either by the method laid down in the first paper of this volume, or they may be derived from LE VERRIER's development of the perturbative function in Tome I of his Researches.* To guard against serious mistakes, I have had most of the computations made by both methods, Mr. PRENTISS using the first method, to which he had become accustomed, and I using LE VERRIER's development. I have been so far satisfied with the general agreement of the results as to deem a careful control of either computation unnecessary, and this for two reasons. Firstly, no attempt has been made to include all the possible terms in either computation, so that differences of the order of magnitude of the omitted terms do not necessarily imply an error in either result. Secondly, the order of approximation aimed at was only that necessary to the special problem in hand.

To enable the degree of uncertainty to be estimated I shall present both sets of results in all cases where the two methods were used. The principal combinations which lead to the argument now under consideration are

$$\begin{gathered} 4\,M - 2\,E - (5\,E - 3\,V) \\ 4\,M - 3\,E - (4\,E - 3\,V) \\ 4\,M - 4\,E - (3\,E - 3\,V) \\ 4\,M - 5\,E - (2\,E - 3\,V) \\ \text{etc.} \qquad\qquad \text{etc.} \end{gathered}$$

In the case of each planet the adjacent terms depending on arguments one of whose coefficients is different by unity are also to be considered.

Designating quantities pertaining to the Earth by one accent and to Mars by two accents, we have

$$4\,n'' - 7\,n' + 3\,n = +4286'',$$

and, for the epoch 1850.0,

$$N_0 = 4\,g'' - 2\,g' + 3\,g = 66^\circ.80.$$

*Annales de l'Observatoire de Paris. Mémoires. At the time the computations of this paper were made (1890–1891) the developments in Part I of the present volume were not in shape for use.

α. Inequality in the motion of Venus.

In R and its derivatives we need the terms arising from the action of the Earth on Venus, which depend on the argument just given, and on the adjacent ones if necessary. To determine what arguments should be included the following approximate coefficients have been computed:

Mars on Earth.			Earth and Venus.				Earth and Venus.			
M., E.	$\delta v'$	$\delta \rho'$	E., V.	$\frac{dR}{dv}$	$\frac{d^2R}{dv\,dv'}$	$\frac{d^2R}{dv\,d\rho'}$	E., V.	$\frac{dR}{dv}$	$\frac{d^2R}{dv\,dv'}$	$\frac{d^2R}{dv\,d\rho'}$
	"	"								
4, −2	.70	.05	5, −3	.006	.03	.03	5, −4	.07	.3	.5
4, −3	.55	.20	4, −3	.060	.15	.28	4, −4	.90	3.6	6.
4, −4	.04	.02	3, −3	.96	2.9	4.7	3, −2	.04	.2	.1
4, −5	.001	.001	2, −3	.005	.02	.18	2, −2	1.05	2.0	2.

The combined effect of the terms depending upon the combination of the arguments 4 M − 2 E with 5 E − 3 V, and 4 M − 3 E with 4 E − 4 V is a quantity whose limit is about 0."025; they have therefore been omitted in the accurate computation.

The principal necessary coefficients in the mutual action of Venus and the Earth are as follows. The symbol D is used for the derivative as to log a.

Venus and the Earth.

i	Values of			$D_{\log a}$ of			$D^2_{\log a}$ of		
	$a' A_i$	$a' \alpha D_a A_i$	$\frac{1}{2} a' \alpha^2 D^2_a A_i$	$a' A_i$	$a' \alpha D_a A_i$	$\frac{1}{2} a' \alpha^2 D^2_a A_i$	$a' A_i$	$a' \alpha D_a A_i$	$\frac{1}{2} a' \alpha^2 D^2_a A_i$
3	0.31957	1.2290	2.257	1.2290	5.743	14.87	5.743	35.48	131.9
4	0.20379	0.9927	2.256	0.9927	5.504	15.75	5.504	37.00	139.9
5	0.13324	0.7844	2.139	0.7844	5.063	16.22	5.063	37.50	148.1

From these I derive the following expressions for R and its derivatives from LeVerrier's development:

Venus and the Earth.

g', g, w', w	R	$\frac{dR}{d\rho}$	$\frac{d^2R}{d\rho^2}$	$\frac{dR}{dv}$	$\frac{d^2R}{dv^2}$	$\frac{d^2R}{dv\,d\rho}$	of $j'w' - jw$ cos	sin
3, −3, 3, −3	+0.3192	+1.2288	+5.753	+0.9576	−2.873	+3.686	+.0472	−.9989
2, −2	+0.0001	−0.0003	−0.006	+0.0002	−0.000	−0.001	+.5270	−.8499
4, −4	+0.0001	+0.0004	−0.001	+0.0004	−0.002	+0.002	−.4446	−.8957
4, −3, 4 −4	−0.0089	−0.0460	−0.277	−0.0356	+0.142	−0.184	−.4446	−.8957
3 −3	+0.0291	+0.1204	+0.636	+0.0873	−0.262	+0.361	+.0472	−.9989
5, −3, 3, −3	+0.0017	+0.0076	+0.0452	+0.0051	−0.015	+0.023	+.047	−.999
4, −4	−0.0011	−0.0058	−0.0384	−0.0042	+0.017	−0.023	−.445	−.896
5, −5	+0.0002	+0.0011	+0.0081	+0.0008	−0.004	+0.005	−.824	−.566
5, −3	+0.0014	+0.0126	+0.131	+0.0044	−0.013	−0.038	+.796	−.605

Reducing these expressions to the form

$$H_c \cos (i' g' + i g) + H_s \sin (i' g' + i g)$$

we have the results found in the first of each pair of lines below. Under each is given the result of the developments made according to the eccentric anomaly and then transformed to mean anomaly:

Earth on Venus.

Argument.	$3g'-3g$		$4g'-3g$	
Function.	cos	sin	cos	sin
R	+ 15.1 + 15.1	+ 319.0 + 319.0	+ 5.4 + 5.3	+ 21.1 + 20.9
$\frac{dR}{d\rho}$	+ 58 + 58	+1228 +1228	+ 26 + 26	+ 79 + 79
$\frac{d^2R}{d\rho^2}$	+ 269 + 269	+5741 +5744	+153 +142	+387 +384
$\frac{dR}{dv} = -\frac{dR}{dv'}$	− 957.1 − 957.2	+ 45.1 + 45.2	− 55.3 − 55.1	+ 19.9 + 19.9
$\frac{d^2R}{dv^2} = \frac{d^2R}{dv'^2}$	− 134 − 135	−2872 −2871	− 74 − 75	−135 −133
$\frac{dR}{d\rho'}$	− 73 − 73	−1547 −1547	− 31 − 31	−100 −100
$\frac{d^2R}{d\rho\, d\rho'}$	− 327 − 327	−6969 −6972	−179 −168	−466 −463
$\frac{d^2R}{dv\, d\rho} = -\frac{d^2R}{dv'\, d\rho}$	−3683 −3705	+ 173 + 168	−196 −196	+ 99 + 97
$\frac{d^2R}{dv\, d\rho'} = -\frac{d^2R}{dv'\, d\rho'}$	+4640 +4662	− 218 − 213	+251 +251	−119 −117
$\frac{d^2R}{d\rho'^2}$	+ 398 + 400	+8514 +8519	+210 +199	+566 +563

The terms depending on $5g' - 3g$ are here omitted as unimportant.

To obtain the inequality in the motion of Venus arising through the action of Mars on the Earth we need the expressions

$$\delta \frac{dR}{dv} = \frac{d^2R}{dv\, dv'}\, \delta v' + \frac{d^2R}{dv\, d\rho'}\, \delta \rho'$$

$$\delta \frac{dR}{d\rho} = \frac{d^2R}{dv'\, d\rho}\, \delta v' + \frac{d^2R}{d\rho\, d\rho'}\, \delta \rho'$$

where $\delta v'$ and $\delta \rho'$ are the perturbations of the Earth by Mars, which are as follows.* The derivatives of these perturbations are added for subsequent use in the motion of the Earth:

* Vol. III, p. 529

Perturbations of the Earth by Mars.

g'', g'	$\delta v'$		$\delta \rho'$		$\frac{1}{n'} D_t \delta v'$		$\frac{1}{n'} D_t \delta \rho'$	
	cos	sin	cos	sin	cos	sin	cos	sin
	"	"	"	"				
4, −5	+0.001	+0.001	0.000	0.000	....	....	....	....
4, −4	+0.011	+0.032	+0.022	−0.008	−0.060	+0.021	+0.015	+0.039
4, −3	−0.131	+0.483	+0.219	+0.060	−0.422	−0.115	−0.052	+0.191
4, −2	+0.526	−0.256	+0.020	+0.046	−0.032	−0.066	+0.006	−0.003

Performing the necessary multiplications we find the following quantities relating to the action of the Earth on Venus:

$$V_0=+0.''304;\ V'_0=+0.''028$$

$$V_1=+2.7;\ V'_1=-0.5$$

$$R_1=+0.7;\ R'_1=+3.4$$

$$h_c=+0.334;\ h_s=+0.024$$

$$\delta^2 l=+0.''268 \cos N+0.''019 \sin N$$

β. Inequality in the motion of the Earth.

The terms which come into consideration have two sources: (1). The action of Mars on the Earth as modified by the perturbations of the Earth by Venus. (2). The action of Venus on the Earth as modified by the perturbations of the Earth by Mars.

The arguments to be combined are

$4\,g''-5\,g'$	$2\,g'-3\,g$
$4\quad -4$	$3\quad -3$
etc.	etc.

The coefficients for forming R and its derivatives for the action of Mars on the Earth belonging to these arguments are given subsequently (p. 74). We have to combine them with the following perturbations of the Earth by Venus (Vol. III, p. 487):

Argument.	$\delta v'$		$\delta \rho'$		$\frac{1}{n'} D_t \delta v'$		$\frac{1}{n'} D_t \delta \rho'$	
g', g	cos	sin	cos	sin	cos	sin	cos	sin
	"	"	"	"	"	"	"	"
2, −3	−0.013	+0.001	+0.001	+0.010	0.00	−0.04	−0.03	0.00
3, −3	−0.676	+0.027	+0.021	+0.506	−0.04	−1.27	−0.95	+0.04
4, −3	+1.531	−0.403	−0.183	−0.701	+0.35	+1.34	+0.61	−0.16
5, −3	+0.775	−0.694	+0.061	+0.071	−0.09	−0.10	+0.01	−0.01

We find, by executing the multiplications:

$$2\,\delta\frac{d\,\mathrm{R}}{d\,\mathrm{v}'}=\frac{m''}{a''}\left\{\begin{array}{l}-4.''55\cos\mathrm{N}-0.''11\sin\mathrm{N}\\+3.''7\cos(\mathrm{N}-g')-2.''2\sin(\mathrm{N}-g')\end{array}\right\}$$

$$2\,\delta\frac{d\,\mathrm{R}}{d\,\rho'}=\frac{m''}{a''}\left\{-2.''7\cos(\mathrm{N}-g')-4.''7\sin(\mathrm{N}-g')\right\}$$

$$2\frac{d\,\mathrm{R}}{d\,\mathrm{v}'}\mathrm{D}_t\,\delta\,v'+2\frac{d\,\mathrm{R}}{d\,\rho'}\mathrm{D}_t\,\delta\,\rho'=\frac{m''\,n'}{a''}\left\{+1.''66\cos\mathrm{N}-0.''17\sin\mathrm{N}\right\}$$

From these coefficients we find

$$h_c=-2.''89$$

$$h_s=-0.''34$$

Also, from the adopted mass of Mars $1\div 3093500$

$$\frac{3}{2}\frac{m''}{\mu''}\frac{a'}{a''}\frac{n'^2}{\mathrm{n}^2}=0.02909$$

Whence

$$\delta^2\,l'=-0.''084\cos\mathrm{N}-0.''010\sin\mathrm{N}$$

We have next to compute the modification in the action of Venus produced by the perturbations of the Earth by Mars. From the data already given we find, in the action of Venus on the Earth, the following terms, each of which is to be multiplied by the factor $\frac{m}{a'}$.

Argument.	$\frac{d^2\,\mathrm{R}}{d\,\mathrm{v}'^2}$		$\frac{d^2\,\mathrm{R}}{d\,\mathrm{v}'\,d\,\rho'}$		$\frac{d^2\,\mathrm{R}}{d\,\rho'^2}$	
$g',\ g$	cos	sin	cos	sin	cos	sin
3, −3	−0.13	−2.87	−4.64	+0.22	+0.40	+8.51
4, −3	−0.074	−0.135	−0.25	+0.12	+0.21	+0.58
5, −3	−0.033	−0.015	−0.05	+0.06		

Multiplying by the perturbations of the Earth by Mars, already given, we find

$$2\,\delta\frac{d\,\mathrm{R}}{d\,\mathrm{v}'}=\frac{m}{a'}\left\{\begin{array}{l}-0.''311\cos\mathrm{N}-0.''022\sin\mathrm{N}\\-2.38\cos(\mathrm{N}+g')-0.''69\sin(\mathrm{N}+g')\end{array}\right\}$$

$$2\,\delta\frac{d\,\mathrm{R}}{d\,\rho'}=\frac{m}{a'}\left\{+1.''16\cos(\mathrm{N}+g')-4.''08\sin(\mathrm{N}+g')\right\}$$

$$2\frac{d\,\mathrm{R}}{d\,\mathrm{v}'}\mathrm{D}_t\,\delta\,v'=\frac{m\,n'}{a'}\left\{-0.''079\cos\mathrm{N}+0.''003\sin\mathrm{N}\right\}$$

$$2\frac{d\,R}{d\,\rho'}\,D_t\,\delta\,\rho'=\frac{m\,n'}{a'}\left\{-0.''078\cos N+0.''009\sin N\right\}$$

$$h_c=-0.''543$$

$$h_s=-0.032$$

$$\frac{3}{2}\frac{m}{\mu}\frac{n'^2}{n^2}=0.3412$$

$$\delta^2\,l'=-0.''185\cos N-0.''011\sin N$$

Adding this to the term arising from the action of Mars, the total perturbation becomes

$$\delta^2\,l'=-0.''269\cos N-0.''021\sin N$$

Le Verrier's value, reduced to Hall's mass of Mars, is

$$-0.276\cos N-0.018\sin N$$

γ. In the motion of Mars.

The only appreciable effect is that resulting from the action of the Earth on Mars as modified by the perturbations of the Earth by Venus. The most important terms for the action of Earth on Mars will be those hereafter given in connection with the action of the Earth and Jupiter on Mars, but terms depending on the following arguments may also be appreciable:

g''	g'
4	−2
3	−4
3	−3
3	−2

From Le Verrier's tables I find the following terms depending on these arguments (units of third place of decimals):

g'' g'	w'' w'	R	$\frac{d\,R}{d\,\rho}$	$\frac{d^2\,R}{d\,\rho^2}$
3, −2,	2, −2	+141.9	+409.4	+1655
	3 −3	− 18.0	− 69.2	− 320
3 −3	2 −2	+ 1.4	+ 0.3	− 24
	3 −3	+211.0	+792.0	+3435
	4 −4	+ 1.4	+ 4.9	+ 9
3 −4	3 −3	+ 4.4	+ 12.3	+ 22
	4 −4	− 14.3	− 53 2	− 159

g''	g'	w''	w'	R	$\frac{dR}{d\rho}$
4	−2	2	−2	+35.4	+112
		3	−3	−9.1	−38
		4	−4	+0.6	+3
		4	−2	+0.3	+2

Reducing each term to a binomial after forming the derivatives, the following values of the derivatives of R are found:

Earth on Mars.

[Units of third place of decimals.]

Argument.	$\frac{d^2 R}{d v' d v''}$		$\frac{d^2 R}{d \rho' d v''}$		$\frac{d^2 R}{d v' d \rho''}$		$\frac{d^2 R}{d \rho' d \rho''}$	
g'' g'	cos	sin	cos	sin	cos	sin	cos	sin
3, −4	+ 232	−106	− 99	− 215	− 124	− 276	− 213	+ 100
3, −3	+1749	+691	−867	−2197	+1096	−2781	−3933	−1556
3, −2	− 307	−404	−863	+ 418	−1155	+ 546	+ 929	+2127
4, −2	− 121	−162	−251	+ 177	----------	----------	----------	----------

Multiplying these expressions by the proper terms in the perturbations of the Earth by Venus we find

$$2\,\delta\frac{d\,R}{d\,v''}=\frac{m'}{a''}\left\{\begin{array}{l}+3.''53\cos(N-g'')+2.''45\sin(N-g'')\\+4.56\cos N-0.05\sin N\\+2.41\cos(N+g'')-0.73\sin(N+g'')\end{array}\right\}$$

$$2\,\delta\frac{d\,R}{d\,\rho''}=\frac{m'}{a''}\left\{\begin{array}{l}+3.''6\cos(N-g'')-6.''3\sin(N-g'')\\-1.0\cos(N+g'')-3.9\sin(N+g'')\end{array}\right\}$$

From these coefficients follow

$$h_c=+5.''22$$
$$h_s=+0.''32$$

We also have

$$\frac{3}{2}\frac{m'}{\mu}\frac{n''^2}{n}=0.1185$$

and hence, in the mean longitude of Mars, the inequality

$$+0.''619\cos N+0.''038\sin N$$

LeVerrier's value, when reduced to $m'=1\div 327000$, is

$$+0.''620\cos N+0.''053\sin N$$

VII.

Terms depending on the argument 3 Jupiter — 8 Mars + 4 Earth.

The existence of a term in the Sun's mean longitude depending on this argument, and therefore having a period of nearly eighteen hundred years, was, I believe, first recognized by HANSEN. To it is due the difference of 6″ in the Sun's mean longitude, as given by the tables of CARLINI and the recent ones of HANSEN and LE VERRIER.

Designating quantities pertaining to Jupiter by two accents, and those pertaining to Mars by one, we have

$$3\,n'' - 8\,n' + 4\,n = -727''.0$$

and for the epoch 1850.0

$$3\,g'' - 8\,g' + 4\,g = 282^{\circ}.94$$

α. In the motion of the Earth.

The only terms in the mean longitude of the Earth depending on this argument which can become appreciable are those which arise from the changes in the mutual action of Mars and the Earth produced by the action of Jupiter. But it is necessary to take into account the action of Jupiter on both planets. Thus, putting R for the perturbative function depending on the action of Mars on the Earth, we have

$$\delta\frac{d\,\mathrm{R}}{d\,\mathrm{v}} = \frac{d^2\,\mathrm{R}}{d\,\mathrm{v}^2}\,\delta\,v + \frac{d^2\,\mathrm{R}}{d\,\mathrm{v}\,d\,\mathrm{v}'}\,\delta\,v' + \frac{d^2\,\mathrm{R}}{d\,\mathrm{v}\,d\,\rho}\,\delta\,\rho + \frac{d^2\,\mathrm{R}}{d\,\mathrm{v}\,d\,\rho'}\,\delta\,\rho'$$

$$\delta\frac{d\,\mathrm{R}}{d\,\rho} = \frac{d^2\,\mathrm{R}}{d\,\rho\,d\,\mathrm{v}}\,\delta\,v + \frac{d^2\,\mathrm{R}}{d\,\rho\,d\,\mathrm{v}'}\,\delta\,v' + \frac{d^2\,\mathrm{R}}{d\,\rho^2}\,\delta\,\rho + \frac{d^2\,\mathrm{R}}{d\,\rho\,d\,\rho'}\,\delta\,\rho'$$

As $\delta\,v'$ and and $\delta\,\rho'$ do not depend on the same argument as $\delta\,v$ and $\delta\,\rho$ they have to be computed separately. The combinations of the mean anomalies which may give rise to appreciable results are

For R and its derivatives			For $\delta\,v'$ and $\delta\,\rho'$
$4\,g' - 4\,g$	$4\,g' - 5\,g$	$4\,g' - 3\,g$	$3\,g'' - 4\,g'$
5 −4	5 −5	5 −3	3 −3
6 −4	6 −5	6 −3	3 −2
7 −4	7 −5	7 −3	3 −1

The following tables show the values of the coefficients which are employed in the computation of R and its derivatives when LE VERRIER's development of R is used for the computation:

Table of coefficients for the action of Mars on the Earth.

i	$a' A_i$	$a' \alpha D_\alpha A_i$	$\frac{1}{2} a' \alpha^2 D^2_\alpha A_i$	$\frac{1}{6} a' \alpha^3 D^3_\alpha A_i$	$\frac{1}{24} a' \alpha^4 D^4_\alpha A_i$
2	0.4048	1.0467	1.2207	1.3861	1.973
3	0.2240	0.8108	1.2919	1.5336	2.026
4	0.12956	0.6012	1.2324	1.6820	2.158
5	0.07688	0.4347	1.0958	1.7559	2.340
6	0.04640	0.3092	0.9275	1.7361	2.511
7	0.02834	0.2174	0.7574	1.6356	2.619
8	0.01747	0.1516	0.6018	1.4798	2.638

i	$D_{\log a}$ of				
	$a' A_i$	$a' \alpha D_\alpha A_i$	$\frac{1}{2} a' \alpha^2 D^2_\alpha A_i$	$\frac{1}{6} a' \alpha^3 D^3_\alpha A_i$	$\frac{1}{24} a' \alpha^4 D^4_\alpha A_i$
2	1.0467	3.4880	6.600	12.048	22.62
3	0.8108	3.3945	7.185	12.703	23.13
4	0.6012	3.0660	7.511	13.677	24.07
5	0.4347	2.6263	7.459	14.629	25.52
6	0.3092	2.1643	7.063	15.254	27.26
7	0.2174	1.7322	6.421	15.384	28.69
8	0.1516	1.3552	5.643	14.992	30.19

i	$D^2_{\log a}$ of				
	$a' A_i$	$a' \alpha D_\alpha A_i$	$\frac{1}{2} a' \alpha^2 D^2_\alpha A_i$	$\frac{1}{6} a' \alpha^3 D^3_\alpha A_i$	$\frac{1}{24} a' \alpha^4 D^4_\alpha A_i$
2	3.4880	16.687	49.345	126.64	301.5
3	3.3946	17.764	52.479	130.62	306.7
4	3.0660	18.088	56.053	137.30	315.2
5	2.6263	17.545	58.806	145.95	328.1
6	2.1643	16.291	59.886	154.80	345.8
7	1.7322	14.575	58.994	161.90	366.3
8	1.3552	12.641	56.26	165.74	387.6

Thence we have, from LE VERRIER's development of R, the values of the coefficients given in the following table:

$g'\ g\ \ w'\ w$	R	$\frac{dR}{d\rho}$	$\frac{d^2R}{d\rho^2}$
4, −3, 2, −2	+ 0.27	− 0.5	−12
3, −3	+107.86	+416.7	+1992
4, −4	− 12.74	− 63.2	− 360
4, −4, 3, −3	+ 2.03	+ 5.0	+ 1
4, −4	+119.16	+562.3	+2958
5, −5	+ 1.51	+ 7.4	+ 33
4, −5, 4, −4	+ 3.33	+ 13.0	+ 44
5, −5	− 11.32	− 55.9	− 255
5, −3, 3, −3	+ 34.13	+140.0	+ 719
4, −4	− 8.43	− 43.0	− 258
5, −5	+ 0.55	+ 3.3	+ 22
5, −4, 3, −3	+ 0.56	+ 1.2	− 4
4, −4	+ 78.01	+378.9	+2090
5, −5	− 8.90	− 53.4	− 360
5, −5, 4, −4	+ 2.21	+ 8.7	+ 28
5, −5	+ 66.52	+382.4	+2379
6, −6	+ 1.43	+ 8.7	+ 52

$g'\ g\ \ w'\ w$	R	$\frac{dR}{d\rho}$	$\frac{d^2R}{d\rho^2}$
6, −3, 3, −3	+ 8.96	+ 38.4	+ 212
4, −4	− 3.45	− 18.1	− 113
5, −5	+ 0.43	+ 2.8	+ 20
6, −6	− 0.02	− 0.1	− 1
6, −4, 4, −4	+ 30.29	+152.3	+ 890
5, −5	− 7.28	− 44.1	− 298
6, −6	+ 0.49	+ 3.4	+ 25
6, −5, 4, −4	+ 0.80	+ 3.0	+ 8
5, −5	+ 54.59	+319.5	+2044
6, −6	− 5.95	− 41.8	− 324
7, −3, 3, −3	+ 2.03	+ 9.3	+ 57
4, −4	− 1.05	− 5.8	− 39
5, −5	+ 0.20	+ 1.3	+ 10
7, −4, 4, −4	+ 9.20	+ 47.9	+ 299
5, −5	− 3.63	− 22.3	− 154
6, −6	+ 0.46	+ 3.2	+ 25
7, −7	− 0.02	− 0.2	− 1
7, −5, 5, −5	+ 25.22	+151.2	+1005
6, −6	− 5.93	− 41.6	− 319
7, −7	+ 0.41	+ 3.2	+ 27

The adopted values of the combinations of w and w' and of their cosines and sines are

	cos	sin
$3\,(w'-w) = -\ 21^\circ\ 10'.4$	+.9325	−.3612
$4\qquad\ = -148\ \ 13.9$	−.8502	−.5265
$5\qquad\ = +\ 84\ \ 42.7$	+.0922	+.9957
$6\qquad\ = -\ 42\ \ 20.8$	+.7391	−.6736

From these data we derive the values of the coefficients of

$$\frac{m'}{a'}\,{\sin \atop \cos}\,(i'\,g' + i\,g)$$

in the various terms of R and its derivatives, as shown in the second sets of numbers in the following table by combining the terms which depend upon each combination of g and g' into a binomial, and using the precepts of Vol. III, pp. 18 and 81. The first sets are obtained by development in terms of the eccentric anomalies, and are computed to detect any serious error in the others:

MARS AND EARTH.

Coefficients for expressing R and its derivatives.

[Units of third place of decimals.]

M., E.	R		D R		D^2 R		$D^2_{\rho v}$ R		D_v R		D^2_v R	
	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin	cos	sin
4, −5	− 4.04	+13.04	− 17.2	+ 63.5	− 74	+ 287	− 310	− 75	− 63.5	− 17.2	+ 74	− 310
	− 3.87	+13.02	− 16.2	+ 62.5	− 60	+ 277	− 305	− 70	− 63.4	− 16.5	+ 71	− 310
−4	− 99.48	+62.19	−473.8	+291.6	−2525	+1537	−1157	−1901	−246.6	−399.9	+1604	− 977
	− 99.28	+61.97	−472.6	+290.5	−2510	+1524	−1152	−1895	−245.5	−399.3	+1599	− 969
−3	+111.44	+32.09	+442.8	+118.5	+2167	+ 559	− 326	+1382	− 89.9	+345.2	−1077	− 244
	+111.33	+31.98	+442.2	+117.8	+2156	+ 542	− 320	+1380	− 89.5	+344.9	−1078	− 244
5, −5	+ 5.36	−64.60	+ 34.4	−373.2	+ 221	−2354	+1865	+ 192	+323.1	+ 29.5	− 162	+1616
	+ 5.31	−64.21	+ 34.2	−370.3	+ 233	−2319	+1850	+ 184	+320.8	+ 29.5	− 161	+1602
−4	− 66.86	+50.25	−328.0	+252.2	−1824	+1441	−1061	−1316	−209.2	−269.0	+1083	− 880
	− 66.62	+50.14	−325.9	+253.1	−1814	+1457	−1065	−1310	−209.2	−267.8	+1076	− 881
−3	+ 38.21	+ 7.31	+161.4	+ 24.6	+ 846	+ 108	− 48	+ 520	− 16.7	+121.7	− 394	− 28
	+ 38.95	+ 7.33	+167.5	+ 24.7	+ 891	+ 102	− 46	+ 540	− 15.4	+124.0	− 400	− 22
6, −5	+ 0.15	−58.24	− 2.8	−346.2	+ 30	−2188	+1760	− 104	+295.6	− 3.0	+ 42	+1504
	− 0.05	−57.95	− 4.1	−344.7	− 58	−2249	+1753	− 38	+294.2	− 3.9	+ 43	+1497
−4	− 25.24	+23.01	−127.4	+111.7	− 736	+ 736	− 533	− 508	− 99.6	−100.8	+ 403	− 435
	− 26.06	+23.52	−131.1	+126.4	− 766	+ 782	− 555	− 523	−102.0	−104.2	+ 416	− 448
−3	+ 10.59	+ 0.99	+ 47.6	+ 1.8	+ 266	+ 2	+ 8	+ 155	− 0.5	+ 34.6	− 113	+ 9
	+ 11.32	+ 0.98	+ 51.5	+ 1.6	+ 296	− 3	+ 10	+ 171	− 0.2	+ 36.9	− 123	+ 11
7, −5	− 2.56	−28.67	− 19.0	−172.8	− 73	−1094	+ 892	+ 51	+145.7	− 17.0	+ 114	+ 752
	− 2.46	−29.04	− 19.9	−177.9	− 171	−1211	+ 916	− 136	+149.1	− 17.5	+ 120	+ 768
−4	− 8.37	+ 8.23	− 38.4	+ 45.5	− 230	+ 291	− 208	− 159	− 36.8	− 34.0	+ 138	− 168
	− 7.79	+ 8.76	− 40.2	+ 49.6	− 248	+ 317	− 225	− 159	− 39.3	− 30.8	+ 120	− 179
−3	+ 2.40	+ 0.16	+ 11.0	+ 0.2	+ 75	− 5	+ 2	+ 36	+ 0.2	+ 7.9	− 26	+ 4
	+ 2.81	− 0.02	+ 13.7	− 1.0	+ 87	− 10	+ 8	+ 46	+ 1.0	+ 9.4	− 31	+ 7

The four second derivatives which enter into the formulæ for the various inequalities are given by the equations

$$\frac{d^2 R}{d v\, d v'} = - D^2_v R$$

$$\frac{d^2 R}{d \rho\, d v'} = - D^2_{\rho, v} R$$

$$\frac{d^2 R}{d\,\text{v}\,d\rho'} = -\text{D}_{\text{v}}\,\text{R} - \text{D}^2_{\rho,\text{v}}\,\text{R}$$

$$\frac{d^2 R}{d\rho\,d\rho'} = -\text{D}_{\rho}\,\text{R} - \text{D}^2_{\rho}\,\text{R}$$

The constant factor $\frac{m'}{a'}$ is omitted, as usual.

For each quantity the upper value is that derived from the development just described, the lower one that from the independent development in terms of eccentric anomalies:

Mars and Earth.

g', g	$\frac{d^2 R}{d\text{v}\,d\text{v}'}$ cos	$\frac{d^2 R}{d\text{v}\,d\text{v}'}$ sin	$\frac{d^2 R}{d\rho\,d\text{v}'}$ cos	$\frac{d^2 R}{d\rho\,d\text{v}'}$ sin	$\frac{d^2 R}{d\text{v}\,d\rho'}$ cos	$\frac{d^2 R}{d\text{v}\,d\rho'}$ sin	$\frac{d^2 R}{d\rho\,d\rho'}$ cos	$\frac{d^2 R}{d\rho\,d\rho'}$ sin
4, −5	− .071	+ .310	+ .305	+ .070	+ .368	+ .087	+ .076	− .339
	− .074	+ .310	+ .310	+ .075	+ .373	+ .092	+ .091	− .350
−4	−1.599	+ .969	+1.152	+1.895	+1.398	+2.294	+2.983	−1.814
	−1.604	+ .977	+1.157	+1.901	+1.404	+2.301	+2.999	−1.829
−3	+1.078	+ .244	+ .320	−1.380	+ .410	−1.725	−2.598	− .660
	+1.077	+ .244	+ .326	−1.382	+ .416	−1.727	−2.610	− .677
5, −5	+ .161	+1.602	−1.850	− .184	−2.171	− .214	− .267	+2.689
	+ .162	+1.616	−1.865	− .192	−2.188	− .222	− .255	+2.727
−4	−1.076	+ .881	+1.065	+1.310	+1.274	+1.578	+2.140	−1.710
	−1.083	+ .880	+1.061	+1.316	+1.270	+1.585	+2.152	−1.693
−3	+ .400	+ .022	+ .046	− .540	+ .061	− .664	−1.059	− .127
	+ .394	+ .028	+ .048	− .520	+ .065	− .642	−1.007	− .133
6, −5	− .043	−1.497	−1.753	+ .038	−2.047	+ .042	+ .062	+2.594
	− .042	−1.504	−1.760	+ .104	−2.056	+ .107	− .027	+2.534
−4	− .416	+ .448	+ .555	+ .523	+ .657	+ .627	+ .897	− .908
	− .403	+ .435	+ .533	+ .508	+ .633	+ .609	+ .863	− .848
−3	+ .123	− .011	− .010	− .171	− .010	− .208	− .347	+ .001
	+ .113	− .009	− .008	− .155	− .008	− .190	− .314	− .004
7, −5	− .120	− .768	− .916	+ .136	−1.065	+ .153	+ .191	+1.389
	− .114	− .752	− .892	− .051	−1.038	− .034	+ .092	+1.267
−4	− .120	+ .179	+ .225	+ .159	+ .264	+ .190	+ .288	− .367
	− .138	+ .168	+ .208	+ .159	+ .245	+ .193	+ .268	− .336
−3	+ .031	− .007	− .008	− .046	− .009	− .055	− .101	+ .010
	+ .026	− .004	− .002	− .036	− .002	− .044	− .064	+ .005

In the continuation of the computation I have used the mean of the two sets of coefficients. A computation of the difference between the results was also made.

The perturbations of Mars by Jupiter, which are to be combined with the above derivatives, are shown in the following table:

Perturbations of Mars by Jupiter.

Argument.	$\delta v'$		$\delta \rho'$		$\frac{1}{n'} D_t \delta v'$		$\frac{1}{n'} D_t \delta \rho'$	
g'' g'	cos	sin	cos	sin	cos	sin	cos	sin
	$''$	$''$	$''$	$''$	$''$	$''$	$''$	$''$
3—0	−0.205	−0.108	+0.054	−0.101	−0.051	+0.097	−0.048	−0.026
—1	−2.821	−1.450	−0.396	+0.818	+0.760	−1.478	−0.429	−0.208
—2	−1.730	−1.949	−1.244	+1.053	+2.970	−2.637	−1.605	−1.896
—3	−1.281	+0.266	+0.242	+0.987	−0.671	−3.233	−2.491	+0.611
—4	−0.201	+0.046	+0.039	+0.156	−0.162	−0.709	−0.550	+0.138
—5	−0.027	+0.006	+0.005	+0.021	−0.027	−0.122	−0.095	+0.023

By multiplying these expressions for $\delta v'$ and $\delta \rho'$ into the second derivatives of R, we find the following terms of the second order:

Put

$$N = 3g'' - 8g' + 4g$$

Then

$$2\,\delta \frac{dR}{dv} = \frac{m'}{a'} \left\{ \begin{array}{l} -2.''09 \cos (N-g) - 0.''13 \sin (N-g) \\ +4.''12 \cos N + 5.''95 \sin N \\ +6.''22 \cos (N+g) - 11.''34 \sin (N+g) \end{array} \right\}$$

$$2\,\delta \frac{dR}{d\rho} = \frac{m'}{a'} \left\{ \begin{array}{l} -\ 0.''03 \cos (N-g) - 3.''09 \sin (N-g) \\ +14.''11 \cos (N+g) + 7.''09 \sin (N+g) \end{array} \right\}$$

The terms from which the inequality principally arises are those depending on the argument N in $2\,\delta \frac{dR}{dv}$. In the case of this term the result of the development through the eccentric anomalies exceeds that through the mean anomalies by the amount

$$+0.''11 \cos N - 0.''12 \sin N$$

so that the separate values would have been:

From mean anomalies,

$$4.''07 \cos N + 6.''01 \sin N$$

From eccentric anomalies,

$$4.''17 \cos N + 5.''89 \sin N$$

The formulæ (13) now give

$$h_c = +4.''27; \quad h_s = +5.''64$$

Whence,

$$\delta_2 l = +4.''32 \cos N + 5.''70 \sin N$$

We have next to determine that portion of the inequality arising from the action of Mars on the Earth as modified by the perturbations of the Earth by Jupiter. The only combinations of arguments which can give rise to appreciable terms are:

Jupiter on Earth.	*Mars on Earth.*
δv or $\delta \rho$.	Derivatives of R.
$3g'' - 3g$	$8g' - 7g$
$3 \quad -2$	$8 \quad -6$
$3 \quad -1$	$8 \quad -5$

The terms of this class are so small that I have deemed it unnecessary to compute those which depend upon other arguments than N in the values of $\delta \frac{dR}{dv}$ and $\delta \frac{dR}{d\rho}$, except the largest. Of this I have made only an approximate computation in order to determine its order of magnitude. The only development used is that in terms of the mean anomaly. The terms which have been computed are shown in the following table:

Action of Mars on the Earth.

$g' \quad g \quad w' \quad w$	R	$\frac{dR}{d\rho}$
$8-7+8-8$	-3.6	-32
$+7-7$	$+25.0$	$+197$
$8-6+8-8$	$+0.3$	$+3$
$+7-7$	-4.3	-37
$+6-6$	$+20.2$	$+141$
$8-5+6-6$	-3.6	-25
$+5-5$	$+10.6$	$+64$

Forming the derivatives as to v and reducing the terms to binomials we have the following results:

Argument.	$\frac{d^2R}{dv^2}$		$\frac{d^2R}{dv\,d\rho}$		$\frac{d^2R}{d\rho^2}$	
$g', \; g$	cos	sin	cos	sin	cos	sin
8, −7	+1.31	−0.43	−0.58	−1.47	−1.60	+0.48
8, −6	−0.75	−0.43	−0.50	+0.89	....	....
8, −5	+0.07	+0.35	+0.43	−0.08	....	....

The value of $\frac{d^2 R}{d\rho^2}$ has been assumed, by induction, as equal to $8\frac{dR}{d\rho}$.

The perturbations of the Earth by Jupiter which enter into the problem are as follows:

Argument.	δv		$\delta \rho$		$\frac{1}{n} D_t \delta v$		$\frac{1}{n} D_t \delta \rho$	
g'' g	cos	sin	cos	sin	cos	sin	cos	sin
	$''$	$''$	$''$	$''$	$''$	$''$	$''$	$''$
3−3	−0.162	+0.027	+0.021	+0.132	−0.07	−0.44	−0.36	+0.06
3−2	+0.071	+0.551	+0.379	−0.050	−0.96	+0.12	+0.09	+0.66
3−1	−0.031	+0.208	+0.082	+0.012	−0.16	−0.02	−0.01	+0.06

We thus find

$$2\,\delta\frac{dR}{dv}=\frac{m'}{a'}\left\{\begin{array}{l}-0.''83\cos N-0.''74\sin N\\ -0.''40\cos(N+g)+1.''09\sin(N+g)\end{array}\right\}$$

$$2\,\delta\frac{dR}{d\rho}=\frac{m'}{a'}\left\{-1.''5\cos(N+g)-0.''3\sin(N+g)\right\}$$

$$\frac{2}{n}\left(\frac{dR}{dv}D_t\delta v+\frac{dR}{d\rho}D_t\delta\rho\right)=\frac{m'}{a'}\{0.''29\cos N+0''21\sin N\}$$

Thence

$$h_c=-0.''55$$
$$h_s=-0.''52$$

The complete values of h_c and h_s are thus found to be

$$h_c=+3.''72$$
$$h_s=+5.''12$$

We also have

$$\frac{3}{2}\frac{m'}{\mu}\frac{a}{a'}\frac{n^2}{\mathrm{n}^2}=1.012$$

and thus, for the definitive inequality in the mean longitude of the Earth

$$\delta^2 l=+3.''76\cos N+5.''18\sin N$$

If, in order to make the motion of the argument positive, we put

$$N'=360^\circ-N=77.^\circ06+727.''0\,(t-1850.0)$$

the inequality becomes

$$\begin{aligned}\delta^2 l&=+3.''76\cos N'-5.''18\sin N'\\ &=6.''40\sin(N'+144.^\circ03)\\ &=6.''40\sin\{221.^\circ09+727''\,(t-1850.0)\}\end{aligned}$$

The values of HANSEN and LE VERRIER, for 1850, when reduced to our mass of Mars are, respectively:

$$\text{HANSEN: } 6.''43 \sin 220.^\circ 72$$

$$\text{LE VERRIER: } 6.''21 \sin 225.^\circ 23$$

β. In the motion of Mars.

Appreciable terms in the motion of Mars depending on the argument 3 Jupiter—8 Mars+4 Earth arise in three ways, namely:

1. Through the action of the Earth on Mars as modified by the perturbations of Mars by Jupiter.

2. Through the same action as modified by the perturbations of the Earth by Jupiter.

3. Through the action of Jupiter on Mars as modified by the perturbations of Mars by the Earth.

I. *Earth on Mars as modified by perturbations of Mars by Jupiter.*—The following are the combinations of arguments which lead to sensible terms of this class:

Earth on Mars.	*Jupiter on Mars.*
$4g'-4g$	$3g''-4g'$
$5\quad -4$	$3\quad -3$
$6\quad -4$	$3\quad -2$
$7\quad -4$	$3\quad -1$

In the action of Earth on Mars the derivatives of R required are

$$\frac{d^2 R}{d\mathrm{v}'^2};\quad \frac{d^2 R}{d\rho'\, d\mathrm{v}'};\quad \frac{d^2 R}{d\rho'^2};\quad \frac{dR}{d\mathrm{v}'};\quad \frac{dR}{d\rho'}$$

The first three are derived from the values already given for the action of Mars on the Earth by the formulæ

$$\frac{d^2 R}{d\mathrm{v}'^2} = -\frac{d^2 R}{d\mathrm{v}\, d\mathrm{v}'}$$

$$\frac{d^2 R}{d\mathrm{v}'\, d\rho'} = -\frac{d^2 R}{d\mathrm{v}\, d\rho'}$$

$$\frac{d^2 R}{d\rho'^2} = -\frac{dR}{d\rho'} - \frac{d^2 R}{d\rho\, d\rho'}$$

The perturbations of Mars by Jupiter depending upon the arguments in question are given on p. 78. By performing the necessary multiplications we find

$$2\,\delta\frac{d\,\mathrm{R}}{d\,\mathrm{v}'}=\frac{m}{a'}\left\{\begin{array}{l}-1.''85\cos(\mathrm{N}-g')-2.''61\sin(\mathrm{N}-g')\\ -4.''12\cos\mathrm{N}-5.''94\sin\mathrm{N}\\ -5.''8\cos(\mathrm{N}+g')-11.''2\sin(\mathrm{N}+g')\\ -3.''5\cos(\mathrm{N}+2g')-14.''8\sin(\mathrm{N}+2g')\end{array}\right\}$$

$$2\,\delta\frac{d\,\mathrm{R}}{d\,\rho'}=\frac{m}{a'}\left\{\begin{array}{l}+3.''8\cos(\mathrm{N}-g')-3.''3\sin(\mathrm{N}-g')\\ +17.''0\cos(\mathrm{N}+g')-8.''9\sin(\mathrm{N}+g')\\ +21.''9\cos(\mathrm{N}+2g')-5.''5\sin(\mathrm{N}+2g')\end{array}\right\}$$

The separate values of the terms in cos N and sin N arising from the two develements are

From mean anomalies: $-4.''07\cos\mathrm{N}-5.''98\sin\mathrm{N}$

From eccentric anomalies: $-4.''18\cos\mathrm{N}-5.''89\sin\mathrm{N}$

We also have arising in the same way

$$2\frac{d\,\mathrm{R}}{d\,\mathrm{v}'}\mathrm{D}_t\,\delta v'+2\frac{d\,\mathrm{R}}{d\,\rho'}\mathrm{D}_t\,\delta\rho'=\frac{m\,n'}{a'}\left\{-3.''26\cos\mathrm{N}-1.''74\sin\mathrm{N}\right\}$$

Forming the complete value of $\mathrm{D}_t\,s_2$ we find

$$2\,\mathrm{D}_t\,s_2=\frac{m\,n'}{a'}\left\{-8.''42\cos\mathrm{N}-9.''88\sin\mathrm{N}\right\}$$

Adopting mass of Earth $=\frac{1}{327000}$ we find

$$\frac{3}{2}\frac{m\,n'^2}{\mu\,\mathrm{n}^2}=4.121$$

and hence

$$\delta_2\,l'=-34.''7\cos\mathrm{N}-40.''7\sin\mathrm{N}$$

II. *Action of Jupiter on Mars as modified by the perturbations of Mars by the Earth.*—The combinations of the arguments are the same as in the last case. The action of Jupiter on Mars depends on $\frac{d\,\mathrm{R}}{d\,\mathrm{v}'}$ and $\frac{d\,\mathrm{R}}{d\,\rho'}$. The additions to these quantities are

$$\delta\frac{d\,\mathrm{R}}{d\,\mathrm{v}'}=\frac{d^2\,\mathrm{R}}{d\,\mathrm{v}'^2}\,\delta\,\mathrm{v}'+\frac{d^2\,\mathrm{R}}{d\,\mathrm{v}'\,d\,\rho'}\,\delta\,\rho'$$

$$\delta\frac{d\,\mathrm{R}}{d\,\rho'}=\frac{d^2\,\mathrm{R}}{d\,\mathrm{v}'\,d\,\rho'}\,\delta\,v'+\frac{d^2\,\mathrm{R}}{d\,\rho'^2}\,\delta\,\rho'$$

The constants for the action of Jupiter on Mars, as derived from those given in Vol. III, p. 553, are

For action of Jupiter on Mars.

i	$a' A_i$	$a' \alpha D_a A_i$	$\frac{1}{2} a' \alpha^2 D^2_a A_i$	$\frac{1}{6} a' \alpha^3 D^3_a A_i$
0	2.04504	0.09474	0.05731	0.01122
1	0.00994	.03096	.03400	.01573
2	0.06674	.13863	.08025	.01228
3	0.016315	.05027	.05374	.02296
4	0.004185	.01709	.02673	.01976

i	$D_{\log a}$ of			
	$a' A_i$	$a' \alpha D_a A_i$	$\frac{1}{2} a' \alpha^2 D^2_a A_i$	$\frac{1}{6} a' \alpha^3 D^3_a A_i$
0	.09474	.2094	.1483	.0520
1	.03096	.0989	.1151	.0629
2	.13863	.2991	.1974	.0566
3	.05027	.1578	.1764	.0867
4	.01709	.0705	.1128	.0881

i	$D^2_{\log a}$ of			
	$a' A_i$	$a' \alpha D_a A_i$	$\frac{1}{2} a' \alpha^2 D^2_a A_i$	$\frac{1}{6} a' \alpha^3 D^3_a A_i$
0	.2094	.5059	.4527	.257
1	.0989	.3292	.4190	.282
2	.2991	.6938	.5645	.277
3	.1578	.5105	.6129	.365
4	.0705	.2960	.4889	.410

From Le Verrier's development of R and these constants I find the following terms of R and its derivatives in the action of Jupiter on Mars:

[Units of third place of decimals.]

g'' g'	w'' w'	R	D R	D^2 R	$j'w''+jw'$ cos	$j'w''+jw'$ sin
3 0	0 0	+ 0.15	+ 0.04	+ 0.4	1.0000	.0000
	1—1	— 0.04	— 0.12	— 0.4	+.7813	+.6242
	2—2	+ 0.25	+ 0.62	+ 1.4	+.2209	+.9753
	3—3	— 0.01	— 0.19	— 0.7	—.4363	+.8998
3—1	1—1	+ 0.16	+ 0.51	+ 1.6	+.7813	+.6242
	2—2	— 3.24	— 6.92	— 15.8	+.2209	+.9753
	3—3	+ 1.03	+ 3.26	+ 10.5	—.4363	+.8998
3—2	1—1	— 0.01	— 0.03	— 0.1	+.7813	+.6242
	2—2	+11.10	+23.39	+ 51.8	+.2209	+.9753
	3—3	— 6.72	—20.84	— 66.4	—.4363	+.8998
	4—4	— 0.04	— 0.14	— 0.6	—.9025	+.4307
3—3	2—2	+ 1.00	+ 2.00	+ 3.9	+.2209	+.9753
	3—3	+15.27	+47.18	+148.4	—.4363	+.8998
	4—4	+ 0.14	+ 0.54	+ 2.1	—.9025	+.4307
3—4	3—3	+ 2.20	+ 6.70	+ 20.5	—.4363	+.8998
	4—4	— 0.29	— 1.18	— 4.6	—.9025	+.4307

We thus have the results of the following table, which include, in some cases for comparison, the results of the developments in Vol. III, Part V:

g', g''	R cos	R sin	$\frac{dR}{dV'}$ cos	$\frac{dR}{dV'}$ sin	$\frac{dR}{d\rho'}$ cos	$\frac{dR}{d\rho'}$ sin
3, 0	+0.21	— 0.17	+ 0.32	+ 0.17	+ 0.18	— 0.36
	+0.20	— 0.17	+ 0.32	+ 0.16	+ 0.16	— 0.31
3, —1	—1.05	+ 2.13	— 3.43	— 2.67	— 2.55	+ 3.50
	—1.03	+ 2.11	— 3.41	— 2.63	— 2.51	+ 3.48
3, —2	+5.41	— 4.74	+ 3.40	+13.84	+14.36	— 3.98
	+5.41	— 4.77	+ 3.48	+13.81	+14.36	— 3.98
3, —3	—6.57	—14.78	+43.42	—20.06	—20.63	—44.64
	—6.60	—14.79	+43.47	—20.15	—20.73	—44.64
3, —4	—0.70	— 1.86	+ 5.46	— 1.74	— 1.86	— 5.52
	—0.66	— 1.87	+ 5.40	— 1.74	— 1.76	— 5.44

g'', g'	$\frac{d^2 R}{d v'^2}$		$\frac{d^2 R}{d v' d \rho'}$		$\frac{d^2 R}{d \rho'^2}$	
	cos	sin	cos	sin	cos	sin
3, 0	− 0.4	+ 1.4	+ 0.6	+ 0.5	+ 0.4	− 0.5
3, −1	+ 6.8	− 4.1	− 4.4	− 6.9	− 6.9	+ 5.0
3, −2	−36.8	− 11.5	− 10.8	+ 38.1	+40.8	+ 9.6
3, −3	+61.1	+128.6	+132.2	− 62.8	−65.7	−138.2
3, −4	+ 4.2	+ 15.9	+ 16.1	− 4.5	− 4.8	− 16.4

The following are the perturbations of Mars by the Earth which are to be combined with the preceding derivatives:

g', g	$\delta v'$		$\delta \rho'$		$\frac{1}{n'} D_t \delta v'$		$\frac{1}{n'} D_t \delta \rho'$	
	cos	sin	cos	sin	cos	sin	cos	sin
	″	″	″	″	″	″	″	″
4, −4	−0.036	−0.054	−0.050	+0.032	+0.19	−0.13	−0.11	−0.18
5, −4	−0.086	−0.112	−0.090	+0.069	+0.28	−0.22	−0.17	−0.23
6, −4	−0.156	−0.188	−0.130	+0.110	+0.29	−0.24	−0.17	−0.20
7, −4	+0.801	+0.712	+0.224	−0.243	−0.37	+0.42	+0.13	+0.12
8, −4	+0.190	+0.148	−0.057	+0.070	+0.07	−0.09	+0.03	+0.03

Performing the multiplications we find

$$2\,\delta\,\frac{d\,R}{d\,v'} = \frac{m''}{a''}\left\{\begin{array}{l} -0.''0093 \cos(N+g') - 0.''0033 \sin(N+g') \\ -0.''0210 \cos N - 0.''0280 \sin N \\ -0.''1105 \cos(N-g') + 0.''0051 \sin(N-g') \\ +0.''175 \cos(N-2g') + 0.''077 \sin(N-2g') \end{array}\right\}$$

$$2\,\delta\,\frac{d\,R}{d\,\rho'} = \frac{m''}{a''}\left\{\begin{array}{l} -0.''004 \cos(N+g') + 0.''011 \sin(N+g') \\ +0.''007 \cos(N-g') + 0.''113 \sin(N-g') \\ +0.''080 \cos(N-2g') - 0.''179 \sin(N-2g') \end{array}\right\}$$

$$2\,\frac{d\,R}{d\,v'} D_t\,\delta\,v' = \frac{m''}{a''\,n'}\left\{+0.''0158 \cos N + 0.''0113 \sin N\right\}$$

$$2\,\frac{d\,R}{d\,\rho'} D_t\,\delta\,\rho' = \frac{m''}{a''\,n'}\left\{+0.''0134 \cos N + 0.''0076 \sin N\right\}$$

And hence

$$h_c = -0.''0047$$
$$h_s = -0.''0070$$

Also

$$\frac{3}{2}\frac{m'' a' n'^2}{a'' n^2} = 376.6$$

Thence

$$\delta^2 l' = -1.''77 \cos N - 2.''65 \sin N$$

The perturbations of Mars being computed with a mass of the Earth $= 1 \div 354936$, we reduce this result to the mass $1 \div 327000$, and thus find

$$\delta^2 l' = -1.''92 \cos N - 2.''87 \sin N$$

The third action to be computed is that of the Earth on Mars as modified by the perturbations of the Earth by Jupiter. The arguments to be combined are as follow:

Jupiter on Earth.	*Earth on Mars.*	
$3g'' - 4g$	$8g' - 8g$	$7g' - 8g$
$3 \quad -3$	$8 \quad -7$	$7 \quad -7$
$3 \quad -2$	$8 \quad -6$	$7 \quad -6$
$3 \quad -1$	$8 \quad -5$	$7 \quad -5$

In the action of Earth on Mars we require the expressions

$$\delta \frac{d\mathrm{R}}{d\mathrm{v}'} = \frac{d^2\mathrm{R}}{d\mathrm{v}'\, d\mathrm{v}}\,\delta v + \frac{d^2\mathrm{R}}{d\mathrm{v}'\, d\rho}\,\delta\rho$$

$$\delta \frac{d\mathrm{R}}{d\rho'} = \frac{d^2\mathrm{R}}{d\mathrm{v}\, d\rho'}\,\delta v + \frac{d^2\mathrm{R}}{d\rho\, d\rho'}\,\delta\rho$$

δv and $\delta \rho$ being the perturbations of the Earth by Jupiter. From LE VERRIER'S development of R, I have computed the coefficients shown in the following table:

Mars on the Earth.

[Units of third place of decimals.]

Argument. g' g w' w	R	D R	D^2 R	$j(w'-w)$ cos	$j(w'-w)$ sin
8—8+7—7	+ 1.6	+ 12	+ 85	—.983	—.184
+8—8	+11.1	+ 98	+ 903	+.446	+.895
+9—9	+ 0.9	+ 8	+ 68	+.446	—.895
8—7+6—6	+ 1.0	+ 6	+ 37	+.739	—.674
+7—7	+25.0	+197	+1629	—.983	—.184
+8—8	— 3.6	— 32	— 288	+.446	+.895
8—6+5—5	+ 0.3	+ 3	+ 25	+.092	+.996
+6—6	+20.0	+140	+1049	+.739	—.674
+7—7	— 5.5	— 43	— 359	—.983	—.184
+8—8	+ 0.3	+ 3	+ 27	+.446	+.895
8—5+5—5	+ 8.9	+ 55	+ 395	+.092	+.996
+6—6	— 3.5	— 25	— 192	+.739	—.674
+7—7	+ 0.5	+ 4	+ 31	—.983	—.184
+8—8	0.0	0	— 2	+.446	+.895
7—7+6—6	+ 1.9	+ 12	+ 72	+.739	—.674
+7—7	+20.2	+158	+1300	—.983	—.184
+8—8	+ 1.0	+ 9	+ 73	+.446	+.895
7—6+5—5	+ 0.9	+ 5	+ 22	+.092	+.996
+6—6	+37.5	+257	+1876	+.739	—.674
+7—7	— 3.8	— 30	— 251	—.983	—.184

From these numbers we derive the following coefficients for the second derivatives of R in the action of Mars on the Earth:

g', g	$\frac{d^2 R}{d v\, d v'}$ cos	$\frac{d^2 R}{d v\, d v'}$ sin	$\frac{d^2 R}{d v'\, d \rho}$ cos	$\frac{d^2 R}{d v'\, d \rho}$ sin	$\frac{d^2 R}{d v\, d \rho'}$ cos	$\frac{d^2 R}{d v\, d \rho'}$ sin	$\frac{d^2 R}{d \rho\, d \rho'}$ cos	$\frac{d^2 R}{d \rho\, d \rho'}$ sin
8, —8	+ .268	—.554	—.627	— .304	—.697	— .336	— .383	+.810
8, —7	—1.289	+.452	+.508	+1.446	+.569	+1.628	+1.908	—.652
8, —6	+ .805	+.410	+.469	— .927	+.539	—1.055	—1.299	—.662
8, —5	— .098	—.300	—.370	+ .111	—.428	+ .126	+1.54	+.585
7, —7	— .893	+.164	+.187	+ .999	+0.21	+1.13	+1.34	—0.25
7, —6	+1.180	+.855	+.965	—1.343	+1.11	—1.54	—1.85	—1.36
7, —5	— .120	—.768	—.916	+ .136	—1.06	+0.15	+0.19	+1.39

In the action of Jupiter on the Earth we have the following terms:

Jupiter on Earth.

Argument.	δv		$\delta \rho$	
g'', g	cos	sin	cos	sin
	$''$	$''$	$''$	$''$
3, −4	−0.005	+.002	.000	+.002
−3	− .162	+.027	+.009	+.057
−2	+ .071	+.551	+.164	−.021
−1	− .031	+.208	+.035	+.005

Forming and adding the necessary products we have

$$2\,\delta\frac{d\,\mathrm{R}}{d\,\mathrm{v}'}=\frac{m}{a'}\left\{\begin{array}{l}+0.''614\cos\mathrm{N}+0.''571\sin\mathrm{N}\\+0.''763\cos(\mathrm{N}+g')+0.''735\sin(\mathrm{N}+g')\end{array}\right\}$$

$$2\,\delta\frac{d\,\mathrm{R}}{d\,\rho'}=\frac{m}{a'}\left\{-0.''97\cos(\mathrm{N}+g')+0.''99\sin(\mathrm{N}+g')\right\}$$

We then find

$$h_c=+0.''731$$

$$h_s=+0.''685$$

$$\frac{3}{2}\frac{m\,n'^2}{\mu\,\mathrm{n}^2}=4.121$$

whence

$$\delta^2 l=+3.'01\cos\mathrm{N}+2.''82\sin\mathrm{N}$$

The combination of the three parts gives for the total value of the inequality

$$\delta^2 l'=-33.''6\cos\mathrm{N}-40.''8\sin\mathrm{N}$$

LE VERRIER's value is, when reduced to the mass $1\div 327000$ of the Earth,

$$\delta^2 l'=-33.''1\cos\mathrm{N}-32.''2\sin\mathrm{N}$$

VIII.

Inequality in the motion of Mars dependent on the argument $J-2M+E$.

This inequality is so minute that it might be omitted but for the ease with which it may be computed, and the fact that it may be multiplied four times in the geocentric

longitude of Mars at opposition. As it is, only an approximate computation is required The combinations of arguments to be considered are

$$
\begin{array}{lll}
g''-g' & \text{and} & g'-g \\
g'' & \text{"} & 2g'-g \\
g''-2g' & \text{"} & g
\end{array}
$$

Much the largest part of the inequality arises from the combination of the first two arguments

We have two classes of terms to deal with, according as the immediate disturbing planet is Jupiter or the Earth, the perturbations of Mars by the other planet being the modifying cause.

For the action of Jupiter on Mars I have computed, from LE VERRIER'S data, the following coefficients:

g'', g'	ϖ'' ϖ'	R	D R	D^2 R
1, −2	0 0	.00	− .01	+ .01
	1 − 1	− .50	− 1.67	− 6.06
	2 − 2	− 1.46	− 2.76	− 4.73
1, −1	0 0	− 0.34	− 0.81	− 2.3
	1 − 1	+10.19	+31.80	+101.9
	2 − 1	+ 0.41	+ 0.76	+ 1.2
1, 0	0 0	+51.62	+ 7.46	+ 17.7
	1 − 1	− 2.39	− 7.56	− 24.7
	2 − 2	− 0.03	− 0.05	− 0.2

Hence are derived the following expressions for the derivatives of R required in the computation.

Jupiter on Mars.

[Units of third place of decimals.]

Argument.	$\frac{d^2 R}{d v'^2}$		$\frac{d^2 R}{d v' d \rho'}$		$\frac{d^2 R}{d \rho'^2}$	
g', g	cos	sin	cos	sin	cos	sin
1, −2	+1.7	−6.0	− 6.4	− 2.5	− 5.8	+ 8.4
1, −1	−8.3	+8.0	+21.3	+25.2	+77.6	−64.8
1, −0	+1.9	−1.6	− 4.8	− 5.9	− 1.9	+15.6

The perturbations of Mars by the Earth to be multiplied by these derivatives are as follows:

Earth on Mars.

Argument.	$\delta v'$		$\delta \rho'$		$\frac{1}{n'} D_t \delta v'$		$\frac{1}{n'} D_t \delta \rho'$	
g', g	cos	sin	cos	sin	cos	sin	cos	sin
	"	"	"	"	"	"	"	"
0, —1	+ 0.67	+0.46	+0.27	—0.40	—0.86	+1.26	+0.75	+0.51
1, —1	+ 6.50	+4.57	+1.73	—2.52	—4.02	+5.72	+2.22	+1.52
2, —1	+12.14	+4.54	—0.50	+1.16	+0.54	—1.45	+0.14	+0.06

Forming the products of the quantities thus indicated we find

$$2\,\delta\frac{d\,R}{d\,v'}=\frac{m''}{a''}\left\{\begin{array}{l}-0.''068\cos(N-g')+0.''033\sin(N-g')\\ -0.''035\cos N+0.''161\sin N\end{array}\right\}$$

$$2\,\delta\frac{d\,R}{d\,\rho'}=\frac{m''}{a''}\left\{+0.''175\cos(N-g')+0.''164\sin(N-g')\right\}$$

$$2\frac{d\,R\,d\,\delta\,v'}{d\,v'\,n'\,d\,t}+2\frac{d\,R\,d\,\delta\,\rho'}{d\,\rho'\,n'\,d\,t}=\frac{m''}{a''}\left\{+0.''045\cos N-0.''150\sin N\right\}$$

We hence have

$$h_c=-0.''003$$

$$h_s=+0.''022$$

and

$$\frac{3}{2}\frac{m''}{\mu}\frac{a'}{a''}\frac{n'^2}{n^2}=0.270$$

Hence, in the motion of Mars, we have the inequality

$$-0.''001\cos N+0.''006\sin N$$

We have next to compute the terms arising through the action of the Earth as modified by the perturbations of Mars by Jupiter. I have computed the quantities found in the following tables:

Earth on Mars.

g' g	w' w	R	D R	D^2 R
0, —1	0 0	— .007	— .024	— .127
	1 —1	+ .344	— .555	—1.87
1, —1	1 —1	—1.517	+5.865	6.045
2, —1	1 —1	+0.06	+0.54	+0.74
	2 —2	—0.02	—0.06	—0.26

Argument.	$\frac{d^2 R}{d v'^2}$		$\frac{d^2 R}{d v' d \rho'}$		$\frac{d^2 R}{d \rho'^2}$	
g' g	cos	sin	cos	sin	cos	sin
0, −1	+0.21	−0.28	+0.17	+0.12	−0.85	+0.87
1, −1	−0.91	+1.19	−3.50	−2.62	−2.51	+3.31
2, −1	+0.02	−0.13	−0.64	−0.32	−1.03	+1.90

The perturbations of Mars by Jupiter are:

Jupiter on Mars.

g'', g'	δv		$\delta \rho$		$\frac{1}{n} D_t \delta v'$		$\frac{1}{n} D_t \delta \rho'$	
	cos	sin	cos	sin	cos	sin	cos	sin
	"	"	"	"	"	"	"	"
1, −2	+ 2.15	+ 2.29	+1.34	−1.27	− 4.22	+ 3.96	+2.34	+2.47
1, −1	+16.66	+19.14	+8.26	−7.21	−16.11	+14.02	+6.07	+6.95
1, 0	+ 3.56	− 1.13	−0.34	+0.64	− 0.18	− 0.57	+0.10	+0.05

We now find

$$2\,\delta \frac{d\,\mathrm{R}}{d\,\mathrm{v}'} = \frac{m}{a'} \left\{ \begin{array}{l} -5.''5 \cos(\mathrm{N}-g') + 13.''0 \sin(\mathrm{N}-g') \\ -2.''4 \cos \mathrm{N} + 10.''0 \sin \mathrm{N} \\ -6.''6 \cos(\mathrm{N}+g') + 0.''3 \sin(\mathrm{N}+g') \end{array} \right\}$$

$$2\,\delta \frac{d\,\mathrm{R}}{d\,\rho'} = \frac{m}{a'} \left\{ \begin{array}{l} -59.''5 \cos(\mathrm{N}-g') - 18.''8 \sin(\mathrm{N}-g') \\ -14.''8 \cos(\mathrm{N}+g') + 12.''8 \sin(\mathrm{N}+g') \end{array} \right\}$$

$$\frac{2}{n'} \left\{ \frac{d\,\mathrm{R}}{d\,\mathrm{v}'} \frac{d\,\delta\,v'}{d\,t} + \frac{d\,\mathrm{R}}{d\,\rho'} \frac{d\,\delta\,\rho'}{d\,t} \right\} = \frac{m}{a'} \left\{ -1.''8 \cos \mathrm{N} + 9.''9 \sin \mathrm{N} \right\}$$

Hence follows

$$h_c = -\ 3.''8$$

$$h_s = +19.''0$$

and

$$\frac{3}{2} \frac{m}{\mu} \frac{n'^2}{\mathrm{n}^2} = 0.00296$$

$$\delta^2 l' = -0.''011 \cos \mathrm{N} + 0.''055 \sin \mathrm{N}$$

We have, in the third place, to consider the effect of the action of Jupiter on the Earth in disturbing Mars. In the present case this is extremely simple, as only a single set of terms need be considered. We have, in the action of Mars on the Earth, the terms

$$\frac{d^2 \mathrm{R}}{d\,\mathrm{v}\,d\,\mathrm{v}'}=\frac{m}{a'}\left(-0.44\cos(2g'-2g)-1.56\sin(2g'-2g)\right)$$

$$\frac{d^2 \mathrm{R}}{d\,\rho\,d\,\mathrm{v}'}=\frac{m}{a'}\left(-2.01\cos(2g'-2g)+0.57\sin(2g'-2g)\right)$$

and in the perturbations of the Earth by Jupiter

$$\delta v=-7.''20\cos(g''-g)+0.''06\sin(g''-g)$$
$$\delta \rho=+0.''03\cos(g''-g)+3.''35\sin(g''-g)$$

which give

$$\delta^2 l=+0.''015\cos \mathrm{N}-0.''053\sin \mathrm{N}$$

We thus have, for the total perturbation of the mean longitude of Mars depending on the argument

$$\mathrm{N}=g''-2g'+g$$

Action of Jupiter:	$\delta^2 l'=-0.''001\cos \mathrm{N}$	$+0.''006\sin \mathrm{N}$
Action of Earth:	$-0.''011$	$+0.''055$
" "	$+0.''015$	$-0.''053$
Total	$\delta^2 l'=+0.''003\cos \mathrm{N}$	$+0.''008\sin \mathrm{N}$

IX.

Addendum.

The preceding work was completed and prepared for the press during the year 1890 except some final arrangement and revision. Meanwhile a very valuable paper on the same subject has appeared from the pen of von HÆRDTL,* which I overlooked until the preceding pages were in type. He finds that the second combination, *Jupiter—Mars—Earth*, on page 59 preceding, gives rise to an important term in the motion of Mars and that there is also an appreciable term depending on the argument 2 *Saturn*—2 *Mars*+*Earth*. Reduced to the forms of the preceding paper by putting

$$\mathrm{N}_1=g''-2g'+g$$
$$\mathrm{N}_2=2g''-6g'+3g$$

I find that von HÆRDTL's results for the two inequalities arising from the action of Jupiter become

$$\delta^2 l'=-0.''003\cos \mathrm{N}_1+0.''022\sin \mathrm{N}_1$$
$$+0.''297\cos \mathrm{N}_2+0.''014\sin \mathrm{N}_2$$

It will be seen that the value of the first inequality, the only one computed in the preceding paper, differs from the value

$$+0.''003\cos \mathrm{N}_1+0.''008\sin \mathrm{N}_1$$

*Bulletin Astronomique; Tisserand: 1892. Tome IX, pp. 409 417.

by a considerable fraction of its whole amount, but not by a quantity of astronomical importance.

The new term discovered by von HÆRDTL is of sufficient importance to merit re-computation. The following steps of the computation seem all that need be presented.

Earth on Mars.

[Units of third place of decimals.]

g' g	w' w	R	D R	D^2 R	$j'w'+jw$ cos	$j'w'+jw$ sin
3, −3	3−3	+211	+792	+3435	+.9325	−.3612
4, −3	3−3	+108.0	+417.0	+1960	+.9325	−.3612
	4−4	− 13.7	− 66.1	− 358	−.8502	−.5265
5, −3	3−3	+ 34.9	+141.3	+ 708	+.9325	−.3612
	4−4	− 8.4	− 44.4	− 257	−.8502	−.5265
	5−5	+ 0.5	+ 3.3	+ 22	+.0922	+.9957

g', g	$\frac{d^2 R}{d v'^2}$ cos	$\frac{d^2 R}{d v'^2}$ sin	$\frac{d^2 R}{d v' d\rho}$ cos	$\frac{d^2 R}{d v' d\rho}$ sin	$\frac{d^2 R}{d\rho'^2}$ cos	$\frac{d^2 R}{d\rho'^2}$ sin
3, −3	−1.75	−0.69	−1.10	+2.78	+4.88	+1.89
4, −3	−1.093	−0.236	−0.401	+1.740	+3.13	+0.78
5, −3	−0.406	−0.031	−0.061	+0.673	+1.26	+0.16

These are to be multiplied by the following perturbations of Mars by Jupiter:

g'', g'	$\delta v'$ cos	$\delta v'$ sin	$\delta\rho'$ cos	$\delta\rho'$ sin	$\frac{1}{n'} D_t \delta v'$ cos	$\frac{1}{n'} D_t \delta v'$ sin	$\frac{1}{n'} D_t \delta\rho'$ cos	$\frac{1}{n'} D_t \delta\rho'$ sin
	"	"	"	"	"	"	"	"
2, −3	− 2.05	−0.46	−0.32	+ 1.41	+1.24	− 5.51	− 3.78	−0.86
2, −2	−15.68	−3.30	−2.18	+10.28	+5.56	−26.40	−17.31	−3.67
2, −1	−21.63	−3.17	−1.14	+ 7.81	+2.16	−14.77	− 5.33	−0.78
2, 0	− 1.44	−0.24	+0.11	− 0.69	−0.08	+ 0.46	− 0.22	−0.04

Jupiter on Mars.

[Units of third place of decimals.]

g'' g'	w'' w'	R	D R	D^2 R	$j''w''+j'w'$ cos	$j''w''+j'w'$ sin
2, 0,	2, —2	+ 1.51	+ 3.28	+ 7.69	+ .2208	+.9753
	1, —1	— 0.35	— 1.13	— 3.83	+ .7813	+.6242
	0, 0	+ 2.58	+ 0.56	+ 1.39	1.	.0
2, —1,	2, —2	—18.83	— 39.67	— 87.96	+ .2208	+.9753
	1, —1	+ 1.47	+ 4.62	+ 15.10	+ .7813	+.6242
2, —2,	3, —3	+ 0.3	+ 0.9	+ 2.8	— .436	+.900
	2, —2	+65.1	+135.3	+292.8	+ .2208	+.9753
	1, —1	— 0.1	— 0.3	— 1.0	+ .7813	+.6242

g'', g'	$\frac{d^2 R}{d v'^2}$ cos	$\frac{d^2 R}{d v'^2}$ sin	$\frac{d^2 R}{d v' d \rho'}$ cos	$\frac{d^2 R}{d v' d \rho'}$ sin	$\frac{d^2 R}{d \rho'^2}$ cos	$\frac{d^2 R}{d \rho'^2}$ sin
2, 0	—.001	+.006	+.006	.000	—.001	—.005
2, —1	+.0154	—.0725	—.0745	—.0139	—.0076	+.0764
2, —2	—.057	+.257	+.266	+.058	+.063	—.287

These are to be multiplied by the following perturbations of Mars by the Earth:

g', g	$\delta v'$ cos	$\delta v'$ sin	$\delta \rho'$ cos	$\delta \rho'$ sin	$\frac{1}{n'} D_t \delta v'$ cos	$\frac{1}{n'} D_t \delta v'$ sin	$\frac{1}{n'} D_t \delta \rho'$ cos	$\frac{1}{n'} D_t \delta \rho'$ sin
	"	"	"	"	"	"	"	"
6, —3	+0.095	—0.895	+0.257	+0.030	—0.320	—0.034	+0.011	—0.092
5, —3	+0.450	—2.409	—0.883	—0.165	+1.545	+0.289	+0.106	—0.566
4, —3	—0.116	+0.308	+0.245	+0.085	—0.506	—0.191	—0.140	+0.402
3, —3	—0.072	+0.187	+0.160	+0.062				

We have also in the action of Jupiter on the Earth the perturbations

$$\delta v = -0.''54 \cos(2g'' - g) + 1.''52 \sin(2g'' - g)$$
$$+0.''14 \cos(2g'' - 2g) + 2.''73 \sin(2g'' - 2g)$$

$$\delta \rho = +0.''66 \cos(2g'' - g) + 0.''23 \sin(2g'' - g)$$
$$+1.''91 \cos(2g'' - 2g) - 0.''10 \sin(2g'' - 2g)$$

which are to be multiplied by the following terms in the action of the Earth on Mars:

$$\frac{d^2 R}{d\text{v}\, d\text{v}'} = \frac{m}{a'}\begin{pmatrix} -0.42 \cos(6g'-4g) + 0.45 \sin(6g-4g) \\ -0.04 \cos(6g'-5g) - 1.50 \sin(6g'-5g) \end{pmatrix}$$

$$\frac{d^2 R}{d\rho\, d\text{v}'} = \frac{m}{a'}\begin{pmatrix} -1.74 \cos(6g'-4g) + 0.04 \sin(6g'-4g) \\ +0.56 \cos(6g'-5g) + 0.52 \sin(6g'-5g) \end{pmatrix}$$

The following are the results from the three actions:

$$\begin{aligned} \delta^2 l' = &+0.''023 \cos N_2 - 0.''006 \sin N_2 \\ &+0.''306 \qquad\quad -0.''015 \\ &-0.''009 \qquad\quad -0.''005 \\ = &+0.''320 \cos N_2 - 0.''026 \sin N_2 \end{aligned}$$

The method of computation adopted by von Hærdtl is so radically different from that of the preceding paper that no common element or datum enters in the same way into the two results. He has, in fact, used Le Verrier's method throughout, in which the variations of the disturbing forces are expressed in terms of the variations of the elements. On this method it may be remarked that the disturbing forces are a function only of the co-ordinates of the disturbing planet at any moment, and that the changes which they may undergo by perturbations of the planet are, in consequence, a function solely of the perturbations of its co-ordinates. But the perturbations of the elements are functions not only of the co-ordinates but of their first derivatives as to the time, and the perturbations of these first derivatives thus enter into the expressions for the forces when elements are used, although their final results must be self compensatory. Thus there are implicitly contained in the equations certain quantities which are to disappear from the final result. This is not of itself a serious inconvenience in cases where the perturbations of the elements are no larger than those of the co-ordinates. But there are cases, one of which occurs in the action of any inner planet upon an outer planet, in which the perturbations of the elements may be many times larger than those of the co-ordinates. In such cases these self-compensatory terms must be computed with a high degree of precision, and perhaps in great numbers, when the final results may be comparatively small.

In the preceding paper are computed only the perturbations of the mean longitude, and in this case only those which contain the square of ν, the ratio of the length of the period to the time of revolution of the planet. Le Verrier's method, as applied by von Hærdtl, would include the first power of ν as well as the second in the mean longitude. I conceive, however, that this term would ordinarily be smaller than the other in the ratio $1:\nu$, and would, therefore, be unimportant in the present case.

What has here been said on the use of perturbations of the elements instead of those of the co-ordinates in the computation has no relation to the question whether the perturbations finally computed shall be those of the co-ordinates or of the elements. Even though the La Grangian method of the variation of elements is used, it is not necessary to express these variations in terms of the partial derivatives of the disturbing force with respect to the elements. In practice it will be found more advantageous

to use the derivatives with respect to v, r, or its logarithm, and the two quantities which fix the plane of the orbit. These quantities, as I have shown in several papers, can be readily derived from the ordinary development of the perturbative function as a function of the elements.

It is to be expected that Mars may undergo not unimportant perturbations of comparatively short period dependent on the product of the masses of Jupiter and Saturn. The investigation of these perturbations must, however, be deferred to a future occasion.

The recomputation, by the preceding method, of von HÆRDTL's inequality depending on the action of Saturn, must also be included in the supplementary investigations of the perturbations of the four inner planets which, it is expected, will form a future part of the present volume

III.

THEORY

OF THE

INEQUALITIES IN THE MOTION OF THE MOON

PRODUCED BY

THE ACTION OF THE PLANETS

BY

SIMON NEWCOMB.

CONTENTS OF PART III.

PREFACE.

The fact that the observed motion of the Moon is apparently affected by inequalities of long period not yet satisfactorily accounted for is well known to all who have interested themselves in the subject. The study of these inequalities was undertaken by the author about the year 1870. The discussion of such observations as seemed best fitted to throw light on the subject was made at the U. S. Naval Observatory, and published in the year 1878, as an appendix to the Washington Observations for 1875. The result of that discussion was that the inequalities in question could not be represented by any existing theory, and might, therefore, be attributed to changes in the Earth's axial rotation, if it could be shown by other independent observations that such changes really existed. In the absence of such independent proof an attempt was made to represent the outstanding differences between theory and observation by an empirical formula; and the curious result was reached that by a quite simple change in what is now known as HANSEN's first inequality of long period a fairly good agreement with observation could be brought about.

The changes indicated by this formula were made in the mean longitude of HANSEN's lunar tables, and the observations up to the year 1863 show that the corrected motion agrees remarkably well with the fact. The simple way in which this agreement was brought about would seem to point very strongly to the theoretical correctness of the proposed change. Yet the most rigid examination into the subject has failed to show the possibility of any error in HANSEN's result.

Meanwhile an independent test of the constancy of the Earth's axis of rotation has been made through the discussion of the observed transits of Mercury, published in the first volume of the present series. The results of this test were inconclusive, in that, while it seemed proved very clearly that the apparent inequalities in the motion of the Moon were not wholly due to changes in the Earth's rotation, yet such changes were indicated in a somewhat inconclusive way, having a certain general analogy to those previously inferred from the motion of the Moon, though much smaller in amount, and therefore not very certain. What is most remarkable in this connection is that a study of the motions of the Moon, Mercury, and the first satellite of Jupiter all agree in indicating that, during some years previous to 1863, there was an increase in the speed of the rotation of the Earth which, however, rapidly subsided about that epoch. Added importance was thus given to the theoretical side of the question.

Among the problems of celestial mechanics which admit of rigorous treatment, it is well understood that that of the action of the planets on the Moon is the most diffi-

cult which presents itself. It was therefore necessary, in commencing the work, to devise a method by which it was supposed these difficulties could best be surmounted, and especially, by which every possible inequality could be detected and investigated. It occurred to the author that a more rigorous and general treatment than that hitherto adopted would be to consider the arbitrary constants which enter into the solution of the problem of three bodies, the Sun, Moon, and Earth, as simultaneously varied by the action of a fourth body, a planet, using LA GRANGE's method of the variation of elements. The determination of the LA GRANGIAN coefficients necessary in the theory proved to be a work of more simplicity and elegance than would have been supposed. Accordingly he took advantage of a visit to Paris in April, 1871, to read, before the French Academy, a preliminary chapter on the subject, which was soon afterward published in *Liouville's Journal de Mathématiques Pures et Appliquées.* In the course of the next two years the results of DELAUNAY's great work were utilized for the purpose of forming the actual values of the expressions which entered into the theory, and the greater part of the numerical developments pertaining to the action of the individual planets was also effected. The computation of the inequalities of long period produced by the action of Venus and Mars was also supposed to be nearly completed.

At this stage of the work a more careful investigation was made of its foundation, leading to the discovery that the author had wrongly measured the importance of quantities in the theory of three bodies by the numerical value of the perturbations to which they gave rise. The motion of the center of gravity of the Earth and Moon around the Sun, as determined by the action of the Sun itself, deviates from an invariable ellipse by quantities of no practical importance in ordinary astronomy. It was, therefore, supposed at the outset that the elements of the orbit of the Moon need not be regarded as contained in this motion. An examination of the problem showed, however, that this conclusion was ill-founded, and that, so far as mere order of magnitude went, the consideration of the lunar elements in this motion was as necessary as in the motion of the Moon itself. It therefore seemed to be necessary to form expressions for the solar elements in terms of the lunar elements, a work which, if completely executed, might be very laborious. Under these circumstances the work was laid aside until the author should find the leisure necessary to carry through the required investigations. Attempts were made from time to time to overcome or evade the difficulty, but none of them were brought to an entirely satisfactory conclusion. Indeed, the author is quite prepared for the conclusion that the rule found to hold in the theories of the planets, that the more general and abstractly rigorous a method of considering perturbations is, the less it is adapted to their actual numerical computation, may hold true in the present case. He has, however, at length deemed it best to publish the work, notwithstanding its lack of completeness, for the following reasons:

1. Its analogy with some of the modern theories of celestial mechanics, and its possible value as illustrating those theories, especially in the form developed by POINCARÉ in his recent treatise *Les méthodes nouvelles de la Mécanique céleste.* The path on which it proceeds to extend the solution of the problem of three bodies to that of four may perhaps lead to the method by which the most general solution of the equations of planetary motion is hereafter to be investigated.

2. The numerical developments for the action of the planets, from Mercury to Jupiter, inclusive, are equally necessary in applying any other method. Indeed, one circumstance which led to the question of publication being considered at the present moment, was the remark that these numerical developments might perhaps have been of use to RADAU in his recent elaborate researches on the subject.

3. Many of the analytic developments of the present method will be found closely analogous to, or identical with, those of other methods, and may thus serve as a control on the latter, or as a basis for their immediate application.

4. We have also to consider the usefulness of even an abstract method of considering the theory in its most general form, in that a more luminous view of the scope and principles of other methods may be obtained.

5. Finally, the author is not without hope that some way of avoiding or overcoming the difficulties he has encountered will be devised by an abler investigator; or, at least, that such an investigator will find the work useful. In any case the problem of applying the method is now merely that of completing the analytic theory of the motion of three bodies.

The first part of the work, relating to the general dynamical theory of the subject, was reconstructed at various times between the years 1873 and 1888, with a view of putting it in such a shape as would best admit of farther development and wider application. This remark applies especially to §§ 2 to 4, which, though first in order, are practically the last that have been prepared. All the rest of the work is now published as originally written in the years 1871–1873, except ordinary revision of details. Being once shaped into a logical and continuous whole, it was scarcely possible to make any extended changes or additions, except by a process which would have been equivalent to a complete reconstruction. The author was the more willing to refrain from such a course because he might then be more readily excused if it should appear that he had not duly considered the advances made within the last twenty years in the theory of the subject. By this course he has made plain an oversight in his conclusions as to the possibility of inequalities being produced by Jupiter, whereby what is now known as the JOVIAN evection was left to be discovered by Mr. NEVILL.

It may not be amiss in this connection to call attention to the fact that the problem has lost none of its importance during the time which has elapsed. There are three possible sources to which may be attributed the apparent inequalities in question:

(α) Defects in the gravitational theory.

(β) Inequalities in the Earth's time of rotation.

(γ) Action on the Moon of other forces than the gravitation of known bodies.

The second of these causes seems to be not at all improbable, in view of recent discoveries respecting variations of latitude, which can be accounted for only by minute changes in the Earth's axis of rotation. But the independent tests of this cause offered by the observed transits of Mercury do not show that it suffices to explain the observed inequalities in the motion of the Moon. Two other tests remain to be applied: the motion of the first satellite of Jupiter and observations of Mercury, Venus, and Mars. So far as evidence on these points has been brought out, the conclusion indicated is

that inequalities of the Earth's rotation are real, but not large enough to produce the observed effects on the Moon's mean longitude. The subject is therefore still involved in doubt, and the first step toward the removal of this doubt is such a determination of the inequalities in the mean motion produced by the action of the planets, as shall place the subject on an undoubted basis.

CHAPTER I.

GENERAL THEORY.

§ I.

FUNDAMENTAL DIFFERENTIAL EQUATIONS FOR THE DISTURBED AND UNDISTURBED MOTION OF THREE BODIES.

For convenient reference, a part of the notation used throughout this paper is given here. We put

m_1, m_2, m_3, m_4, the masses of the Sun, Earth, Moon, and planet, respectively;

x, y, z, the rectangular coordinates of the Moon relative to the Earth;

x', y', z', the rectangular coordinates of the Sun relative to the center of gravity of the Earth and Moon;

$$\mu = \frac{m_3}{m_2+m_3}$$

$$\mu' = \frac{m_2+m_3}{m_1+m_2+m_3}$$

$$\mu_1 = m_1 \mu'$$

$$\mu_2 = m_2 \mu$$

X_4, Y_4, Z_4, etc., the rectangular coordinates of the planet relatively to the center of gravity of the Sun, Earth, and Moon;

x_4, y_4, z_4, r_4, the rectangular coordinates and radius vector of the planet referred to the Sun;

r, the radius vector of the Moon, so that

$$r^2 = x^2 + y^2 + z^2;$$

r', the Sun's radius vector from center of gravity of Earth and Moon;

$$r'^2 = x'^2 + y'^2 + z'^2;$$

ρ, the distance of the planet from the center of gravity of the Earth and Moon;

a, e, π, etc., the elements of the Moon's orbit round the Earth, the major axes being determined by the condition $a^3 n^2 =$ sum of masses;

a', e', π', etc., the elements of the relative orbit of the Sun round the center of gravity of the Earth and Moon;

n, n', the mean motions of the Moon and Sun, respectively, as actually observed;

Ω, the potential, or sum of the products of each pair of masses divided by the distance, when only three bodies are considered;

R, the terms of the potential added by the planet;

$P = \Omega + R$, the entire potential;

x_1, y_1, z_1, x'_1, etc., the derivatives of x, y, z, x', etc., as to the time.

If we suppose, for the present, ξ_i, η_i, and ζ_i (i having in succession the values 1, 2, 3, 4), to represent the rectangular coordinates of the four bodies referred to any arbitrary fixed origin, and ξ_0, η_0, and ζ_0 to represent the coordinates of the center of gravity of the Sun, Earth and Moon, the differential equations of the motion of the three bodies will be

$$m_i \frac{d^2 \xi_i}{d t^2} = \frac{d P}{d \xi_i}; \quad m_i \frac{d^2 \eta_i}{d t^2} = \frac{d P}{d \eta_i}, \text{ etc.} \tag{1}$$

$$(i = 1, 2, 3)$$

Both sides of these equations admit of being completely expressed in terms of the three relative coordinates already defined. The expressions for the relative coördinates x, y, z, x', y', z' in terms of the absolute coordinates are

$$\begin{aligned} x &= \xi_3 - \xi_2 \\ y &= \eta_3 - \eta_2 \\ z &= \zeta_3 - \zeta_2 \\ x' &= \xi_1 - (1-\mu)\,\xi_2 - \mu\,\xi_3 \\ y' &= \eta_1 - (1-\mu)\,\eta_2 - \mu\,\eta_3 \\ z' &= \zeta_1 - (1-\mu)\,\zeta_2 - \mu\zeta_3 \end{aligned} \tag{2}$$

Differentiating these equations twice, and substituting the values of the second derivatives in (1) we find

$$\begin{aligned} \frac{d^2 x}{d t^2} &= \frac{1}{m_3} \frac{d P}{d \xi_3} - \frac{1}{m_2} \frac{d P}{d \xi_2} \\ \frac{d^2 x'}{d t^2} &= \frac{1}{m_1} \frac{d P}{d \xi_1} - \frac{1}{m_2 + m_3} \left(\frac{d P}{d \xi_2} + \frac{d P}{d \xi_3} \right) \end{aligned} \tag{3}$$

with similar equations in y, y', z and z'.

The four absolute coordinates ξ_1, ξ_2, ξ_3, ξ_4 admit of being completely expressed as a function of the four quantities x, x', ξ_0, and X_4. If, therefore, we suppose P expressed in terms of the last four quantities, we may put, in (3)

$$\frac{d\,P}{d\,\xi_i}=\frac{d\,P}{d\,x}\frac{d\,x}{d\,\xi_i}+\frac{d\,P}{d\,x'}\frac{d\,x'}{\partial\,\xi_i}+\frac{d\,P}{d\,X_4}\frac{d\,X_4}{d\,\xi_i}+\frac{d\,P}{d\,\xi_0}\frac{d\,\xi_0}{d\,\xi_i} \tag{4}$$

the last term, however, being identically zero, because ξ_0 disappears from P.

The complete expressions for the coordinates ξ are

$$\begin{aligned}\xi_1&=\xi_0+\mu' x'\\ \xi_2&=\xi_0-(1-\mu')\,x'-\mu x\\ \xi_3&=\xi_0-(1-\mu')\,x'+(1-\mu)\,x\\ \xi_4&=\xi_0+X_4\end{aligned} \tag{5}$$

Representing the distance of the bodies numbered i and j by $\rho_{i,j}$, the complete expression for P is

$$P=\Sigma^2_{i,j}\frac{m_i\,m_j}{\rho_{ij}}$$

the general value of $\rho_{i,j}$ being derived from

$$\rho^2_{i,j}=(\xi_i-\xi_j)^2+(\eta_i-\eta_j)^2+(\zeta_i-\zeta_j)^2$$

The expressions for the six values of $\rho_{i,j}$ in terms of the relative coordinates are from (5)

$$\begin{aligned}\rho^2_{12}&=(x'+\mu\,x)^2+(y'+\mu y)^2+(z'+\mu\,z)^2\\ \rho^2_{13}&=(x'-(1-\mu)\,x)^2+(y'-(1-\mu)\,y)^2+(z'-(1-\mu)\,z)^2\\ \rho^2_{14}&=(X_4-\mu'\,x')^2+(Y_4-\mu'\,y')^2+(Z_4-\mu'\,z')^2\\ \rho^2_{23}&=r^2=x^2+y^2+z^2\\ \rho^2_{24}&=(X_4+(1-\mu')\,x'+\mu\,x)^2+(Y_4+(1-\mu')\,y'+\mu\,y)^2\\ &\quad+(Z_4+(1-\mu')\,z'+\mu\,z)^2\\ \rho^2_{34}&=(X_4+(1-\mu')\,x'-(1-\mu)\,x)^2+(Y_4+(1-\mu')\,y'-(1-\mu)\,y)^2\\ &\quad+(Z^4+(1-\mu')\,Z'-(1-\mu)\,z)^2\end{aligned} \tag{6}$$

From (2), combined with the value of X_4 derived from the expression (5), namely,

$$X_4=\xi_4-\frac{m_1\,\xi_1+m_2\,\xi_2+m_3\,\xi_3}{m_1+m_2+m_3}$$

we find the partial derivatives

$$\frac{d\,x}{d\,\xi_1}=0 \qquad \frac{d\,x}{d\,\xi_2}=-1 \qquad \frac{d\,x}{d\,\xi_3}=1$$

$$\frac{d\,x'}{d\,\xi_1}=1 \qquad \frac{d\,x'}{d\,\xi_2}=\mu-1 \qquad \frac{d\,x'}{d\,\xi_3}=-\mu$$

$$\frac{d\,X_4}{d\,\xi_1}=-\frac{m_1}{m_1+m_2+m_3} \qquad \frac{d\,X_4}{d\,\xi_2}=-\frac{m_2}{m_1+m_2+m_3} \qquad \frac{d\,X_4}{d\,\xi_3}=-\frac{m_3}{m_1+m_2+m_3}$$

These values being substituted in the general expression (4), we find for the values of the derivatives in (3)

$$\frac{d\,P}{d\,\xi_1}=\frac{d\,P}{d\,x'}-\frac{m_1}{m_1+m_2+m_3}\frac{d\,P}{d\,X_4}$$

$$\frac{d\,P}{d\,\xi_2}=-\frac{d\,P}{d\,x}+(\mu-1)\frac{d\,P}{d\,x'}-\frac{m_2}{m_1+m_2+m_3}\frac{d\,P}{d\,X_4}$$

$$\frac{d\,P}{d\,\xi_3}=\frac{d\,P}{d\,x}-\mu\frac{d\,P}{d\,x'}-\frac{m_3}{m_1+m_2+m_3}\frac{d\,P}{d\,X_4}$$

By the substitution of these values and the corresponding ones for the other two coordinates equations (3) become

$$\left.\begin{aligned}
\mu_2\frac{d^2x}{d\,t^2}&=\frac{d\,P}{d\,x}\\
\mu_2\frac{d^2y}{d\,t_2}&=\frac{d\,P}{d\,y}\\
\mu_2\frac{d^2z}{d\,t^2}&=\frac{d\,P}{d\,z}\\
\mu_1\frac{d^2x'}{d\,t^2}&=\frac{d\,P}{d\,x'}\\
\mu_1\frac{d^2y'}{d\,t^2}&=\frac{d\,P}{d\,y'}\\
\mu_1\frac{d^2z'}{d\,t^2}&=\frac{d\,P}{d\,z'}
\end{aligned}\right\}\qquad(7)$$

where P is to be conceived as expressed in the form

$$P=f\,(x, y, z, x', y', z', X_4, Y_4, Z_4)$$

by the substitution (5).

These are the fundamental differential equations of the problem. As thus presented, the coordinates of the planet X_4, Y_4, and Z_4, are supposed to be given in terms of the time. The differential equations for these coordinates, supposing the planet to be acted upon only by the Sun, Earth, and Moon, would be of the same form, and if joined to (7), the whole would represent the differential equations of the problem of four bodies. In reality, however, we should add nothing to the astronomical completeness of our problem by considering it as a problem of four bodies, because, if we regard the coordinates of the planet as depending on the attraction of the Earth and Moon, we must equally regard them as depending on the action of the other planets. We shall therefore regard our problem as that of the disturbed motion of three bodies, the position of the disturbing body being supposed to be given independently of the integrations by which the relative coordinates of the three bodies are determined.

§ 2.

REDUCED FORM OF THE POTENTIAL.

Let us first consider the principal portion Ω of the potential P, namely,

$$\Omega=\frac{m_1 m_2}{\rho_{1,2}}+\frac{m_1 m_3}{\rho_{1,3}}+\frac{m_2 m_3}{\rho_{2,3}}$$

the three denominators having the values given in equations (6). The coordinates x, y, and z being very small in comparison with x', y', and z', the first two terms of Ω may be developed in powers of their ratios. The first approximation will be the value which these terms assume when we suppose x, y, and z to vanish. Putting

$$r'^2=x'^2+y'^2+z'^2$$
$$r^2=x^2+y^2+z^2=\rho^2_{2,3},$$

we find for a portion of Ω, which we shall call Ω_0,

$$\Omega_0=\frac{m_1(m_2+m_3)}{r'}+\frac{m_2 m_3}{r} \tag{8}$$

while the remaining terms of Ω will be

$$\Omega_1=m_1\left\{m_2\left(\frac{1}{\rho_{1,2}}-\frac{1}{r'}\right)+m_3\left(\frac{1}{\rho_{1,3}}-\frac{1}{r'}\right)\right\}$$

Now let us put

W, the angular distance of the Sun and Moon, as seen from the center of gravity of the Earth and Moon. We shall then have from (6)

$$\rho^2_{1,2}=r'^2\left\{1+2\,\mu\frac{r}{r'}\cos W+\mu^2\frac{r^2}{r'^2}\right\}$$

$$\rho^2_{1,3}=r'^2\left\{1-2\,(1-\mu)\frac{r}{r'}\cos W+(1-\mu)^2\frac{r^2}{r'^2}\right\}$$

Developing the power of $-\frac{1}{2}$ of these expressions to terms of the second order in $\frac{r}{r'}$ we find

$$\frac{1}{\rho_{1,2}}=\frac{1}{r'}\left\{1-\mu\frac{r}{r'}\cos W+\frac{1}{2}\mu^2\frac{r^2}{r'^2}(3\cos^2 W-1)\right\}$$

$$\frac{1}{\rho_{1,3}}=\frac{1}{r'}\left\{1+(1-\mu)\frac{r}{r'}\cos W+\frac{1}{2}(1-\mu)^2\frac{r^2}{r'^2}(3\cos^2 W-1)\right\}$$

Substituting these values in Ω_1, and noticing that by the expression for μ, namely,

$$\mu=\frac{m_3}{m_2+m_3}$$

we have

$$m_2 \mu - m_3 (1 - \mu) = 0$$

$$m_2 \mu^2 + m_3 (1 - \mu)^2 = \frac{m_2 m_3}{m_2 + m_3} = \mu_2$$

we find

$$\Omega_1 = \frac{1}{2} \frac{m_1 m_2 m_3}{m_2 + m_3} \frac{r^2}{r'^3} (3 \cos^2 W - 1) \tag{9}$$

The expressions (8) and (9) make up the complete value of Ω when terms of the third and higher orders with respect to $\frac{r}{r'}$ are omitted.

The remaining terms of P are those depending on the mass of the disturbing planet, the expression being

$$R = m_4 \left(\frac{m_1}{\rho_{1,4}} + \frac{m_2}{\rho_{2,4}} + \frac{m_3}{\rho_{3,4}} \right)$$

For convenience in subsequent developments we shall substitute for the first term the integral of its derivatives with respect to the variables x', y', z', which we do because only these derivatives enter into the fundamental differential equations, and $\rho_{1,4}$ does not contain any of the quantities x, y, or z. This term of R may then be replaced by the expression

$$m_1 m_4 \int \left(\frac{d \frac{1}{\rho_{1,4}}}{d x'} d x' + \frac{d \frac{1}{\rho_{1,4}}}{d y'} d y' + \frac{d \frac{1}{\rho_{1,4}}}{d z'} d z' \right)$$

Substituting the value of $\rho_{1,4}$ in (6), and putting after differentiation

$$x_4, y_4, z_4, r_4$$

for the coordinates of the planet relative to the Sun, or

$$x_4 = X_4 - \mu' x', \text{ etc.},$$

this integral becomes

$$\int \frac{m_1 m_4 \mu'}{\rho^3_{1,4}} (x_4 \, d x' + y_4 \, d y' + z_4 \, d z')$$

and may be expressed in the form

$$m_1 m_4 \mu' \frac{x_4 x' + y_4 y' + z_4 z'}{r^3_4} \tag{10}$$

provided that in forming the partial derivatives we regard the coordinates of the planet relative to the Sun as independent of x', y', and z'.

The remaining two terms of R may be developed in the same way that we have

just developed Ω_1. If we put

ρ, the distance of the planet from the center of gravity of the Earth and Moon, and for brevity,

ξ, η, ζ, the coordinates of the planet relative to the same origin, we have

$$\rho^2_{2,4} = \rho^2 + 2\,\mu\,(\xi\,x + \eta\,y + \zeta\,z) + \mu^2\,r^2$$
$$\rho^2_{3,4} = \rho^2 - 2\,(1-\mu)\,(\xi\,x + \eta\,y + \zeta\,z) + (1-\mu)^2\,r^2$$

Taking the power $-\frac{1}{2}$ and putting, for brevity,

$$\varDelta = \xi\,x + \eta\,y + \zeta\,z$$

we find for the principal terms

$$\frac{1}{\rho_{2,4}} = \frac{1}{\rho} - \frac{1}{\rho^3}\left(\mu\,\varDelta + \frac{1}{2}\,\mu^2\,r^2\right) + \frac{3\,\mu^2\,\varDelta^2}{2\,\rho^5} + \quad . \quad .$$

$$\frac{1}{\rho_{3,4}} = \frac{1}{\rho} - \frac{1}{\rho^3}\left((\mu-1)\,\varDelta + \frac{1}{2}\,(1-\mu)^2\,r^2\right) + \frac{3\,(1-\mu)^2\,\varDelta^2}{2\,\rho^5} + \quad . \quad .$$

Multiplying the first of these by $m_4\,m_2$, and the second by $m_4\,m_3$, and adding, we find for the corresponding terms of R

$$m_4\left\{\frac{m_2+m_3}{\rho} - \frac{\mu_2\,r^2}{2\,\rho^3} + \frac{3\,\mu_2\,\varDelta^2}{2\,\rho^5} + \quad . \quad . \right\} \tag{11}$$

The first term is subject to the inconvenience that ρ is there to be regarded in differentiating with respect to x' as a function of the coordinates of the Earth and planet relative to the center of gravity of the Sun, Earth, and Moon, and therefore as a function of x'. To avoid this inconvenience, and introduce the coordinates of the planet relative to the Sun, we shall make the same transformation we made on the first term of R, putting instead of $\frac{1}{\rho}$ the expression

$$\int\left(\frac{d\frac{1}{\rho}}{dx'}\,d\,x' + \frac{d\frac{1}{\rho}}{d\,y'}\,d\,y' + \frac{d\frac{1}{\rho}}{d\,z'}\,d\,z'\right) = -\int\frac{1-\mu'}{\rho^3}\,(\,\xi\,d\,x' + \eta\,d\,y' + \zeta\,d\,z'\,)$$

When we introduce instead of X_4, Y_4, Z_4 the coordinates of the planet relatively to the Sun we have

$$X_4 = x_4 - (1-\mu')\,x', \text{ etc.,}$$

and hence

$$\xi = x_4 + x',\ \eta = y_4 + y',\ \zeta = z_4 + z'$$

Substituting these values in the integral it becomes

$$-\int\frac{1-\mu'}{\rho^3}\,\rho\,d\,\rho = \frac{1-\mu'}{\quad} \tag{12}$$

so that the first term of (11) is simply to be multiplied by $1-\mu'$, and ρ to be regarded as a function of the heliocentric position of the planet only.

Collecting the various terms of R in (10) and (11) as modified by (12) we have its complete transformed expression. The fundamental differential equations (7) now become

$$
\left.\begin{aligned}
\mu_1 \frac{d^2 x'}{d t^2} &= \frac{d\Omega}{d x'} + \frac{d\mathrm{R}}{d x'} \\
\mu_1 \frac{d^2 y'}{d t^2} &= \frac{d\Omega}{d y'} + \frac{d\mathrm{R}}{d y'} \\
\mu_1 \frac{d^2 z'}{d t^2} &= \frac{d\Omega}{d z'} + \frac{d\mathrm{R}}{d z'} \\
\mu_2 \frac{d^2 x}{d t^2} &= \frac{d\Omega}{d x} + \frac{d\mathrm{R}}{d x} \\
\mu_2 \frac{d^2 y}{d t^2} &= \frac{d\Omega}{d y} + \frac{d\mathrm{R}}{d y} \\
\mu_2 \frac{d^2 z}{d t^2} &= \frac{d\Omega}{d z} + \frac{d\mathrm{R}}{d z}
\end{aligned}\right\} \qquad (13)
$$

Where

$$
\left.\begin{aligned}
\Omega &= \Omega_0 + \Omega_1 \\
\Omega_0 &= \frac{m_1(m_2+m_3)}{r'} + \frac{m_2 m_3}{r} \\
\Omega_1 &= \frac{1}{2} m_1 \mu_2 \frac{r^2}{r'^3}\left(3\cos^2 \mathrm{W} - 1\right) \\
\mathrm{R} &= m_4 \mu_1 \left(\frac{1}{\rho} + \frac{x_4 x' + y_4 y' + z_4 z'}{r_4^3}\right) + \frac{1}{2} m_4 \mu_2 \left(\frac{3\Delta^2}{\rho^5} - \frac{r^2}{\rho^3}\right)
\end{aligned}\right\} \qquad (14)
$$

in which we have put

$$\mu_1 = \frac{m_1(m_2+m_3)}{m_1+m_2+m_3} = m_1 \mu'$$

$$\mu_2 = \frac{m_2 m_3}{m_2+m_3} = m_2 \mu$$

$$\Delta = x(x_4+x') + y(y_4+y') + z(z_4+z')$$

It will be remarked that the variable part of Ω_1 is identical with the force function for the action of the Sun in the lunar theory, and that R is closely analogous to the disturbing function for the action of a planet in DELAUNAY'S theory. It is also to be noted that Ω_1 and R are developed in powers of the ratio of the mean distance of the

Moon to that of the Sun or planet, and that only the first terms of the development are included in the expressions (14).

§ 3.

COMPARISON OF TWO METHODS OF INTEGRATING THE PRECEDING EQUATIONS.

The differential equations (13) of the preceding chapter, or some transformation of them, form the necessary starting point of all methods of treating the theory of the Moon's motion. It may also be held that the most rigorous method hitherto developed of treating the problem analytically, and now well known, is that resting upon the results of DELAUNAY's system of integrating the equations in question, or upon some process equivalent to it.

The problem as actually considered by DELAUNAY was that of three bodies only, considered as material points. Even with this restriction he did not fully complete the solution, because he considered the relative elliptic orbit of the Sun to be performed around the center of the Earth, instead of the center of gravity of the Earth and Moon. So far as the Moon's motion is concerned, the additions required to reduce his solution to the actual case are extremely simple, so that for all practical purposes we may consider DELAUNAY's theory to be complete as regards the Moon. The deviations of the path described by the center of gravity of the Earth and Moon from an elliptic orbit around the Sun are, from a theoretical point of view, necessary to complete the solution, but are too minute to be of importance in observational astronomy.

To complete the solution of the problem of the Moon's motion subsidiary actions have to be taken into account, such as those arising from the ellipticity of the Earth and from the action of the planets. It is to the two methods of effecting these subsidiary solutions that the attention of the reader is now called. One of these was doubtless that intended by DELAUNAY himself, and consists essentially in continuing the method adopted by DELAUNAY, considering the actions in question simply as perturbations additional to those produced by the Sun. This method has been followed by G. W. HILL on DELAUNAY's lines in Vol. III of these papers. The methods of other investigators, as RADAU and NEVILL, may also be considered as founded substantially upon the same idea. This I shall designate hereafter as the first method.

In the second method, which it is the object of the present paper to develop, all the constants of integration of the problem of three bodies are considered as simultaneously variable under the action of a fourth body, a planet. In the actual application of this second method there is so close an analogy to the first that the two must be mutually helpful to each other. Their relations can be clearly shown by a comparative statement of the general plan followed in them.

In both methods the solution of the last three equations (13) is obtained by three successive steps. In the first step, which is common to both methods, we have, instead of these equations, those which result from putting Ω_0 for $\Omega + \mathrm{R}$, the value of the force function then used being

$$\Omega_0 = \frac{m_1(m_2+m_3)}{r'} + \frac{m_2 m_3}{r}$$

The results of this step are the theories of elliptic motion; the moon revolves around the Earth in an ellipse, as does the center of gravity of the Earth and Moon around the Sun. Thus in the solution the Moon's coordinates, x, y, and z, are represented as functions of six arbitrary constants, say

$$a_1, a_2, \quad . \quad . \quad . \quad a_6$$

and of the time; while the Sun's coordinates, x', y', and z', appear as functions of six other arbitrary constants, say

$$a'_1, a'_2, \quad . \quad . \quad . \quad a'_6$$

and of the time.

In this step the effect of the solution is that the twelve arbitrary constants of motion are completely separated into two classes, six appearing only in the expressions for the Moon's coordinates, and the other six only in those of the Sun.

In the second step $\Omega_0+\Omega_1$ is substituted for Ω_0 in the second members of the differential equations which determine the Moon's motion. Ω_1 is here a function of the coordinates of both bodies, and the value of its principal term is given in (14). The complete solution of this step is effected by regarding the twelve elements as variable. Since at the first step the Moon's coordinates contain only unaccented elements, which we may call lunar elements, and the Sun only the accented elements, which we may call solar elements, it follows that the La Grangian coefficients which pertain to the Moon will contain only the six lunar elements, and those which pertain to the Sun only the solar ones. The solution might then be made by successive approximations, the lunar elements being first regarded as constant in integrating the equations for the Sun's motion, and the solar elements as constant in integrating the equations for the Moon's motion. The values of the twelve elements obtained in this first approximation may then be substituted in the second members of all the equations, a second approximation made, and so on until a solution of the requisite accuracy is obtained.

In Delaunay's theory, as in all others hitherto published, the attention being confined to the Moon's coordinates, the solution is effected only for the lunar elements, and moreover the solar elements are regarded as constant, it being practically unnecessary to consider the variations of the solar elements due to Ω_1. Successive approximations are, however, made in the case of the lunar elements. These elements, and hence the lunar coordinates, thus become functions of twelve arbitrary constants, six of which pertain to the Sun alone, and are not constants of integration, but are considered as entering into the differential equations of the Moon's motion. The other six are the constants of integration, and may be called lunar constants. We may represent these six lunar constants by the symbols

$$\alpha_1, \alpha_2 \quad . \quad . \quad . \quad . \quad \alpha_6$$

So far we have only the solution of the problem of three bodies, which, in the second method, is considered to be solved in advance. The divergence of the two methods

now commences in the continuation of the solution. As a matter of fact, the solar elements, say

$$\alpha'_1, \alpha'_2 \quad . \quad . \quad . \quad . \quad \alpha'_6$$

become variable through the action of the planet. The determination of the variations offers no difficulty, their value being the same whichever method be adopted. But in the first method, when the coordinates of the Moon are considered, these solar elements are not regarded as constants of integration, but as quantities given *a priori*, and appearing in the differential equations. The LA GRANGIAN coefficients

$$[\alpha_i, \alpha_j] = \frac{d\,x}{d\,\alpha_i}\frac{d\,x_1}{d\,\alpha_j} - \frac{d\,x_1}{d\,\alpha_i}\frac{d\,x}{d\,\alpha_j} + \frac{d\,y}{d\,\alpha_i}\frac{d\,y_1}{d\,\alpha_j} - \text{etc.}$$

are therefore formed only for the combinations of the six lunar elements, and the perturbative function R appears in two parts, the one being the value of R, as found in (14), the other the value of $\delta\,\Omega_1$ arising from the perturbations of the solar elements through the action of the planet, which we may conceive expressed in the form

$$\delta\,\Omega_1 = \frac{d\,\Omega_1}{d\,\alpha'_1}\delta\,\alpha'_1 + \frac{d\,\Omega_1}{d\,\alpha'_2}\delta\,\alpha'_2 + \quad . \quad . \quad .$$

Thus in the final solution obtained in this way the Moon's coordinates will not appear as functions of the actual variable elements of the Sun, but of mean elements at some epoch, the entire effect of the perturbations produced by the planet being thrown upon the lunar elements.

In the second method, all twelve elements are regarded as simultaneously variable, and the LA GRANGIAN coefficients are formed accordingly, so that the lunar coordinates are in the first plane considered as functions of all twelve varying elements.

The final forms of the solution by the two methods may now be seen and compared.

In the results of DELAUNAY's work, where only three bodies are considered, the Moon's coordinates are expressed as a function of the twelve constants

$$\alpha_1, \alpha_2 \quad . \quad . \quad . \quad . \quad \alpha_6, \alpha'_1, \alpha'_2 \quad . \quad . \quad . \quad . \quad \alpha'_6$$

When the action of the planet is included by HILL's continuation of DELAUNAY's process, the Moon's coordinates are regarded as still the same functions of these twelve elements, the accented ones remaining constant, and the others becoming variable.

In the new method the coordinates are also regarded as still the same functions of these elements, but all twelve of the elements are regarded as variable.

We may now construct the equations of condition, which must be satisfied in order that the two methods may give the same result. Let us represent by the symbol δ' the perturbations of the elements by the first method, and by δ those by the second. The corresponding perturbations of any coordinate v will then be

$$\text{I. } \delta v = \frac{dv}{d\alpha_1}\delta'\alpha_1 + \frac{dv}{d\alpha_2}\delta'\alpha_2 + \quad . \quad . \quad . \quad + \frac{dv}{d\alpha_6}\delta'\alpha_6$$

$$\text{II. } \delta v = \frac{dv}{d\alpha_1}\delta\alpha_1 + \quad . \quad . \quad . \quad + \frac{dv}{d\alpha_6}\delta\alpha_6 + \frac{dv}{d\alpha'_1}\delta\alpha'_1 + \quad . \quad . \quad . \quad + \frac{dv}{d\alpha'_6}\delta\alpha'_6$$

The equality of these perturbations gives the relation

$$\Sigma \frac{dv}{d\alpha_i}(\delta'\alpha_i - \delta\alpha_i) = \Sigma \frac{dv}{d\alpha'_i}\delta\alpha'_i \tag{15}$$

$$(i = 1, 2, \quad . \quad . \quad . \quad . \quad 6)$$

which is the required relation.

§ 4.

FUNDAMENTAL EQUATIONS OF THE METHOD.

The present paper being devoted wholly to the development of the second of the preceding methods, we have to consider further the general method of proceeding. We recall that in the general LA GRANGIAN method, which alone will be used, it is presupposed that a solution is obtained giving the values of the coordinates in terms of the requisite number of arbitrary quantities, such as shall completely satisfy the differential equations when certain terms of the potential are omitted. A solution of the case when the omitted terms are taken account of is then obtained by a system of ordinary differential equations, in which the variables are the arbitrary elements introduced by the preceding solution. There is no limit to the number of times by which this process may be successively repeated. The effect of each new term of the potential introduced is taken account of by the integration of a system of differential equations in which the arbitrary constants of the preceding solution appear as the variables. DELAUNAY'S classic work forms a beautiful example of the operations by which the equations of the Moon's motion, as affected by the action of the Sun, may thus be integrated.

The problem with which the present paper is ultimately concerned is not that of the motion of the Moon as affected by the action of the Sun, but as affected by the action of the planets. This action is involved only in the term R of the potential. Hence the sole object of the present paper is to to determine the variations of the lunar, and incidentally of the solar elements produced by the term in question, the equations being completely integrated for the case when R is omitted. The general nature of the process is so well known that no development is necessary. We shall therefore merely present a résumé of the necessary equations.

Firstly. We presuppose that the values of the six coordinates x, y, z, x', y', z', are expressed in terms of the time and of twelve arbitrary constants, which we may represent in the most general way by the symbols

$$\alpha_1, \alpha_2, \quad . \quad . \quad . \quad . \quad \alpha_{12},$$

in such way as to completely satisfy the six equations

$$
\begin{aligned}
&\mu_2 \frac{d^2 x}{d t^2} = \frac{d \Omega}{d x} && \mu_1 \frac{d^2 x'}{d t^2} = \frac{d \Omega}{d x'} \\
&\mu_2 \frac{d^2 y}{d t^2} = \frac{d \Omega}{d y} && \mu_1 \frac{d^2 y'}{d t^2} = \frac{d \Omega}{d y'} \\
&\mu_2 \frac{d^2 z}{d t^2} = \frac{d \Omega}{d z} && \mu_1 \frac{d^2 z'}{d t^2} = \frac{d \Omega}{d z'}
\end{aligned}
\tag{16}
$$

Let us represent these expressions in the form

$$
\begin{aligned}
&x = f_1 && x' = \mathrm{F}_1 \\
&y = f_2 && y' = \mathrm{F}_2 \\
&z = f_3 && z' = \mathrm{F}_3
\end{aligned}
\tag{17}
$$

where f and F are functions of the twelve arbitrary constants and of t.

Secondly. We construct the conjugate functions x_1, y_1, z_1, x'_1, y'_1, z'_1 by the equations

$$
\begin{aligned}
&x_1 = \mu_2 \frac{d f_1}{d t} && x'_1 = \mu_1 \frac{d \mathrm{F}_1}{d t} \\
&y_1 = \mu_2 \frac{d f_2}{d t} && y'_1 = \mu_1 \frac{d \mathrm{F}_2}{d t} \\
&z_1 = \mu_2 \frac{d f_3}{d t} && z'_1 = \mu_1 \frac{d \mathrm{F}_3}{d t}
\end{aligned}
\tag{18}
$$

The variations of the twelve elements when R is taken into account must then satisfy a simultaneous system of twelve differential equations, of which four are as follows:

$$
\begin{aligned}
&\frac{d x}{d \alpha_1}\frac{d \alpha_1}{d t} + \frac{d x}{d \alpha_2}\frac{d \alpha_2}{d t} + \quad \cdots \quad + \frac{d x_{12}}{d \alpha_{12}}\frac{d \alpha_{12}}{d t} = 0 \\
&\frac{d x_1}{d \alpha_1}\frac{d \alpha_1}{d t} + \frac{d x_1}{d \alpha_2}\frac{d \alpha_2}{d t} + \quad \cdots \quad + \frac{d x_1}{d \alpha_{12}}\frac{d \alpha_{12}}{d t} = \frac{d \mathrm{R}}{d x} \\
&\frac{d x'}{d \alpha_1}\frac{d \alpha_1}{d t} + \frac{d x'}{d \alpha_2}\frac{d \alpha_2}{d t} + \quad \cdots \quad + \frac{d x'}{d \alpha_{12}}\frac{d \alpha_{12}}{d t} = 0 \\
&\frac{d x'_1}{d \alpha_1}\frac{d \alpha_1}{d t} + \frac{d x'_1}{d \alpha_2}\frac{d \alpha_2}{d t} + \quad \cdots \quad + \frac{d x'_1}{d x_{12}}\frac{d \alpha_{12}}{d t} = \frac{d \mathrm{R}}{d x'}
\end{aligned}
\tag{19}
$$

while the remaining eight are formed by writing y and z in the place of x, and need not therefore be actually written.

From these equations are derived, by a well known linear combination, the La Grangian equations for the variation of elements in the following form. We put

$$
[a_i, a_j] = \Sigma \left(\frac{d x}{d a_i}\frac{d x_1}{d \alpha_j} - \frac{d x}{d a_j}\frac{d x_1}{d a_i} \right)
\tag{20}
$$

the sign Σ indicating a summation of the terms which arise by writing x, y, z, x', y', z' in succession for x.

We also suppose that R becomes a function of the elements α_i, and of the time by the substitution of the values of the coordinates as functions of these quantities. The La Grangian equations for the variation of the elements then take the following form:

$$[\alpha_i\,\alpha_1]\frac{d\,\alpha_1}{d\,t}+[\alpha_i,\,\alpha_2]\frac{d\,\alpha_2}{d\,t}+\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad+[\alpha_i,\,\alpha_{12}]\frac{d\,\alpha_{12}}{d\,t}=\frac{d\,\mathrm{R}}{d\,\alpha_i}\qquad(21)$$

$$(i=1;\ 2;\ 3;\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad 12)$$

In determining these La Grangian coefficients I make use of the known theorem that they are constants. This theorem rests, however, upon the supposition that the expressions for the coordinates actually satisfy the differential equations. As a matter of fact they are developed in series, so that it is impossible that this condition shall be rigorously fulfilled. It is therefore important to determine finally to what degree of approximation the constancy of the coefficients in question is assured. For our present purpose I have deemed it sufficient to develop the constant portions of the coefficients so far as the degree of approximation to which the solution is carried would admit.

This theorem of the constancy of the coefficients enables their values to be formed with much greater facility than if it were necessary to actually perform all the multiplications which enter into the equation (20).

In considering the general theory a yet farther simplification may be made. It is known that if we take six of the elements, which we may call

$$l_1,\ l_2,\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad l_6$$

at pleasure, we can always substitute for the other six certain functions of them,

$$c_1,\ c_2\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad c_6$$

such that the La Grangian coefficients shall reduce to zero in all the combinations except those six in which an l_i is coordinated with its corresponding c_i, in which case the combination will be reduced to unity. We shall make frequent use of this theorem in the course of the present investigation.

Such a selection of elements I shall hereafter call *canonical.*

§ 5.

THE PROBLEM OF THREE BODIES.

The preliminary step now is to express the six coordinates, x, y, z, x', y', z', in terms of twelve arbitrary constants, so as to satisfy the differential equations (16).

We recall that two processes of integration are necessary in the preliminary investigation, namely:

1. We integrate omitting Ω_1 in the equations (13), writing Ω_0 for $\Omega+\mathrm{R}$.
2. We include Ω_1, but omit R, as in the equations (16).

The first step involves no difficulty, since it gives rise only to the equations of elliptic motion—

(*a*) Of the Moon around the Earth;

(*b*) Of the center of gravity of the Earth and Moon around the Sun.

Putting in a general way

$$c_1,\ c_2,\ c_3,\ l_1,\ l_2,\ l_3$$

for a set of canonical elliptic elements pertaining to the Moon, and

$$c'_1,\ c'_2,\ c'_3,\ l'_1,\ l'_2,\ l'_3$$

for those pertaining to the Sun, we shall then have, as the result of putting $\Omega_1 = 0$

$$\left.\begin{aligned} &x,\ y,\ \text{and}\ z = f\,(c_1,\ c_2,\ c_3,\ l_1,\ l_2,\ l_3,\ t)\ \text{and} \\ &x',\ y',\ \text{and}\ z' = f\,(c'_1,\ c'_2,\ c'_3,\ l'_1,\ l'_2,\ l'_3,\ t) \end{aligned}\right. \qquad (22)$$

Next, to take account of Ω_1, we substitute in Ω_1, for $x,\ y,\ z,\ x',\ y',\ z'$, their expressions (22). Then, using canonical elements, the form (21) will become

$$\left.\begin{aligned} &\frac{d\,c_i}{d\,t} = \frac{d\,\Omega_1}{d\,l_i} \qquad &\frac{d\,l_i}{d\,t} = -\frac{d\,\Omega_1}{d\,c_i} \\ &\frac{d\,c'_i}{d\,t} = \frac{d\,\Omega_1}{d\,l'_i} \qquad &\frac{d\,l'_i}{d\,t} = -\frac{d\,\Omega_1}{d\,c'_i} \end{aligned}\right\} \quad (i = 1,\ 2,\ 3) \qquad (23)$$

These equations are now to be integrated by successive approximations. We have first to integrate the equations for c and l on the supposition that c' and l' are constant. Next, by substituting these values of c and l in the equations for c' and l' and integrating, we obtain approximate expressions for these quantities. If necessary these new values of c' and l' can be employed to reintegrate the first set of equations, and so on.

The first integration has been performed by DELAUNAY in his great work *Théorie du mouvement de la Lune.* We may consider that, in this work, DELAUNAY has implicitly formed expressions c and l which substituted in (22) give values of $x,\ y$, and z that completely satisfy the equations

$$\mu_2 \frac{d^2\,x}{d\,t^2} = \frac{d\,(\Omega_0 + \Omega_1)}{d\,x}\ \text{etc.} \qquad (24)$$

on the supposition that c'_i and l'_i are treated as constants. More exactly, if we put, in (14)

$$\left.\begin{aligned} \Omega'_0 &= \frac{m_2\,m_3}{r} \\ \Omega''_0 &= \frac{m_1\,(m_2 + m_3)}{r'} \end{aligned}\right. \qquad (25)$$

DELAUNAY'S values of the coordinates, x, y, and z, of the Moon satisfy the differential equations

$$\mu_2 \frac{d^2 x}{d t^2} = \frac{d(\Omega'_0 + \Omega_1)}{d x}$$

with similar equations in y and z, provided that we regard Ω'_0 and Ω_1 as functions of x, y, z, x', y', z', before differentiating as to x, and afterwards substitute for x, y, z, their expressions in terms of c_i and l_i.

Since Ω''_0, when regarded as a function of the six coordinates, does not contain x, y, or z, we may add it to $\Omega'_0 + \Omega_1$ in the equations without altering the result. We thus find that the equations

$$\mu_2 \frac{d^2 x}{d t^2} = \frac{d\Omega}{d x}$$

will be identically satisfied with regard to all twelve constants in DELAUNAY'S theory on the supposition above made.

It must be noted that c and l in these equations have not the same signification as in (22), but are now the arbitrary constants introduced by the integration of (23).

To complete the solution of the problem of three bodies we have next to integrate the second set of equations (23). We therefore substitute in Ω_1 the values of x, y, z, x', y', z' as functions of all twelve elements c, c', l, l'. But it is to be noted that at this stage we are to differentiate Ω_1 with respect to c' and l' only so far as these quantities are introduced through x', y', and z', because, from the mode of forming the equations (23) we have

$$\frac{d\Omega_1}{dc'} = \frac{d\Omega_1}{dx'}\frac{dx'}{dc'} + \frac{d\Omega_1}{dy'}\frac{dy'}{dc'} + \frac{d\Omega_1}{dz'}\frac{dz'}{dc'} \tag{26}$$

c' being any accented element.

Since Ω_1 contains the lunar elements, c and l, the integration of the equations

$$\frac{dc'_i}{dt} = \frac{d\Omega_1}{dl'_i} \text{ etc.} \tag{27}$$

will give c'_i and l'_i as functions of c_i, l_i and six new arbitrary constants, which we may again represent by c'_i and l'_i. By substituting these values of c'_i and l'_i in (22) our three equations

$$\mu \frac{d^2 x'}{d t^2} = \frac{d\Omega}{d x'} \text{ etc.}$$

will then be identically satisfied with respect to all twelve constants, provided that in representing the identity we first regard Ω as a function of the coordinates expressed in terms of the twelve constants.

By this process we have for the first time introduced the lunar elements c and l into the expressions for the Sun's coordinates. Hence, in strictness, the equations (23)

will now no longer be satisfied with these new values of x', y', etc. It may be shown, however, that the error thus introduced is too small to affect the solution of the problem which now concerns us.

The result of integrating the equations in question may be expressed in the following way: By completing the solution of the problem of three bodies in the manner here proposed we shall implicitly obtain values of the six coordinates x, y, z, x' y', and z' as functions of the quantities

$$c_1, c_2, c_3;\ l_1, l_2, l_3;\ c'_1, c'_2, c'_3;\ l'_1, l'_2, l'_3;\ t \qquad (a)$$

which completely satisfy the differential equations

$$\mu_2 \frac{d^2 x}{d t^2} = \frac{d \Omega}{d x}$$

$$\vdots \qquad \vdots$$

$$\mu_1 \frac{d^2 x'}{d t^2} = \frac{d \Omega}{d x'}$$

$$\text{etc.} \qquad \text{etc.}$$

the two members becoming identical if, after differentiating Ω, we substitute for the coordinates their expressions in terms of the elements (a).

Regarding Ω as a function of the constants c, l, c', l', and of the time, we may indicate the satisfaction of the equations by expressing them in the form

$$\mu_2 \frac{d^2 x}{d t^2} = \Sigma \left\{ \frac{d \Omega}{d c_i} \frac{d c_i}{d x} + \frac{d \Omega}{d l_i} \frac{d l_i}{d x} + \frac{d \Omega}{d c'_i} \frac{d c'_i}{d x} + \frac{d \Omega}{d l'_i} \frac{d l'_i}{d x} \right\} \qquad (28)$$

with the five additional equations formed by writing successively y, z, x', y', z' in place of x in this equation, the seventy-two partial derivatives of c_i, l_i, etc., with respect to x, x', etc., being determined by the seventy-two equations each of one of the forms

$$\sum_{i=1}^{i=3} \left\{ \frac{d x}{d c_i} \frac{d c_i}{d x} + \frac{d x}{d l_i} \frac{d l_i}{d x} + \frac{d x}{d c'_i} \frac{d c'_i}{d x} + \frac{d x}{d l'_i} \frac{d l'_i}{d x} \right\} = 1$$

$$(29)$$

$$\sum_{i=1}^{i=3} \left\{ \frac{d x_1}{d c_i} \frac{d c_i}{d x} + \frac{d x_1}{d l_i} \frac{d l_i}{d x} + \frac{d x_1}{d c_i} \frac{d c_i}{d x} + \frac{d x_1}{d l'_i} \frac{d l'_i}{d x} \right\} = 0$$

the remaining seventy equations being formed by writing in succession x', x_1, x'_1, y, etc., in place of x or x_1 in the numerators found in these equations, and y, z, x', y', and z' for x in the denominators.

§ 6.

CONSIDERATIONS ON THE FORM AND ORDER OF MAGNITUDE OF THE EXPRESSIONS WHICH ENTER INTO THE PRECEDING THEORY.

The fifteen coefficients $[\alpha_i, \alpha_j]$ which enter into the equations (21) are each divisible into two parts, one of which is formed by differentiating the lunar coordinates x, y, and z, the other by differentiating the solar coordinates x', y', and z'. If we put

$$\begin{aligned}[\alpha_i\, \alpha_j]_1 = &\frac{d\,x}{d\,\alpha_i}\frac{d\,x_1}{d\,\alpha_j}+\frac{d\,y}{d\,\alpha_i}\frac{d\,y_1}{d\,\alpha_j}+\frac{d\,z}{d\,\alpha_i}\frac{d\,z_1}{d\,\alpha_j}\\ &-\frac{d\,x}{d\,\alpha_j}\frac{d\,x_1}{d\,\alpha_i}-\frac{d\,y}{d\,\alpha_j}\frac{d\,y_1}{d\,\alpha_i}-\frac{d\,z}{d\,\alpha_j}\frac{d\,z_1}{d\,\alpha_i}\end{aligned} \tag{30}$$

$$\begin{aligned}[\alpha_i\, \alpha_j]_2 = &\frac{d\,x'}{d\,\alpha_i}\frac{d\,x'_1}{d\,\alpha_j}+\frac{d\,y'}{d\,\alpha_i}\frac{d\,y'_1}{d\,\alpha_j}+\frac{d\,z'}{d\,\alpha_i}\frac{d\,z'_1}{d\,\alpha_j}\\ &-\frac{d\,x'}{d\,\alpha_j}\frac{d\,x'_1}{d\,\alpha_i}-\frac{d\,y'}{d\,\alpha_j}\frac{d\,y'_1}{d\,\alpha_i}-\frac{d\,z'}{d\,\alpha_j}\frac{d\,z'_1}{d\,\alpha_i}\end{aligned} \tag{31}$$

$$\left(\frac{d\,\mathrm{R}}{d\,\alpha_i}\right)_1=\frac{d\,\mathrm{R}}{d\,x}\frac{d\,x}{d\,\alpha_i}+\frac{d\,\mathrm{R}}{d\,y}\frac{d\,y}{d\,\alpha_i}+\frac{d\,\mathrm{R}}{d\,z}\frac{d\,z}{d\,\alpha_i} \tag{32}$$

$$\left(\frac{d\,\mathrm{R}}{d\,\alpha_i}\right)_2=\frac{d\,\mathrm{R}}{d\,x'}\frac{d\,x'}{d\,\alpha_i}+\frac{d\,\mathrm{R}}{d\,y'}\frac{d\,y'}{d\,\alpha_i}+\frac{d\,\mathrm{R}}{d\,z'}\frac{d\,z'}{d\,\alpha_i} \tag{33}$$

we may form by means of the expression (17) the two equations

$$\sum_{j=1}^{j=12}[\alpha_i, \alpha_j]_1\frac{d\,\alpha_j}{d\,t}=\left(\frac{d\,\mathrm{R}}{d\,\alpha_i}\right)_1 \tag{34}$$

$$\sum_{j=1}^{j=12}[\alpha_i, \alpha_j]_2\frac{d\,\alpha_j}{d\,t}=\left(\frac{d\,\mathrm{R}}{d\,\alpha_i}\right)_2 \tag{35}$$

and the sum of these two equations will make up the equation (21).

We may divide the twelve elements α into two classes, which may be designated *lunar* and *solar* elements. The first class comprise those which, when the Sun's disturbing force vanishes, merge into the elements of the Moon's elliptic motion around the Earth; the second, those which, when the mass of the Moon vanishes, merge into the elements of the Earth's elliptic motion around the Sun.

Let us next investigate the order of magnitude of these several coefficients and of $\left(\frac{d\,\mathrm{R}}{d\,\alpha}\right)_1$ and $\left(\frac{d\,\mathrm{R}}{d\,\alpha}\right)_2$. In the expressions (17) for the solar coordinates we are to distinguish two portions, one a purely elliptic part containing only the six elements of

the elliptic motion of the Sun, the other part containing the terms added when we take into account the distance of the Moon as compared with that of the Sun. The first part arises from the integration of the differential equations for x', y', z' when only the part Ω_0 of the potential is considered, and is equal to the mean distance of the Sun, a', multiplied by purely numerical ratios.

In the expressions for x, y, and z, all the terms containing solar elements will be multiplied by $\frac{n'}{n}$.

We shall use the symbols S, S′, etc., to represent in the most general way the sum of a series of sines and cosines of angles, each multiplied into purely numerical ratios. These symbols will not twice represent the same quantity, and may, in special cases, reduce to purely numerical constants. If we put for brevity

$$m = \frac{n'}{n}$$
$$k = \mu \frac{a^2}{a'^2} \tag{36}$$

the expressions of the form (17) and (18) for x, x_1, etc., and their derivatives may be put in the form

$$\begin{aligned} x &= a\ (\mathrm{S} + m\,\mathrm{S}') \\ x' &= a'\ (\mathrm{S}' + k\,\mathrm{S}) \\ x_1 &= \mu_2\, a\, n\, (\mathrm{S} + m\,\mathrm{S}') \\ x'_1 &= \mu_1\, a'\, n'\ (\mathrm{S}' + k\,\mathrm{S}) \\ &\text{etc.} \qquad \text{etc.} \end{aligned} \tag{37}$$

If, in forming the derivatives of these expressions as to the elements α_1 . . . a_{12}, we choose for the latter purely numerical quantities, which we may always do by replacing a line element by its logarithm, then the derivatives in question will be of the same form and the same order of magnitude with x, x', etc., themselves, or more exactly with the terms of those quantities which contain the elements. If, then, as before we distinguish the solar elements by accents, the order of magnitude of the largest terms of several derivatives will be as follow:

$$\begin{aligned} \frac{dx}{d\alpha} &= a\,\mathrm{S} \\ \frac{dx_1}{d\alpha} &= \mu_2\, a\, n\,\mathrm{S} \\ \frac{dx'}{d\alpha} &= k\, a'\,\mathrm{S} \\ \frac{dx'_1}{d\alpha} &= \mu_1\, k\, a'\, n'\,\mathrm{S} \end{aligned} \tag{38}$$

$$
\left.
\begin{aligned}
\frac{dx}{d\alpha'} &= m\,a\,S' \\
\frac{dx_1}{d\alpha'} &= \mu_2\,m\,a\,n\,S' \\
\frac{dx'}{d\alpha'} &= a'\,S' \\
\frac{dx'_1}{d\alpha'} &= \mu_1\,a'\,n'\,S'
\end{aligned}
\right\} \tag{38}
$$

The various classes of coefficients $[a_i, a_j]$ formed by (30) and (31) will therefore be of the following orders of magnitude:

$$
\begin{aligned}
[\alpha,\ \alpha_i\,]_1 &= \mu_2\,a^2\,n\,S \\
[\alpha',\ \alpha_i]_1 &= \mu_2\,m\,a^2\,n\,S \\
[\alpha,\ \alpha'_i]_1 &= \mu_2\,m\,a^2\,n\,S \\
[\alpha',\ \alpha'_i]_1 &= \mu_2\,m^2\,a^2\,n\,S \\
[\alpha,\ \alpha_i\,]_2 &= \mu_1 k^2\,a'^2\,n'\,S \\
[\alpha,\ \alpha'_i]_2 &= \mu_1\,k\,a'^2\,n'\,S \\
[\alpha',\ \alpha_i\,]_2 &= \mu_1\,k\,a'^2\,n'\,S \\
[\alpha',\ \alpha'_i]_2 &= \mu_1\,a'^2\,n'\,S
\end{aligned} \tag{39}
$$

Let us next revert to the perturbative function R as found in (14). We see that it is composed of two parts of very different magnitude. Considering all linear heliocentric coordinates as of the same order of magnitude, a', and the coordinates of the Moon as of the order of a, the order of magnitude of the two terms of R will be

$$R = \frac{m_4\,\mu_1}{a'}\,S + \frac{m_4\,\mu_2\,a^2}{a'^3}\,S$$

or, if we use coordinates instead of mean distances, the orders of magnitude will be represented by

$$R = \frac{m_4\,\mu_1}{x'}\,S + \frac{m_4\,\mu_2\,x^2}{x'^3}\,S$$

where the first term does not contain the Moon's coordinates.

From this equation we shall have, using the sign $=$ only to indicate identity of form and order of magnitude,

$$\frac{dR}{dx} = \frac{m_4\,\mu_2\,x}{x'^3}\,S = \frac{m_4\,\mu_2\,a}{a'^3}\,S$$

$$\frac{dR}{dx'} = \frac{m_4\,\mu_1}{x'^2}\,S + \frac{m_4\,\mu_2\,x^2}{x'^4}\,S = \frac{m_4\,\mu_1}{a'^2}\,S + \frac{m_4\,\mu_2\,a^2}{a'^4}\,S$$

and thence, from (32) and (33), having reference to the orders of magnitude of $\frac{dx}{d\alpha}$, etc.,

$$\left(\frac{d\,R}{d\,\alpha}\right)_1 = \frac{m_4\,\mu_2\,a^2}{a'^3}\,S = \frac{m_4\,\mu_1\,k}{a'}\,S$$

$$\left(\frac{d\,R}{d\,\alpha}\right)_2 = \frac{m_4\,\mu_1\,k}{a'}\,S$$

$$\left(\frac{d\,R}{d\,\alpha'}\right)_1 = \frac{m_4\,\mu_2\,m\,a^2}{a'^3}\,S = \frac{m_4\,\mu_1\,m\,k}{a'}\,S \qquad (40)$$

$$\left(\frac{d\,R}{d\,\alpha'}\right)_2 = \frac{m_4\,\mu_1}{a'}\,S$$

The approximate values of μ_1 and μ_2 are

$$\mu_1 = m_2$$

$$\mu_2 = m_3$$

by which we shall replace them in R. The relations between the mean motions and mean distances give, approximately,

$$a^2\,n = \frac{m_2}{a\,n}$$

$$a'^2\,n' = \frac{m_1}{a'\,n'}$$

Substituting these values in the expressions (39) for the coefficients $[\alpha_1, \alpha_i]$, and representing by S_1, S_2, etc., U_1, U_2, etc., numerical quantities like those we have hitherto represented by the general symbol S, we have

$$[\alpha,\ \alpha_i\,]_1 = \frac{m_2\,m_3}{a\,n}\,U_i^{(4)}$$

$$[\alpha,\ \alpha'_i]_1 = \frac{m_2\,m_3}{a\,n}\,m\,U_i^{(3)} \qquad (i = 1, 2\ \ .\ \ .\ \ .\ \ 6)$$

$$[\alpha',\ \alpha_i\,]_1 = \frac{m_2\,m_3}{a\,n}\,m\,U_i^{(2)}$$

$$[\alpha',\ \alpha'_i]_1 = \frac{m_2\,m_3}{a\,n}\,m^2\,U_i^{(1)} \qquad (41)$$

$$[\alpha,\ \alpha_i\,]_2 = \frac{m_1\,m_2}{a'\,n'}\,k^2\,S_i^{(4)}$$

$$[\alpha,\ \alpha'_i]_2 = \frac{m_1\,m_2}{a'\,n'}\,k\,S_i^{(3)}$$

$$[\alpha',\ \alpha_i\,]_2 = \frac{m_1\,m_2}{a'\,n'}\,k\,S_i^{(2)}$$

$$[\alpha',\ \alpha'_i]_2 = \frac{m_1\,m_2}{a'\,n'}\,S_i^{(1)}$$

The expressions thus formed are to be substituted in the pair of general equations (34) and (35) which, it must be remembered, may belong to either of two classes, according as α_j there represents a solar (accented) element, or a lunar (unaccented) one. Let us consider the former case first. The right-hand members of these equations, taken from (40), may then be put in the form

$$\left(\frac{d\,\mathrm{R}}{d\,\alpha'}\right)_1 = m\,\frac{m_2\,m_4}{a'}\,k\,\mathrm{U}^{(5)} = \frac{m_3\,m_4}{a}\,m\,\frac{a^3}{a'^3}\,\mathrm{U}^{(5)}$$

$$\left(\frac{d\,\mathrm{R}}{d\,\alpha'}\right)_2 = \frac{m_2\,m_4}{a'}\left(\mathrm{S}^{(5)} + k\,\mathrm{S}^{(6)}\right)$$

The pair of equations then become

$$\left.\begin{aligned}
&\frac{m_1\,m_2}{a'\,n'}\left\{\mathrm{S}_1^{(1)}\frac{d\,\alpha'_1}{d\,t} + \mathrm{S}_2^{(1)}\frac{d\,\alpha'_2}{d\,t} + \;.\;\;.\; + \mathrm{S}_6^{(1)}\frac{d\,\alpha'_6}{d\,t} + k\left(\mathrm{S}_1^{(2)}\frac{d\,\alpha_1}{d\,t} + \;.\;\;.\; + \mathrm{S}_6^{(2)}\frac{d\,\alpha_6}{d\,t}\right)\right\}\\
&\qquad\qquad = \frac{m_2\,m_4}{a'}\left(\mathrm{S}^{(5)} + k\,\mathrm{S}^{(6)}\right)\\
&\frac{m_2\,m_3}{a\,n}\left\{m^2\left(\mathrm{U}_1^{(1)}\frac{d\,\alpha'_1}{d\,t} + \;.\;\;.\; + \mathrm{U}_6^{(1)}\frac{d\,\alpha'_6}{d\,t}\right) + m\left(\mathrm{U}_1^{(2)}\frac{d\,\alpha_1}{d\,t} + \;.\;\;.\; + \mathrm{U}_1^{(2)}\frac{d\,\alpha_6}{d\,t}\right)\right\}\\
&\qquad\qquad = \frac{m_3\,m_4}{a}\,m\,\frac{a^3}{a'^3}\,\mathrm{U}^{(5)}
\end{aligned}\right\} \quad (42)$$

The sum of these two equations gives one of the rigorous **La Grangian** equations for the variations of the elements, in which each coefficient of the differential of an element is a constant. The coefficient of $\frac{d\,\alpha'_i}{d\,t}$ in the sum is

$$\frac{m_1\,m_2}{a'\,n'}\,\mathrm{S}_i^{(1)} + \frac{m_2\,m_3}{a\,n}\,m^2\,\mathrm{U}_i^{(1)} = [\alpha', \alpha'_i] = \text{constant.}$$

Here the coefficient of S_i is greater than that of U_i in the ratio $\frac{m_1}{m_3} \div \frac{a\,n^3}{a'\,n'^3}$, or about a hundred millions to unity. The constant terms of $\mathrm{S}_i^{(1)}$ must therefore be to the periodic terms in at least this ratio. The coefficient of $\mathrm{U}_i^{(1)}$ may therefore be entirely neglected along side that of $\mathrm{S}_i^{(1)}$.

Again, the approximate value of the coefficient k is

$$k = \frac{m_3}{m_2}\,\frac{a^2}{a'^2} = \frac{1}{12{,}000{,}000}$$

the terms multiplied by this quantity may therefore be entirely neglected. The variations of the solar elements may therefore be determined entirely from the first of the pair of equations by a system of six equations of the form

$$S_1^{(1)}\frac{d\,\alpha'_1}{d\,t}+S_2^{(1)}\frac{d\,\alpha'_2}{d\,t}+\quad\ldots\quad+S_6^{(1)}\frac{d\,\alpha'_6}{d\,t}=\frac{m_4}{m_1}n'\,S^{(5)} \tag{43}$$

which will, in fact, be the usual equations obtained on the supposition that the masses of the Earth and Sun are combined in their center of gravity. This hypothesis is known to suffice even for the action of the Sun, *a fortiori* must it suffice for the disturbing action of a planet.

Let us next consider that in the pair of equations (34) and (35), α_j represents a lunar element. If we then express the first two equations (40) in the form

$$\begin{aligned}\left(\frac{d\,R}{d\,\alpha}\right)_1&=\frac{m_2\,m_4}{a'}k\,U^{(7)}\\ \left(\frac{d\,R}{d\,\alpha}\right)_2&=\frac{m_2\,m_4}{a'}k\,S^{(7)}\end{aligned} \tag{44}$$

if, also, we substitute for the coefficients their values in (41) the pair of equations (34) and (35) assume the form

$$\frac{m_2\,m_3}{a\,n}\left(U_1^{(4)}\frac{d\,\alpha_1}{d\,t}+U_2^{(4)}\frac{d\,\alpha_2}{d\,t}+\quad\ldots\right)+\frac{m_2\,m_3}{a\,n}m\left(U_1^{(3)}\frac{d\,\alpha'_1}{d\,t}+U_2^{(3)}\frac{d\,\alpha'_2}{d\,t}+\quad\ldots\right)$$

$$=\frac{m_2\,m_4}{a'}k\,U^{(7)}$$

$$\frac{m_1\,m_2}{a'\,n'}k^2\left(S_1^{(4)}\frac{d\,\alpha_1}{d\,t}+S_2^{(4)}\frac{d\,\alpha_2}{d\,t}+\quad\ldots\right)+\frac{m_1\,m_2}{a'\,n'}k\left(S_1^{(3)}\frac{d\,\alpha'_1}{d\,t}+S_2^{(3)}\frac{d\,\alpha'_2}{d\,t}+\quad\ldots\right)$$

$$=\frac{m_2\,m_4}{a'}k\,S^{(7)}$$

The sum of these two equations forms the final equation for the variations of the lunar elements after the variations of the solar elements are determined from (43). If we multiply their terms by $a\,n\div m_2\,m_3$, putting for k its value

$$k=\frac{m_3}{m_2+m_3}\frac{a^2}{a'^2}$$

using the relations

$$a^3\;n^2\;=m_2+m_3$$

$$a'^3\,n'^2=m_1+m_2+m_3$$

and regarding as unity the ratio

$$m_1\div m_1+m_2+m_3$$

the equations become

$$U_1^{(4)} \frac{d\alpha_1}{dt} + U_2^{(4)} \frac{d\alpha_2}{dt} + \quad . \quad . \quad . + m\, U_1^{(3)} \frac{d\alpha'_1}{dt} + m\, U_2^{(3)} \frac{d\alpha'_2}{dt} + \quad . \quad . \quad . \quad = \frac{m_4 n}{m_2 + m_3} \frac{a^3}{a'^3} U^{(7)}$$

$$m\, k\, (S_1^{(4)} \frac{d\alpha_1}{dt} + S_2^{(4)} \frac{d\alpha_2}{dt} + \quad . \quad . \quad) + m\, S_1^{(3)} \frac{d\alpha'_1}{dt} + m\, S_2^{(3)} \frac{d\alpha'_2}{dt} + \quad . \quad . \quad = \frac{m_4 n}{m_2 + m_3} \frac{a^3}{a'^3} S^{(7)} \qquad (45)$$

The sum of these two equations will give one of the LA GRANGIAN equations for the variations of the elements, in which the coefficients

$$U_i^{(4)} + m\, k\, S_{(i)}^4,$$

and

$$m\, (U_i^{(3)} + S_i^{(3)})$$

are functions of the elements alone, and do not contain the time explicitly.

In these sums, however, the coefficient of $S^{(4)}$ in the second equation is less than that of U^4 in the first in the ratio of $m\, k$ to 1, and since the periodic terms in the two coefficients destroy each other, the quantities $U_1^{(4)}$, $U_2^{(4)}$, etc., must be constant to their 150,000,000 part, and may therefore be regarded as absolute constants. In fact these coefficients $U_i^{(4)}$ are the same as those which determine the variations of the lunar elements in what I have called the first method, and which is that employed by HILL.

Thus the final equations for the variations of the lunar elements may be written in the form

$$\begin{aligned} U_1^{(4)} \frac{d\alpha_1}{dt} + U_2^{(4)} \frac{d\alpha_2}{dt} + \quad . \quad . \quad . \quad + U_4^{(6)} \frac{d\alpha_6}{dt} &= \frac{m_4 n}{m_2 + m_3} \frac{a^3}{a'^3} (U^{(7)} + S^{(7)}) \\ -(U_1^{(3)} + S_1^{(3)})\, m \frac{d\alpha'_1}{dt} &- (U_2^{(3)} + S_2^{(3)})\, m \frac{d\alpha'_2}{dt} - \text{etc.} \end{aligned} \qquad (46)$$

those of the solar elements being first derived from (43).

§ 7.

METHOD OF EXPRESSING THE COORDINATES IN TERMS OF THE ELEMENTS.

Our next step is to investigate the formation of the coefficients $[\alpha_i, \alpha_j]$ in (20). This presupposes that the rectangular coordinates x, y, z, x', y', z' are expressed in terms of the twelve arbitrary constants $\alpha_1 \quad . \quad . \quad . \quad \alpha_{12}$ and of the time, in such a form as to be a complete solution of the differential equations (16). This requires that the coordinate planes and lines of reference be arbitrary, but no loss of astronomical rigor will be incurred if we suppose the fundamental plane to differ from that of the ecliptic at some epoch only by a quantity which may be treated as an infinitesimal.

The theory of DELAUNAY gives the Moon's longitude and latitude referred to the plane of the ecliptic and expressed in terms of the arbitrary constants

$$a,\ e,\ \gamma,\ a',\ e',\ h,\ g,\ l,\ h',\ g',\ l',$$

of which, however, one is superfluous.

The notation which I shall use is, however, different from that of DELAUNAY, the symbols given at the end of each line being those of DELAUNAY. I put

v;	the Moon's true longitude, from an arbitrary origin . . .	(V in DELAUNAY).
β;	its latitude above the ecliptic	(U " ").
l;	its mean longitude	($h+g+l$ ").
g;	its mean anomaly	(l ").
θ;	the longitude of its node on the ecliptic	(h ").
γ;	the sine of half of the inclination of the Moon's orbit to the ecliptic	(γ ").

The corresponding accented quantities refer to the coordinates x', y', z' of the Sun.

DELAUNAY'S work now enables us to express the spherical coordinates of the Moon in the form

$$
\begin{aligned}
v &= l+\Sigma k \sin \mathrm{N}. \\
r &= \Sigma k' \cos \mathrm{N}. \qquad (47) \\
\beta &= \Sigma c \sin \mathrm{N}'.
\end{aligned}
$$

The coefficients k, k' and c are functions of five independent elements

$$a,\ e,\ \gamma,\ a'\ e',$$

while N and N$'$ are linear functions with integral coefficients of four angles,

$$g,\ g',\ \omega,\ \omega'$$

ω and ω' being the mean distances of the lunar and solar perigees from the Moon's ascending node. Thus the coordinates of the Moon are expressed in terms of ten independent quantities, of which, however, one is only a constant additive to l or to v.

It is to be remarked that, so far as the problem of three bodies is concerned, the position of the ecliptic is to be regarded as invariable.

If we represent by ξ, η, ζ, the rectangular coordinates of the Moon referred to the ecliptic, and put, for brevity

$$\delta v = v - l = \Sigma k \sin \mathrm{N}$$

the expressions ξ, η and ζ may be written

$$
\begin{aligned}
\xi &= r \cos \beta \cos v = r \cos \beta \cos \delta v \cos l - r \cos \beta \sin \delta v \sin l \\
\eta &= r \cos \beta \sin v = r \cos \beta \cos \delta v \sin l + r \cos \beta \sin \delta v \cos l \qquad (48) \\
\zeta &= r \sin \beta.
\end{aligned}
$$

Since β and δv are small quantities, the limit of whose values is little more than 0·1, the three quantities $r \cos \beta \cos \delta v$, $r \cos \beta \sin \delta v$ and $r \sin \beta$ each admit of being developed in a convergent series, as follows:

We first substitute for the sines and cosines of β and δv their developments

$$\sin \delta v = \delta v \left(1 - \frac{\delta v^2}{3!} + \frac{\delta v^4}{5!} - \frac{\delta v^6}{7!} + \text{etc.} \quad . \quad . \quad . \right)$$

$$\cos \delta v = 1 - \frac{\delta v^2}{2!} + \frac{\delta v^4}{4!} - \frac{\delta v^6}{6!} + \text{etc.}$$

$$\sin \beta = \beta \left(1 - \frac{\beta^2}{3!} + \frac{\beta^4}{5!} - \text{etc.} \quad . \quad . \quad . \right)$$

$$\cos \beta = 1 - \frac{\beta^2}{2!} + \frac{\beta^4}{4!} - \text{etc.} \quad . \quad . \quad . \quad .$$

Forming the powers of δv and β and substituting for the powers and products of the sines and cosines their expressions in terms of the sines and cosines of the multiple angles, we see that the three quantities in question can be represented in the same general form with δv, r, and β, that is, in the form

$$r \cos \beta \cos \delta v = \Sigma h \cos \text{N}$$

$$r \cos \beta \sin \delta v = \Sigma h' \sin \text{N}$$

$$r \sin \beta = \Sigma c' \sin \text{N}'$$

h, h' and N being of the same form as k and N in the values of v, β and r in (47). Thus the values of ξ, η, and ζ in (48) assume the form

$$\begin{aligned} \xi &= \Sigma k \cos (l + \text{N}) \\ \eta &= \Sigma k \sin (l + \text{N}) \\ \zeta &= \Sigma c \sin \text{N}' \end{aligned} \tag{49}$$

where

$$k = h \pm h'$$

and N is still a linear function of the four quantities

$$g, \; g', \; \omega, \; \omega'.$$

The corresponding coordinates of the Sun may be expressed in the same general form, which we may write

$$\begin{aligned} \xi' &= \Sigma k' \cos (l' + \text{N}) \\ \eta' &= \Sigma k' \sin (l' + \text{N}) \\ \zeta' &= \Sigma c' \sin \text{N}' \end{aligned} \tag{50}$$

It is however to be remarked that, in the case of the Sun, the principal values of k' will correspond to cases in which N depends on g' alone, and that when g, ω or ω' enter into N or N′, the values of k' and c' will contain the minute factor k of § 6.

Let us now suppose the positions of the Sun and Moon referred to any system of rectangular coordinates whatever. If we represent by

$$a, b, c; a', b', c'; a'', b'', c'',$$

the nine direction cosines for transforming ξ, η, ζ into x, y, z, and if we put

θ', the ascending node of the ecliptic on the plane of XY counted from the axis of X,

γ', the sine of half the inclination of these planes,

τ, the distance of the same node from the axis of Ξ,

we shall have

$$
\begin{aligned}
a &= (1-\gamma'^2)\cos(\theta'-\tau)+\gamma'^2\cos(\theta'+\tau) \\
b &= -(1-\gamma'^2)\sin(\theta'-\tau)+\gamma'^2\sin(\theta'+\tau) \\
c &= 2\gamma'\sqrt{1-\gamma'^2}\sin\theta' \\
a' &= (1-\gamma'^2)\sin(\theta'-\tau)+\gamma'^2\sin(\theta'+\tau) \\
b' &= (1-\gamma'^2)\cos(\theta'-\tau)-\gamma'^2\cos(\theta'+\tau) \\
c' &= -2\gamma'\sqrt{1-\gamma'^2}\cos\theta' \\
a'' &= -2\gamma'\sqrt{1-\gamma'^2}\sin\tau \\
b'' &= 2\gamma'\sqrt{1-\gamma'^2}\cos\tau \\
c'' &= 1-2\gamma'^2
\end{aligned}
\qquad (51)
$$

By means of the values of ξ, η, and ζ given in (49), and the above values of a, b, etc., we find for the general rectangular coordinates

$$
\begin{aligned}
x = {} & \Sigma k(1-\gamma'^2)\cos(\theta'-\tau+l+\mathrm{N}) \\
& +\Sigma k\gamma'^2\cos(\theta'+\tau-l-\mathrm{N}) \\
& -\Sigma c\gamma'\sqrt{1-\gamma'^2}\cos(\theta'+\mathrm{N}') \\
& +\Sigma c\gamma'\sqrt{1-\gamma'^2}\cos(\theta'-\mathrm{N}') \\
y = {} & \Sigma k(1-\gamma'^2)\sin(\theta'-\tau+l+\mathrm{N}) \\
& +\Sigma k\gamma'^2\sin(\theta'+\tau-l-\mathrm{N}) \\
& -\Sigma c\gamma'\sqrt{1-\gamma'^2}\sin(\theta'+\mathrm{N}') \\
& +\Sigma c\gamma'\sqrt{1-\gamma'^2}\sin(\theta'-\mathrm{N}')
\end{aligned}
\qquad (52)
$$

$$
\left.\begin{aligned}
z &= 2\Sigma k\gamma' \sqrt{1-\gamma'^2}\sin(l-\tau+\mathrm{N}) \\
&\quad +\Sigma(1-2\gamma'^2)\, c\sin\mathrm{N}' \\
x' &= \Sigma k'(1-\gamma'^2)\cos(\theta'-\tau+l'+\mathrm{N}) \\
&\quad +\Sigma k'\gamma'^2\cos(\theta'+\tau-l'-\mathrm{N}) \\
&\quad -\Sigma c'\gamma'\sqrt{1-\gamma'^2}\cos(\theta'+\mathrm{N}') \\
&\quad +\Sigma c'\gamma'\sqrt{1-\gamma'^2}\cos(\theta'-\mathrm{N}') \\
y' &= \Sigma k'(1-\gamma'^2)\sin(\theta'-\tau+l'+\mathrm{N}) \\
&\quad +\Sigma k'\gamma'^2\sin(\theta'+\tau-l'-\mathrm{N}) \\
&\quad -\Sigma c'\gamma'\sqrt{1-\gamma'^2}\sin(\theta'+\mathrm{N}') \\
&\quad +\Sigma c'\gamma'\sqrt{1-\gamma'^2}\sin(\theta'-\mathrm{N}') \\
z' &= 2\Sigma k'\gamma'\sqrt{1-\gamma'^2}\sin(l'-\tau+\mathrm{N}) \\
&\quad +\Sigma(1-2\gamma'^2)\, c'\sin\mathrm{N}'
\end{aligned}\right\} \qquad (52)
$$

Examining these equations, we see that the coefficients of the varying angles are now functions of the six constants

$$a,\ e,\ \gamma,\ a',\ e',\ \gamma'$$

and that the angles under the signs *sin* and *cos* may be expressed in terms of the time and six other arbitrary constants. For these angles contain only θ', $\tau-l$, N, and N$'$. The two latter can be expressed as a function of the four quantities already mentioned, $\tau-l$ need contain but one constant, and thus, with θ', we have the required number of six.

To reduce the preceding values of x, y, and z to a common form exhibiting these six arbitrary constants, let us represent the four varying angles of which N and N$'$ are functions by λ, λ', ϖ, ϖ'. These quantities are not to be considered as having a perfectly defined geometrical signification, being only certain analytical expressions which substituted in the equations (47) will give numerical values of the Moon's coordinates. But, we may consider that they represent

λ, the mean distance of the Moon from her node on the ecliptic.

ϖ, the distance of the perigee from the same node.

λ', ϖ', the same quantities for the Sun, counted from the same node.

We shall now express all the quantities found under the sign *sin* or *cos* in x, y, and z in terms of the six arbitrary quantities

$$\varepsilon,\ \pi,\ \theta,\ \varepsilon',\ \pi',\ \theta'$$

the first five of which are determined by the conditions

$$\varepsilon = \theta' + l - \tau$$
$$\pi = \theta + \omega$$
$$\theta = \varepsilon - \lambda$$
$$\varepsilon' = \lambda' - \lambda + \varepsilon, = \theta' + l' - \tau$$
$$\pi' = \theta + \omega'$$

They are therefore introduced by putting in the expressions for x, y, and z,

$$\begin{aligned} l - \tau &= \varepsilon - \theta' \\ l' - \tau &= \varepsilon' - \theta' \\ \lambda &= \varepsilon - \theta \\ \lambda' &= \varepsilon' - \theta \\ \omega &= \pi - \theta \\ \omega' &= \pi' - \theta \end{aligned} \tag{53}$$

The expressions for x, y, and z thus take the form

$$\begin{aligned} x &= \Sigma k \cos (i\,\varepsilon + i^{\text{I}}\pi + i^{\text{II}}\theta + i^{\text{III}}\varepsilon' + i^{\text{IV}}\pi' + i^{\text{V}}\theta') \\ y &= \Sigma k \sin (i\,\varepsilon + i^{\text{I}}\pi + i^{\text{II}}\theta + i^{\text{III}}\varepsilon' + i^{\text{IV}}\pi' + i^{\text{V}}\theta') \\ z &= \Sigma c \sin (j\,\varepsilon + j^{\text{I}}\pi + j^{\text{II}}\theta + j^{\text{III}}\varepsilon' + j^{\text{IV}}\pi' + j^{\text{V}}\theta') \end{aligned} \tag{54}$$

the coefficients i and j being positive or negative integers satisfying the conditions

$$i + i^{\text{I}} + i^{\text{II}} + i^{\text{III}} + i^{\text{IV}} + i^{\text{V}} = 1$$
$$j + j^{\text{I}} + j^{\text{II}} + j^{\text{III}} + j^{\text{IV}} + j^{\text{V}} = 0$$

and k and c being functions of the six quantities a, a', e, e', γ, γ'.

The values of x', y', and z' in (52) can be reduced to the same form, so that it is unnecessary to write them.

§ 8.

METHOD OF FORMING THE LA GRANGIAN COEFFICIENTS.

It will be seen by what precedes that our problem requires as a preliminary work the complete solution of the problem of three bodies by obtaining not only the relative coordinates of the Moon and the Earth, but the coordinates of the Sun relative to the center of gravity of the Earth and Moon, which has not yet been done. Instead of effecting the solution of this preliminary problem at the present stage, I shall proceed on the supposition that this part of the work is done, and proceed with the investigation growing out of it, in order that more exact information may thus be obtained as to the degree of approximation to which the solution must be carried.

It is shown in my paper of 1875, on the General Integrals of Planetary Motion,* that when the solution in question is carried on by successive approximations, the expressions for such coordinates as x, y, z, x', y', z', in terms of the twelve arbitrary constants which were required in the solution of the problem of three bodies, could always be reduced to a general form of the kind found in the last section, namely

$$x \text{ or } x' = \Sigma\, k \cos \mathrm{N}$$
$$y \text{ or } y' = \Sigma\, k \sin \mathrm{N}$$
$$z \text{ or } z' = \Sigma\, c \sin \mathrm{N}'$$

in which each value of N and of N′ is of the form

$$\mathrm{N} = i_1\, \lambda_1 + i_2\, \lambda_2 + \quad \ldots \quad + i_6\, \lambda_6$$
$$\mathrm{N}' = j_1\, \lambda_1 + j_2\, \lambda_2 + \quad \ldots \quad + j_6\, \lambda_6$$

i_1, i_2, ... i_6, j_1 ... j_6, being integers satisfying the conditions

$$i_1 + i_2 + \quad \ldots \quad + i_6 = 0$$
$$j_1 + j_2 + \quad \ldots \quad + j_3 = 1$$

and λ_1 ... λ_6 expressions of the form

$$\lambda_i = l_i + b_i\, t$$

l_1, l_2, ... l_6 being arbitrary constants, and b_1, b_2, ... b_6 as well as the various values of k and of c, functions of six other arbitrary constants, which, if used as in the theory of DELAUNAY, may be called

$$a,\ e,\ \gamma,\ a',\ e',\ \gamma'$$

The symbol Σ indicates a summation extended to all combinations of integral values of the indices i_1, i_2, etc.

As special values of the six constants l we may take those which in the ordinary theory are designated as follows:

θ, the longitude of the Moon's node, or DELAUNAY's h;

π, the longitude of the Moon's perigee, or DELAUNAY's $h+g$;

l, the Moon's mean longitude, or DELAUNAY's $h+g+l$;

θ', π', l', the corresponding quantities relating to the Sun.

It is to be remarked that the assignment of the usual astronomical designation to these quantities does not adequately define them, and may therefore be misleading. No precise definition is possible except by their analytic expressions as integrals, each of them being regarded as a function of the coordinates, their first derivatives, and the time. This definition might, however, be considered as replaced by any complete solution expressing the values of the coordinates in terms of the arbitrary constants and of the time. We shall therefore simply suppose that expressions of the preceding

*Smithsonian Contributions to Knowledge, Vol. XXI.

form for the rectangular coordinates are obtained in terms of two sets of six arbitrary constants each, which satisfy the differential equations for the case of three bodies to the necessary degree of approximation.

The danger of error in assigning geometrical significations to the elements will be seen by considering the special quantity which DELAUNAY has called the longitude of the Moon's node. His entire lunar theory is constructed on the supposition that the plane of the ecliptic is invariable, an hypothesis which is sufficiently correct when the action of the planets is omitted, so far, at least, as any secular variation is concerned. But when we consider the complete problem of three bodies we shall find that there are two planes, either of which may be called that of the ecliptic. One of these is the plane of maximum areal velocity, which is really invariable. The other is the plane determined at each instant by the relative motion of the Sun and of the center of gravity of the Earth and Moon, which plane is variable.

Since all general equations between physical quantities are necessarily homogeneous when each quantity is expressed in units of fundamental or arbitrary elements, it will generally conduce to clearness to write them in such a way that this homogeneousness shall always be manifest. In physics the three fundamental elements whose units are arbitrary are mass, length, and time. But in astronomy it will generally be simpler, as it is more common, to regard the unit of mass as a derived one, and determined by the condition that two bodies of unit mass shall attract each other with unit force when at unit distance. If we represent by the letters M, L, and T the dimensions mass, length, and time respectively, the equation for M to which we are thus led is

$$\mathrm{M} = \mathrm{L}^3\, \mathrm{T}^{-2}$$

In a theory like the present it will conduce to clearness to write all equations in such a way that the homogenity shall be apparent. Since N and N′ are angles, and therefore pure numbers, it follows that λ, l, and $b\,t$ are also pure numbers. Hence b is of dimensions T^{-1}, as are also n and n', the mean motions. Since the coordinates are lines all the coefficients k are of the form

$$a \text{ or } a' \times \text{a pure number}$$

while $b_1, b_2 \quad . \quad . \quad . \quad b_6$ are each of the form $n\,s$, or $n'\,s$, s being a pure number.

Hence each conjugate variable of the kind

$$x_1 = \mu_2 \frac{d\,x}{d\,t}$$

is of the dimensions $\mathrm{M\,L\,T}^{-1}$ or $\mathrm{L}^4\,\mathrm{T}^{-3}$ according as we do or do not retain M, and may be written in the form

$$m\,a\,n \times \text{a pure number.}$$

The six pairs of conjugate variables from which the LA GRANGIAN coefficients are to be formed may therefore be written in the form

$$\begin{aligned} x &= \Sigma\, k_i \cos \mathrm{N}_i & \qquad x_1 &= -\Sigma\, k'_j \sin \mathrm{N}_j \\ y &= \Sigma\, k_i \sin \mathrm{N}_i & y_1 &= \ \ \Sigma\, k'_j \cos \mathrm{N}_j \\ z &= \Sigma\, c_i \sin \mathrm{N}'_i & z_1 &= \ \ \Sigma\, c'_j \cos \mathrm{N}'_j \end{aligned} \qquad (55)$$

where the indices i and j are used to distinguish the terms belonging to the various linear combinations which make up the values of N. We have, in each case,

$$k' = \mu b k; \quad c' = \mu b' c$$

μ representing either μ_1 or μ_2, according as the coordinates are those of the Moon or of the Sun, and b and b' being the coefficients of t in N and N′, so that in general

$$\begin{aligned} b &= i_1 b_1 + i_2 b_2 + \quad . \quad . \quad . \quad + i_6 b_6 \\ b' &= j_1 b_1 + j_2 b_2 + \quad . \quad . \quad . \quad + j_6 b_6 \end{aligned} \tag{56}$$

Now, let α and β be any two elements for which we are to form the LA GRANGIAN coefficient $[\alpha, \beta]$. Putting no restriction as to the class to which either of these elements may belong, we have from (55) the following values of the derivatives from which $[\alpha_i, \beta]$ is to be formed

$$\frac{dx}{d\alpha} = \Sigma \left(\frac{dk_i}{d\alpha} \cos N_i - k_i \frac{dN_i}{d\alpha} \sin N_i \right)$$

$$\frac{dx_1}{d\beta} = \Sigma \left(-\frac{dk'_j}{d\beta} \sin N_j - k'_j \frac{dN_j}{d\beta} \cos N_j \right)$$

$$\frac{dx}{d\beta} = \Sigma \left(\frac{dk_i}{d\beta} \cos N_i - k_i \frac{dN_i}{d\beta} \sin N_i \right)$$

$$\frac{dx_1}{d\alpha} = \Sigma \left(-\frac{dk'_j}{d\alpha} \sin N_j - k'_j \frac{dN_j}{d\alpha} \cos N_j \right)$$

$$\frac{dy}{d\alpha} = \Sigma \left(\frac{dk_i}{d\alpha} \sin N_i + k_i \frac{dN_i}{d\alpha} \cos N_i \right)$$

$$\frac{dy_1}{d\beta} = \Sigma \left(\frac{dk'_j}{d\beta} \cos N_j - k'_j \frac{dN_j}{d\beta} \sin N_j \right)$$

$$\frac{dy}{d\beta} = \Sigma \left(\frac{dk_i}{d\beta} \sin N_i + k_i \frac{dN_i}{d\beta} \cos N_i \right)$$

$$\frac{dy_1}{d\alpha} = \Sigma \left(\frac{dk'_j}{d\alpha} \cos N_j - k'_j \frac{dN_j}{d\alpha} \sin N_j \right)$$

$$\frac{dz}{d\alpha} = \Sigma \left(\frac{dc_i}{d\alpha} \sin N'_i + c_i \frac{dN'_i}{d\alpha} \cos N'_i \right)$$

$$\frac{dz_1}{d\beta} = \Sigma \left(\frac{dc'_j}{d\beta} \cos N'_j - c'_j \frac{dN'_j}{d\beta} \sin N'_j \right)$$

$$\frac{dz}{d\beta} = \Sigma \left(\frac{dc_i}{d\beta} \sin N'_i + c_i \frac{dN'_i}{d\beta} \cos N'_i \right)$$

$$\frac{dz_1}{d\alpha} = \Sigma \left(\frac{dc'_j}{d\alpha} \cos N'_j - c'_j \frac{dN'_j}{d\alpha} \sin N'_j \right)$$

Substituting these values in the expressions of the form (30), p. 122, and putting for brevity

$$[\varphi, \theta] = \frac{d\varphi}{d\alpha}\frac{d\theta}{d\beta} - \frac{d\varphi}{d\beta}\frac{d\theta}{d\alpha}$$

we find

$$
\begin{aligned}
[\alpha, \beta]_1 = \; & \Sigma^2\{[k_i, k'_j] + k_i k'_j [\mathrm{N}_i, \mathrm{N}_j]\} \sin(\mathrm{N}_i - \mathrm{N}_j) \\
& + \Sigma^2\{k_i [\mathrm{N}_i, k'_j] - k'_j [k_i, \mathrm{N}_j]\} \cos(\mathrm{N}_i - \mathrm{N}_j) \\
& + \frac{1}{2}\Sigma^2\{[c_i, c'_j] + c_i c'_j [\mathrm{N}'_i, \mathrm{N}'_j]\} \sin(\mathrm{N}'_i - \mathrm{N}'_j) \\
& + \frac{1}{2}\Sigma^2\{[c_i, c'_j] - c_i c'_j [\mathrm{N}'_i, \mathrm{N}'_j]\} \sin(\mathrm{N}'_i + \mathrm{N}'_j) \\
& + \frac{1}{2}\Sigma^2\{c_i [\mathrm{N}'_i, c'_j] - c'_j [c_i, \mathrm{N}'_j]\} \cos(\mathrm{N}'_i - \mathrm{N}'_j) \\
& + \frac{1}{2}\Sigma^2\{c_i [\mathrm{N}'_i, c'_j] + c'_j [c_i, \mathrm{N}'_j]\} \cos(\mathrm{N}'_i + \mathrm{N}'_j)
\end{aligned} \tag{57}
$$

It will be understood that the two indices i, j, have each the same system of values, and their combination under the sign Σ^2 indicates that each value of N is to be combined with all the other values, itself included.

In order to find the function $[\alpha, \beta]$ of which we are in quest, it is necessary to omit all the periodic terms in (57). Substituting for N_i and N_j their values $f_i + b_i t$ and $f_j + b_j t$, we see that all the terms will be periodic, unless we have either

$$b_i - b_j = 0$$

or

$$b'_i \pm b'_j = 0$$

The values of b_i and b_j being from (22)

$$
\begin{aligned}
b_i &= i_{1,i} b_1 + i_{2,i} b_2 + \quad \ldots \quad + i_{6,i} b_6 \\
b_j &= i_{1,j} b_1 + i_{2,j} b_2 + \quad \ldots \quad + i_{6,j} b_6 \\
b'_i &= j_{1,i} b_1 + j_{2,i} b_2 + \quad \ldots \quad + j_{6,i} b_6 \\
b'_j &= j_{1,j} b_1 + j_{2,j} b_2 + \quad \ldots \quad + j_{6,j} b_6
\end{aligned}
$$

it is necessary that we have either

$$(i_{1,i} - i_{1,j}) b_1 + (i_{2,i} - i_{2,j}) b_2 + \quad \ldots \quad + (i_{6,i} - i_{6,j}) b_6 = 0$$

or

$$(j_{1,i} \pm j_{1,j}) b_1 + (j_{2,i} \pm j_{2,j}) b_2 + \quad \ldots \quad + (j_{6,i} \pm j_{6,j}) b_6 = 0$$

But $b_1 \quad \ldots \quad b_6$ being incommensurable quantities, these equations can not be satisfied except we have

$$i_{1,i}= \quad i_{i,j} \qquad i_{2,i}= \quad i_{2,j} \quad . \quad . \quad . \quad i_{6,i}= \quad i_{6,j}$$

$$j_{1,i}=\mp j_{2,j} \qquad j_{2,i}=\mp j_{2,j} \quad . \quad . \quad . \quad j_{6,i}=\pm j_{6,j}$$

from which follows

$$f_i=f_j \qquad N_i=\mp N_j$$

Therefore we retain in the equation (57) only those terms which satisfy the condition

$$i=j$$

We have, therefore, omitting the index i,

$$[\alpha,\beta]_1= \Sigma\{k[N,k']-k'[k,N]\}$$
$$+\frac{1}{2}\Sigma\{c[N',c']-c'[c,N']\}$$

or, placing for k' its value $\mu_2 b k$, and $\mu_2 b' c$ for c',

$$[\alpha,\beta]_1= \Sigma\left[\frac{d(\mu_2 b k^2)}{d\beta}\frac{dN}{d\alpha}-\frac{d(\mu_2 b k^2)}{d\alpha}\frac{dN}{d\beta}\right]$$
$$+\frac{1}{2}\Sigma\left[\frac{d(\mu_2 b' c^2)}{d\beta}\frac{dN'}{d\alpha}-\frac{d(\mu_2 b' c^2)}{d\alpha}\frac{dN'}{d\beta}\right]$$

Omitting in $\frac{dN}{d\alpha}$ the terms which contain t as a factor, we then have

$$[\alpha,\beta]_1=\mu_2\,\Sigma\left[\frac{df}{d\alpha}\frac{d(bk^2)}{d\beta}-\frac{df}{d\beta}\frac{d(bk^2)}{d\alpha}\right]$$
$$\frac{1}{2}\mu_2\,\Sigma\left[\frac{df'}{d\alpha}\frac{d(b'c^2)}{d\beta}-\frac{df}{d\beta}\frac{d(b'c^2)}{d\alpha}\right] \qquad (58)$$

It will be remembered that α and β here represent any pair of the twelve arbitrary constants which enter into the expressions (55) for x, y, and z. These constants are divisible into two classes.

Class (1) comprises the elements

$$a,\ e,\ \gamma,\ a',\ e',\ \gamma',$$

which enter into, b, b', k, and c, but not into f.

Class (2) comprises the elements

$$l_1,\ l_2,\ \ .\ \ .\ \ .\ \ .\ \ l_6$$

which enter into f and f', but not into b, b', c, or k.

It is evident from the form of the preceding expression for $[\alpha\ \beta]$ that it will vanish whenever the elements α and β belong to the same class. We have therefore only to consider the combinations in which they belong to different classes, of which the number is 18.

Let us now conceive the general quantities λ_1 . . . . λ_6 to be replaced by the six elements

$$\varepsilon,\ \pi,\ \theta,\ \varepsilon',\ \pi',\ \theta',$$

already defined, and let us put

$$n;\ \pi_1,\ \ .\ \ .\ \ .\ \ .\ \ \theta_1',$$

for the motions of the corresponding elements in unit time. Each value of b or b' may then be written

$$b=i\,n+i^{\mathrm{I}}\,\pi_1+i^{\mathrm{II}}\,\theta_1+i^{\mathrm{III}}\,n'+i^{\mathrm{IV}}\,\pi'_1+i^{\mathrm{V}}\,\theta'_1$$
$$b'=j\,n+j^{\mathrm{I}}\,\pi_1+j^{\mathrm{II}}\,\theta_1+j^{\mathrm{III}}\,n'+j^{\mathrm{IV}}\,\pi'_1+j^{\mathrm{V}}\,\theta'_1$$

We then have

$$\frac{d\,f}{d\,\varepsilon}=\frac{d\,\mathrm{N}}{d\,\varepsilon}=i \qquad \frac{d\,f'}{d\,\varepsilon}=\frac{d\,\mathrm{N}'}{d\,\varepsilon}=j$$
$$\frac{d\,f}{d\,\pi}=\frac{d\,\mathrm{N}}{d\,\pi}=i' \qquad \frac{d\,f'}{d\,\pi}=\frac{d\,\mathrm{N}'}{d\,\pi}=j'$$
$$\cdot\ \cdot \qquad\qquad \cdot\ \cdot$$
$$\cdot\ \cdot \qquad\qquad \cdot\ \cdot$$
$$\frac{d\,f}{d\,\theta'}=\frac{d\,\mathrm{N}}{d\,\theta'}=i^{\mathrm{v}} \qquad \frac{d\,f'}{d\,\theta'}=\frac{d\,\mathrm{N}'}{d\,\theta'}=j^{\mathrm{v}}$$

We have therefore, supposing β to represent in succession the six elements of the second class and restoring the sign Σ,

$$[\alpha,\,\varepsilon]_2=-\mu_2\,\Sigma\,i\,\frac{d\,(b\,k^2)}{d\,\alpha}-\frac{1}{2}\,\mu_2\Sigma\,j\,\frac{d\,(b'\,c^2)}{d\,\alpha}$$
$$[\alpha,\pi]_2=-\mu_2\,\Sigma\,i^{\mathrm{I}}\frac{d\,(b\,k^2)}{d\,\alpha}-\frac{1}{2}\,\mu_2\,\Sigma\,j^{\mathrm{I}}\,\frac{d\,(b'\,c^2)}{d\,\alpha} \tag{59}$$
$$\cdot\ \cdot \qquad \cdot\ \cdot \qquad \cdot$$
$$\cdot\ \cdot \qquad \cdot\ \cdot$$
$$[\alpha,\,\theta]_2=-\mu_2\Sigma\,i^{\mathrm{v}}\,\frac{d\,(b\,k^2)}{d\,\alpha}-\frac{1}{2}\,\mu_2\Sigma\,j^{\mathrm{v}}\,\frac{d\,(b'\,c^2)}{d\,\alpha}$$

The sign Σ here refers to the several terms which enter into the expresssions (54). The entire number of sensible coefficients $[\alpha,\,\beta]_1$ is now reduced to thirty-six.

To complete the required expressions we still require the values of $[\alpha\ \beta]_2$. The rigorous values of x', y', and z' can be developed in precisely the same way with those of x, y, and z. As we have seen all the terms containing a, e, γ, ε, π, or θ will contain the minute factor k in (36), but for the sake of generality we shall suppose them all retained. If we put

$$x'=\Sigma\,k\,\cos\,(\mathrm{I}\,\varepsilon+\mathrm{I}^{\mathrm{I}}\,\pi+\mathrm{I}^{\mathrm{II}}\theta+\mathrm{I}^{\mathrm{III}}\,\varepsilon'+\mathrm{I}^{\mathrm{IV}}\,\pi'+\mathrm{I}^{\mathrm{V}}\,\theta')$$
$$y'=\Sigma\,k\,\sin\,(\mathrm{I}\,\varepsilon+\mathrm{I}^{\mathrm{I}}\,\pi+\mathrm{I}^{\mathrm{II}}\theta+\mathrm{I}^{\mathrm{III}}\,\varepsilon'+\mathrm{I}^{\mathrm{IV}}\,\pi'+\mathrm{I}^{\mathrm{V}}\,\theta')$$
$$z'=\Sigma\,c\,\sin\,(\mathrm{J}\,\varepsilon+\mathrm{J}^{\mathrm{I}}\,\pi+\mathrm{J}^{\mathrm{II}}\theta+\mathrm{J}^{\mathrm{III}}\,\varepsilon'+\mathrm{J}^{\mathrm{IV}}\,\pi'+\mathrm{J}^{\mathrm{V}}\,\theta') \tag{60}$$
$$\mathrm{B}=\mathrm{I}\,n+\mathrm{I}^{\mathrm{I}}\,\pi_1+\mathrm{I}^{\mathrm{II}}\,\theta_1+\mathrm{I}^{\mathrm{III}}n'+\mathrm{I}^{\mathrm{IV}}\,\pi'_1+\mathrm{I}^{\mathrm{V}}\,\theta'_1$$
$$\mathrm{B}'=\mathrm{J}\,n+\mathrm{J}^{\mathrm{I}}\,\pi_1+\mathrm{J}^{\mathrm{II}}\,\theta_1+\mathrm{J}^{\mathrm{III}}n'+\mathrm{J}^{\mathrm{IV}}\,\pi'_1+\mathrm{J}^{\mathrm{V}}\,\theta'_1$$

we have as in the case of the lunar coordinates

$$[\alpha,\ \varepsilon]_2 = -\mu_1 \Sigma\, \mathrm{I}\ \frac{d(\mathrm{BK}^2)}{d\,\alpha} - \frac{1}{2}\mu_1 \Sigma\, \mathrm{J}\ \frac{d\,(\mathrm{B}'\,\mathrm{C}^2)}{d\,\alpha}$$

$$[\alpha,\ \pi]_2 = -\mu_1 \Sigma\, \mathrm{I}^{\mathrm{I}}\ \frac{d(\mathrm{BK}^2)}{d\,\alpha} - \frac{1}{2}\mu_1 \Sigma\, \mathrm{J}^{\mathrm{I}}\ \frac{d\,(\mathrm{B}'\,\mathrm{C}^2)}{d\,\alpha} \qquad (61)$$

$$\cdot \qquad \cdot \qquad \cdot$$

$$\cdot \qquad \cdot \qquad \cdot$$

$$\cdot \qquad \cdot \qquad \cdot$$

$$[\alpha,\ \theta']_2 = -\mu_1 \Sigma\, \mathrm{I}^{\mathrm{V}} \frac{d(\mathrm{BK}^2)}{d\,\alpha} - \frac{1}{2}\mu_1 \Sigma\, \mathrm{J}^{\mathrm{V}} \frac{d\,(\mathrm{B}'\,\mathrm{C}^2)}{d\,\alpha}$$

Finally, we have for the complete La Grangian coefficient

$$[\alpha,\ \beta] = [\alpha,\ \beta]_1 + [\alpha,\ \beta]_2$$

We conclude that the values of the La Grangian coefficients may be obtained as follows. Let us determine the six quantities k_ε, k_π, k_θ, k_ε', k_π', k_θ' from the equations:

$$k_\varepsilon\ = \mu_1 \Sigma\, \mathrm{I}\ \ \mathrm{BK}^2 + \frac{1}{2}\mu_1 \Sigma\, \mathrm{J}\ \ \mathrm{B}'\,\mathrm{C}^2 + \mu_2 \Sigma\, i\ \ b\,k^2 + \frac{1}{2}\mu_2 \Sigma\, j\ \ b'\,c^2$$

$$k_\pi\ = \mu_1 \Sigma\, \mathrm{I}^{\mathrm{I}}\ \mathrm{BK}^2 + \frac{1}{2}\mu_1 \Sigma\, \mathrm{J}^{\mathrm{I}}\ \mathrm{B}'\,\mathrm{C}^2 + \mu_2 \Sigma\, i^{\mathrm{I}}\ b\,k^2 + \frac{1}{2}\mu_2 \Sigma\, j^{\mathrm{I}}\ b'\,c^2$$

$$k_\theta\ = \mu_1 \Sigma\, \mathrm{I}^{\mathrm{II}}\, \mathrm{BK}^2 + \frac{1}{2}\mu_1 \Sigma\, \mathrm{J}^{\mathrm{II}}\, \mathrm{B}'\,\mathrm{C}^2 + \mu_2 \Sigma\, i^{\mathrm{II}}\, b\,k^2 + \frac{1}{2}\mu_2 \Sigma\, j^{\mathrm{II}}\, b'\,c^2$$

$$k_\varepsilon' = \mu_1 \Sigma\, \mathrm{I}^{\mathrm{III}}\mathrm{BK}^2 + \frac{1}{2}\mu_1 \Sigma\, \mathrm{J}^{\mathrm{III}}\, \mathrm{B}'\,\mathrm{C}^2 + \mu_2 \Sigma\, i^{\mathrm{III}}\, b\,k^2 + \frac{1}{2}\mu_2 \Sigma\, j^{\mathrm{III}} b'\,c^2 \qquad (62)$$

$$k_\pi' = \mu_1 \Sigma\, \mathrm{I}^{\mathrm{IV}}\mathrm{BK}^2 + \frac{1}{2}\mu_1 \Sigma\, \mathrm{J}^{\mathrm{IV}}\, \mathrm{B}'\,\mathrm{C}^2 + \mu_2 \Sigma\, i^{\mathrm{IV}}\, b\,k^2 + \frac{1}{2}\mu_2 \Sigma\, j^{\mathrm{IV}}\, b'\,c^2$$

$$k_\theta' = \mu_1 \Sigma\, \mathrm{I}^{\mathrm{V}}\, \mathrm{BK}^2 + \frac{1}{2}\mu_1 \Sigma\, \mathrm{J}^{\mathrm{V}}\, \mathrm{B}'\,\mathrm{C}^2 + \mu_2 \Sigma\, i^{\mathrm{V}}\, b\,k^2 + \frac{1}{2}\mu_2 \Sigma\, j^{\mathrm{V}}\, b'\,c^2$$

We shall then have

$$[a,\ \varepsilon] = -\frac{d\,k_\varepsilon}{d\,a}; \qquad [a,\ \pi] = -\frac{d\,k_\pi}{d\,a} \quad \cdot\ \cdot\ \cdot$$

$$[e,\ \varepsilon] = -\frac{d\,k_\varepsilon}{d\,e}; \qquad [e,\ \pi] = -\frac{d\,k_\pi}{d\,e} \quad \cdot\ \cdot\ \cdot$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \qquad (63)$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$[\gamma'\ \varepsilon] = -\frac{d\,k_\varepsilon}{d\,\gamma'}; \qquad [\gamma'\ \pi] = -\frac{d\,k_\pi}{d\,\gamma'} \quad \cdot$$

§ 9.

REDUCTION OF THE EQUATIONS TO THE CANONICAL FORM.

Let us now resume the equations (21), supposing the symbols α to represent in succession the twelve arbitrary elements

$$a,\ e,\ \gamma,\ a',\ e',\ \gamma',\ \varepsilon_0,\ \pi_0,\ \theta_0,\varepsilon'_0,\ \pi'_0,\ \theta'_0$$

where the six elements ε_0, π_0 . . . . . represent the constant parts of ε, π . . . . Observing that all pairs of constants, both members of which are found either among the first six or the last six of these constants, will make the combination (α_i, α) vanish, and that we have

$$\frac{d\mathrm{R}}{d\varepsilon_0}=\frac{d\mathrm{R}}{d\varepsilon} \qquad \frac{d\mathrm{R}}{d\pi_0}=\frac{d\mathrm{R}}{d\pi}, \text{ etc.}$$

the equations (21) by substituting the values of the coefficients in (63) become

$$\begin{aligned}
&\frac{dk_\varepsilon}{da}\frac{da}{dt}+\frac{dk_\varepsilon}{de}\frac{de}{dt}+\frac{dk_\varepsilon}{d\gamma}\frac{d\gamma}{dt}+\frac{dk_\varepsilon}{da'}\frac{da'}{dt}+\frac{dk_\varepsilon}{de'}\frac{de'}{dt}+\frac{dk_\varepsilon}{d\gamma'}\frac{d\gamma'}{dt}=\frac{d\mathrm{R}}{d\varepsilon}\\
&\frac{dk_\pi}{da}\frac{da}{dt}+\frac{dk_\pi}{de}\frac{de}{dt}+\frac{dk_\pi}{d\gamma}\frac{d\gamma}{dt}+\frac{dk_\pi}{da'}\frac{da'}{dt}+\frac{dk_\pi}{de'}\frac{de'}{dt}+\frac{dk_\pi}{d\gamma'}\frac{d\gamma'}{dt}=\frac{d\mathrm{R}}{d\pi}\\
&\frac{dk_\theta}{da}\frac{da}{dt}+\frac{dk_\theta}{de}\frac{de}{dt}+\frac{dk_\theta}{d\gamma}\frac{d\gamma}{dt}+\frac{dk_\theta}{da'}\frac{da'}{dt}+\frac{dk_\theta}{de'}\frac{de'}{dt}+\frac{dk_\theta}{d\gamma'}\frac{d\gamma'}{dt}=\frac{d\mathrm{R}}{d\theta}\\
&\frac{dk_\varepsilon'}{da}\frac{da}{dt}+\frac{dk_\varepsilon'}{de}\frac{de}{dt}+\frac{dk_\varepsilon'}{d\gamma}\frac{d\gamma}{dt}+\frac{dk_\varepsilon'}{da'}\frac{da'}{dt}+\frac{dk_\varepsilon'}{de'}\frac{de'}{dt}+\frac{dk_\varepsilon'}{d\gamma'}\frac{d\gamma'}{dt}=\frac{d\mathrm{R}}{d\varepsilon'}\\
&\frac{dk_\pi'}{da}\frac{da}{dt}+\frac{dk_\pi'}{de}\frac{de}{dt}+\frac{dk_\pi'}{d\gamma}\frac{d\gamma}{dt}+\frac{dk_\pi'}{da'}\frac{da'}{dt}+\frac{dk_\pi'}{de'}\frac{de'}{dt}+\frac{dk_\pi'}{d\gamma'}\frac{d\gamma'}{dt}=\frac{d\mathrm{R}}{d\pi'}\\
&\frac{dk_\theta'}{da}\frac{da}{dt}+\frac{dk_\theta'}{de}\frac{de}{dt}+\frac{dk_\theta'}{d\gamma}\frac{d\gamma}{dt}+\frac{dk_\theta'}{da'}\frac{da'}{dt}+\frac{dk_\theta'}{de'}\frac{de'}{dt}+\frac{dk_\theta'}{d\gamma'}\frac{d\gamma'}{dt}=\frac{d\mathrm{R}}{d\theta'}\\
&\frac{dk_\varepsilon}{da}\frac{d\varepsilon_0}{dt}+\frac{dk_\pi}{da}\frac{d\pi_0}{dt}+\frac{dk_\theta}{da}\frac{d\theta_0}{dt}+\frac{dk_\varepsilon'}{da}\frac{d\varepsilon'_0}{dt}+\frac{dk_\pi'}{da}\frac{d\pi'_0}{dt}+\frac{dk_\theta'}{da}\frac{d\theta'_0}{dt}=-\frac{d\mathrm{R}}{da}\\
&\frac{dk_\varepsilon}{de}\frac{d\varepsilon_0}{dt}+\frac{dk_\pi}{de}\frac{d\pi_0}{dt}+\frac{dk_\theta}{de}\frac{d\theta_0}{dt}+\frac{dk_\varepsilon'}{de}\frac{d\varepsilon'_0}{dt}+\frac{dk_\pi'}{de}\frac{d\pi'_0}{dt}+\frac{dk_\theta'}{de}\frac{d\theta'_0}{dt}=-\frac{d\mathrm{R}}{de}\\
&\frac{dk_\varepsilon}{d\gamma}\frac{d\varepsilon_0}{dt}+\frac{dk_\pi}{d\gamma}\frac{d\pi_0}{dt}+\frac{dk_\theta}{d\gamma}\frac{d\theta_0}{dt}+\frac{dk_\varepsilon'}{d\gamma}\frac{d\varepsilon'_0}{dt}+\frac{dk_\pi'}{d\gamma}\frac{d\pi'_0}{dt}+\frac{dk_\theta'}{d\gamma}\frac{d\theta'_0}{dt}=-\frac{d\mathrm{R}}{d\gamma}\\
&\frac{dk_\varepsilon}{da'}\frac{d\varepsilon_0}{dt}+\frac{dk_\pi}{da'}\frac{d\pi_0}{dt}+\frac{dk_\theta}{da'}\frac{d\theta_0}{dt}+\frac{dk_\varepsilon'}{da'}\frac{d\varepsilon'_0}{dt}+\frac{dk_\pi'}{da'}\frac{d\pi'_0}{dt}+\frac{dk_\theta'}{da'}\frac{d\theta'_0}{dt}=-\frac{d\mathrm{R}}{da'}\\
&\frac{dk_\varepsilon}{de'}\frac{d\varepsilon_0}{dt}+\frac{dk_\pi}{de'}\frac{d\pi_0}{dt}+\frac{dk_\theta}{de'}\frac{d\theta_0}{dt}+\frac{dk_\varepsilon'}{de'}\frac{d\varepsilon'_0}{dt}+\frac{dk_\pi'}{de'}\frac{d\pi'_0}{dt}+\frac{dk_\theta'}{de'}\frac{d\theta'_0}{dt}=-\frac{d\mathrm{R}}{de'}\\
&\frac{dk_\varepsilon}{d\gamma'}\frac{d\varepsilon_0}{dt}+\frac{dk_\pi}{d\gamma'}\frac{d\pi_0}{dt}+\frac{dk_\theta}{d\gamma'}\frac{d\theta_0}{dt}+\frac{dk_\varepsilon'}{d\gamma'}\frac{d\varepsilon'_0}{dt}+\frac{dk_\pi'}{d\gamma'}\frac{d\pi'_0}{dt}+\frac{dk_\theta'}{d\gamma'}\frac{d\theta'_0}{dt}=-\frac{d\mathrm{R}}{d\gamma'}
\end{aligned}\tag{64}$$

Since k_ϵ, k_π, etc., are functions of the six elements a, e, γ, a', e', γ' the left hand member of the first of these equations is equal to $\frac{d\,k_\epsilon}{d\,t}$ that of the second to $\frac{d\,k_\pi}{d\,t}$ and so on through the first six equations. The latter thus reduce to the six left hand equations in (65) below.

The quantities k_ϵ, k_π, etc., being six in number, we may conceive the six elements a, e, γ, a', e', γ', and R to be expressed in terms of these six new elements. If, then, we multiply the seventh of the preceding equations by $\frac{d\,a}{d\,k_\epsilon}$ the eighth by $\frac{d\,e}{d\,k_\epsilon}$ and so on to the twelfth, which we multiply by $\frac{d\,\gamma'}{d\,k_\epsilon}$ and notice that the theory of differentiation of inverse functions gives us

$$\frac{d\,k_\epsilon}{d\,a}\frac{d\,a}{d\,k_\epsilon}+\frac{d\,k_\epsilon}{d\,e}\frac{d\,e}{d\,k_\epsilon}+\frac{d\,k_\epsilon}{d\,\gamma}\frac{d\,\gamma}{d\,k_\epsilon}+\frac{d\,k_\epsilon}{d\,a'}\frac{d\,a'}{d\,k_\epsilon}+\frac{d\,k_\epsilon}{d\,e'}\frac{d\,e'}{d\,k_\epsilon}+\frac{d\,k_\epsilon}{d\,\gamma'}\frac{d\,\gamma'}{d\,k_\epsilon}=1$$

$$\frac{d\,k_\pi}{d\,a}\frac{d\,a}{d\,k_\epsilon}+\frac{d\,k_\pi}{d\,e}\frac{d\,e}{d\,k_\epsilon}+\frac{d\,k_\pi}{d\,\gamma}\frac{d\,\gamma}{d\,k_\epsilon}+\frac{d\,k_\pi}{d\,a'}\frac{d\,a'}{d\,k_\epsilon}+\frac{d\,k_\pi}{d\,e'}\frac{d\,e'}{d\,k_\epsilon}+\frac{d\,k_\pi}{d\,\gamma'}\frac{d\,\gamma'}{d\,k_\epsilon}=0$$

. . . . . .

. . . . . .

. . . . . .

with similar equations for every combination of two k elements, we find

$$\frac{d\,\epsilon_0}{d\,t}=-\frac{d\,\mathrm{R}}{d\,k_\epsilon}$$

In the same way we obtain

$$\frac{d\,\pi_0}{d\,t}=-\frac{d\,\mathrm{R}}{d\,k_\pi}$$

etc. etc.

The equations (64) may therefore be reduced to the canonical form

$$\left.\begin{aligned}
\frac{d\,k_\epsilon}{d\,t}&=\frac{d\,\mathrm{R}}{d\,\epsilon} & \frac{d\,\epsilon_0}{d\,t}&=-\frac{d\,\mathrm{R}}{d\,k_\epsilon}\\
\frac{d\,k_\pi}{d\,t}&=\frac{d\,\mathrm{R}}{d\,\pi} & \frac{d\,\pi_0}{d\,t}&=-\frac{d\,\mathrm{R}}{d\,k_\pi}\\
\frac{d\,k_\theta}{d\,t}&=\frac{d\,\mathrm{R}}{d\,\theta} & \frac{d\,\theta_0}{d\,t}&=-\frac{d\,\mathrm{R}}{d\,k_\theta}\\
\frac{d\,k_\epsilon'}{d\,t}&=\frac{d\,\mathrm{R}}{d\,\epsilon'} & \frac{d\,\epsilon'_0}{d\,t}&=-\frac{d\,\mathrm{R}}{d\,k_\epsilon'}\\
\frac{d\,k'_\pi}{d\,t}&=\frac{d\,\mathrm{R}}{d\,\pi'} & \frac{d\,\pi'_0}{d\,t}&=-\frac{d\,\mathrm{R}}{d\,k'_\pi}\\
\frac{d\,k'_\theta}{d\,t}&=\frac{d\,\mathrm{R}}{d\,\theta'} & \frac{d\,\theta'_0}{d\,t}&=-\frac{d\,\mathrm{R}}{d\,k'_\theta}
\end{aligned}\right\} \quad (65)$$

We may regard these as the rigorous differential equations for the solution of the

problem of four bodies after the solution of the problem is effected for the case of three bodies.

In deriving the equations (64) from (21), we have limited ourselves by no hypothesis respecting the signification of the elements ε, π, etc., which appear under the signs *sin* and *cos*. We have only supposed the variable angles to be linear functions of six variables each of the form $a_0+b\,t$, a_0 being an arbitrary constant and b a function of the six arbitrary constants which form the coefficients k. We may therefore substitute for any of the quantities ε, π, θ, ε', π', θ', any linear function of other independent quantities completely determined in terms of ε, π, etc., by equations of the first degree. For every such system of new variables we shall have a system of canonical equations corresponding to (65). The number of these canonical forms is infinite.

The transformation in question can indeed be performed immediately on the equations (65). Let us determine six new variables l_1, l_2, l_3, l_4, l_5, l_6, by the conditions

$$\left.\begin{aligned}
p_1\,l_1+p_2\,l_2+p_3\,l_3+p_4\,l_4+p_5\,l_5+p_6\,l_6&=\varepsilon_0\\
q_1\,l_1+q_2\,l_2+q_3\,l_3+q_4\,l_4+q_5\,l_5+q_6\,l_6&=\pi_0\\
r_1\,l_1+r_2\,l_2+r_3\,l_3+r_4\,l_4+r_5\,l_5+r_6\,l_6&=\theta_0\\
s_1\,l_1+s_2\,l_2+s_3\,l_3+s_4\,l_4+s_5\,l_5+s_6\,l_6&=\varepsilon'_0\\
t_1\,l_1+t_2\,l_2+t_3\,l_3+t_4\,l_4+t_5\,l_5+t_6\,l_6&=\pi'_0\\
u_1\,l_1+u_2\,l_2+u_3\,l_3+u_4\,l_4+u_5\,l_5+u_6\,l_6&=\theta'_0
\end{aligned}\right\} \tag{66}$$

p, q, r, etc., being any constants whatever of which the determinant does not vanish. Let us put also

$$\left.\begin{aligned}
c_1&=p_1\,k_\varepsilon+q_1\,k_\pi+r_1\,k_\theta+s_1\,k_\varepsilon'+t_1\,k_\pi'+u_1\,k_\theta'\\
c_2&=p_2\,k_\varepsilon+q_2\,k_\pi+r_2\,k_\theta+s_2\,k_\varepsilon'+t_2\,k_\pi'+u_2\,k_\theta'\\
c_6&=p_6\,k_\varepsilon+q\,k_\pi+r_6\,k_\theta+s_6\,k_\varepsilon'+t_6\,k_\pi'+u_6\,k_\theta'
\end{aligned}\right\} \tag{67}$$

Multiply the first of the equations (65) on the left by p_1, the second by q_1, etc., and add the six products thus formed. The sum reduces by (67) to

$$\frac{d\,c_1}{d\,t}=p_1\frac{d\,\mathrm{R}}{d\,\varepsilon_0}+q_1\frac{d\,\mathrm{R}}{d\,\pi_0}+r_1\frac{d\,\mathrm{R}}{d\,\theta_0}+s_1\frac{d\,\mathrm{R}}{d\,\varepsilon'_0}+t_1\frac{d\,\mathrm{R}}{d\,\pi'_0}+u_1\frac{d\,\mathrm{R}}{d\,\theta'_0}$$

The equations (66) give

$$p_1=\frac{d\,\varepsilon_0}{d\,l_1};\quad q_1=\frac{d\,\pi_0}{d\,l_1}\text{ etc.}$$

Substituting these values of p_1, q_1, etc., in the last equation it becomes

$$\frac{d\,c_1}{d\,t}=\frac{d\,\mathrm{R}}{d\,l_1}$$

We have in the same way

$$\frac{d\,c_2}{d\,t}=\frac{d\,\mathrm{R}}{d\,l_2};\quad \frac{d\,c_3}{d\,t}=\frac{d\,\mathrm{R}}{d\,l_3}\text{ etc.}$$

To obtain the values of $\frac{d\,l}{d\,t}$ let us suppose that we solve the equations (66) with respect to l_1, l_2, etc., so as to obtain the values of these quantities in terms of ε_0, π_0, θ_0, etc. These values will be of the form

$$l_1 = P_1\,\varepsilon_0 + Q_1\,\pi_0 + R_1\,\theta_0 + S_1\,\varepsilon'_0 + T_1\,\pi'_0 + U_1\,\theta'_0$$

$$l_2 = P_2\,\varepsilon_0 + Q_2\,\pi_0 + R_2\,\theta_0 + S_2\,\varepsilon'_0 + T_2\,\pi'_0 + U_2\,\theta'_0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$l_6 = P_6\,\varepsilon_0 + Q_6\,\pi_0 + R_6\,\theta_0 + S_6\,\varepsilon'_0 + T_6\,\pi'_0 + U_6\,\theta'_0$$

P, Q, etc., being functions of p, q, etc., which are readily formed by the theory of determinants. From the theorem that any two determinants are identical when the horizontal lines of the one are identical with the vertical lines of the other it follows that the values of k_ε, k_π, etc., derived from (67) will be

$$k_\varepsilon = P_1\,c_1 + P_2\,c_2 + P_3\,c_3 + P_4\,c_4 + P_5\,c_5 + P_6\,c_6$$

$$k_\pi = Q_1\,c_1 + Q_2\,c_2 + Q_3\,c_3 + Q_4\,c_4 + Q_5\,c_5 + Q_6\,c_6$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$k_\theta' = U_1\,c_1 + U_2\,c_2 + U_3\,c_3 + U_4\,c_4 + U_5\,c_5 + U_6\,c_6$$

Let us now multiply the first of the right hand equations (65) by P_i, the second by Q_i, etc., and add the products. We shall have

$$\frac{d\,l_i}{d\,t} = -\frac{d\,R}{d\,c_i}$$

We may therefore replace the equations (65) by the more general ones

$$\left.\begin{array}{ll}
\dfrac{d\,c_1}{d\,t} = \dfrac{d\,R}{d\,l_1} & \dfrac{d\,l_1}{d\,t} = -\dfrac{d\,R}{d\,c_1} \\[2ex]
\dfrac{d\,c_2}{d\,t} = \dfrac{d\,R}{d\,l_2} & \dfrac{d\,l_2}{d\,t} = -\dfrac{d\,R}{d\,c_2} \\[2ex]
\dfrac{d\,c_3}{d\,t} = \dfrac{d\,R}{d\,l_3} & \dfrac{d\,l_3}{d\,t} = -\dfrac{d\,R}{d\,c_3} \\[2ex]
\dfrac{d\,c_4}{d\,t} = \dfrac{d\,R}{d\,l_4} & \dfrac{d\,l_4}{d\,t} = -\dfrac{d\,R}{d\,c_4} \\[2ex]
\dfrac{d\,c_5}{d\,t} = \dfrac{d\,R}{d\,l_5} & \dfrac{d\,l_5}{d\,t} = -\dfrac{d\,R}{d\,c_5} \\[2ex]
\dfrac{d\,c_6}{d\,t} = \dfrac{d\,R}{d\,l_6} & \dfrac{d\,l_6}{d\,t} = -\dfrac{d\,R}{d\,c_6}
\end{array}\right\} \qquad (68)$$

Let us, for the sake of clearness, recapitulate the process by which we have arrived at these equations. Considering first only the mutual action of the Sun, the Earth, and the Moon, we obtain the rectangular coordinates of the Moon relatively to the Earth (x, y, z) and those of the Sun relatively to the center of gravity of the Earth and Moon (x', y', z') in the form

$$\left.\begin{aligned} x &= \Sigma\, k \cos(\lambda + b\,t) \\ y &= \Sigma\, k \sin(\lambda + b\,t) \\ z &= \Sigma\, c \sin(\lambda' + b'\,t) \\[6pt] x' &= \Sigma\, \mathrm{K} \cos(\varLambda + \mathrm{B}\,t) \\ y' &= \Sigma\, \mathrm{K} \sin(\varLambda + \mathrm{B}\,t) \\ z' &= \Sigma\, \mathrm{C} \sin(\varLambda' + \mathrm{B}'\,t) \end{aligned}\right\} \tag{69}$$

k, c, b, b', K, C, B, and B$'$ being functions of the six constants

$$a,\ e,\ \gamma,\ a',\ e',\ \gamma'$$

and λ, λ', $\varLambda$, and $\varLambda'$ being linear functions of six other constants

$$l_1, l_2, l_3, l_4, l_5, l_6,$$

which fix the mean longitudes of the bodies, and the positions of the lines of apsides and of the nodes by any system of linear equations like (66) which we choose to adopt. Of the infinite series of values of λ, λ', $\varLambda$, $\varLambda'$, represented by the sign Σ each will be of the form

$$\lambda \text{ or } \varLambda = \mathrm{I}_1\, l_1 + \mathrm{I}_2\, l_2 + \mathrm{I}_3\, l_3 + \mathrm{I}_4\, l_4 + \mathrm{I}_5\, l_5 + \mathrm{I}_6\, l_6$$

(2) For each term of x or each value of k we form the product

$$\frac{m_2\, m_3}{m_2 + m_3}\, b\, k^2$$

which, for the moment, we represent by k'.

For each term of z, or each value of c we form the product

$$\frac{1}{2}\,\frac{m_2\, m_3}{m_2 + m_3}\, b'\, c^2$$

which we may represent by c'.

For each term of x', or each value of $\varLambda$ we form the product

$$\frac{m_1\,(m_2 + m_3)}{m_1 + m_2 + m_3}\, \mathrm{B}\, \mathrm{K}_2 = \mathrm{K}'$$

For each term of z', or each value of $\varLambda'$ form the product

$$\frac{1}{2}\,\frac{m_1\,(m_2 + m_3)}{m_1 + m_2 + m_3}\, \mathrm{B}'\, \mathrm{C}^2 = \mathrm{C}'$$

(3) Multiply each k' thus formed by the corresponding value of i_1, or the coefficient of l_1 in the corresponding value of λ; each c' by the coefficient of l_1 in the corresponding value of λ'; each K$'$ by the coefficient of l_1 in the corresponding value of $\varLambda$, and each C$'$ by the coefficient of l_1 in the corresponding value of $\varLambda'$. Represent the sum of all the products thus formed by c_1.

In the same way form the value of c_2 from the coefficients of l_2, and so on to c_6. These six values of c_i will then be functions of the six elements

$$a, e, \gamma, a', e', \gamma',$$

and may therefore be conceived to replace them in the expressions for the coordinates.

(4) The disturbing function R being a function of the coordinates x, y, z, x', y', z', let these quantities be replaced by their values in term of the elements $c_1, c_2, \ldots c_6$, $l_1 l_2 \ldots l_6$. Then determine the values of these elements as variables by the integration of the equations

$$\frac{d\,c_i}{d\,t}=\frac{d\,\mathrm{R}}{d\,l_i} \qquad \frac{d\,l_i}{d\,t}=-\frac{d\,\mathrm{R}}{d\,c_i}$$

The values of c_i, l_i thus obtained being substituted in the expressions for the coordinates we shall have the values of the latter under the influence of the disturbing force.

§ 10.

TRANSFORMATION OF THE PRECEDING EQUATIONS.

The values of the elements deduced from the equations (65) or (68) are to be substituted in (69) to obtain the coordinates of the Moon relatively to the Earth under the influence of the attraction of the planet. If we take the equations in their literal acceptation, we shall find terms in which the coefficients of the sines or cosines of the varying angles have the time as a factor. These terms are introduced in the following way: We have assumed that in the disturbed motion the coordinates are expressed in terms of the elements in the same form as in the undisturbed motion, only that the elements are variable. In the undisturbed motion each of the variables

$$\varepsilon, \pi, \theta, \varepsilon', \pi'$$

is expressed in the form

$$l+b\,t,$$

l being an arbitrary constant of the second class, and b a function of the arbitrary constants of the first class. In the disturbed motion the derivative of this expression will be

$$\frac{d\,l}{d\,t}+t\frac{d\,b}{d\,t}+b$$

The expression $\frac{d\,l}{d\,t}$ being formed by taking the derivative of R with respect to quantities which are multiplied by the time in the varying angles, will contain the factor t outside the signs sin and cos.

We shall now show that these terms are destroyed by the term $t\frac{d\,b}{d\,t}$. Let us, as before, represent the angles by which replace ε, π, etc., in the general equations (68) by the general symbols

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$$

where

$$\lambda_1 = l_1 + b_1\, t$$

$$\lambda_2 = l_2 + b_2\, t$$

etc. etc.

We then have for any one λ and the corresponding c

$$\frac{d\,c_i}{d\,t} = \frac{d\,\mathrm{R}}{d\,\lambda_i} \qquad \frac{d\,l_i}{d\,t} = -\frac{d\,\mathrm{R}}{d\,c_i} \tag{70}$$

and

$$\frac{d\,\lambda_i}{d\,t} = \frac{d\,l_i}{d\,t} + t\,\frac{d\,b_i}{d\,t} + b \tag{71}$$

The terms of R in question are of the form

$$\mathrm{R} = h \cos\,(i_1\,\lambda_1 + i_2\,\lambda_2 + i_3\,\lambda_3 + i_4\,\lambda_4 + i_5\,\lambda_5 + i_6\,\lambda_6) = h \cos \mathrm{N}.$$

Consider any one of them. Since λ_1, λ_2, etc., are functions of c_i, through b_i, the value of $\frac{d\,\mathrm{R}}{d\,c_i}$ will contain the terms

$$\frac{d\,\mathrm{R}}{d\,\lambda_1}\frac{d\,\lambda_1}{d\,c_i} + \frac{d\,\mathrm{R}}{d\,\lambda_2}\frac{d\,\lambda_1}{d\,c_i} + \quad \cdot \quad \cdot \quad \cdot$$

But

$$\frac{d\,\mathrm{R}}{d\,\lambda_1} = -i_1\,h \sin \mathrm{N}; \quad \frac{d\,\mathrm{R}}{d\,\lambda_2} = -i_2\,h \sin \mathrm{N}, \text{ etc.}$$

and

$$\frac{d\,\lambda_1}{d\,c_i} = t\,\frac{d\,b_1}{d\,c_i}; \quad \frac{d\,\lambda_2}{d\,c_i} = t\,\frac{d\,b_2}{d\,c_i}, \text{ etc.}$$

Therefore the value of $-\frac{d\,\mathrm{R}}{d\,c_i}$, and consequently of $\frac{d\,l_i}{d\,t}$, will contain the term

$$\left(i_1\,\frac{d\,b_1}{d\,c_i} + i_2\,\frac{d\,b_2}{d\,c_i} + \quad \cdot \quad \cdot \quad + i_6\,\frac{d\,b_6}{d\,c_i}\right) t\,h \sin \mathrm{N}$$

To find the second term of $\frac{d\,\lambda_i}{d\,t}$ in (71) we have

$$\frac{d\,b_i}{d\,t} = \frac{d\,b_i}{d\,c_1}\frac{d\,c_1}{d\,t} + \frac{d\,b_i}{d\,c_2}\frac{d\,c_2}{d\,t} + \quad \cdot \quad \cdot \quad \cdot \quad + \frac{d\,b_i}{d\,c_6}\frac{d\,c_6}{d\,t}$$

Substituting for $\frac{d\,c_1}{d\,t}$, $\frac{d\,c_2}{d\,t}$, etc., their values in (70), and for $\frac{d\,\mathrm{R}}{d\,\lambda}$ the expression just found, we have

$$t\,\frac{d\,b_i}{d\,t} = -\left(i_1\,\frac{d\,b_i}{d\,c_1} + i_2\,\frac{d\,b_i}{d\,c_2} + \quad \cdot \quad \cdot \quad \cdot \quad + i_6\,\frac{d\,b_i}{d\,c_6}\right) t\,h \sin \mathrm{N}$$

Each term of R will therefore produce in $\frac{d\lambda_i}{dt}$ the term

$$\left\{ i_1 \left(\frac{d b_1}{d c_i} - \frac{d b_i}{d c_1}\right) + i_2 \left(\frac{d b_2}{d c_i} - \frac{d b_i}{d c_2}\right) + \quad . \quad . \quad . \quad \right\} h\, t \sin \mathrm{N}.$$

It is known that the values of b are so related to those of c that for all combinations of the indices i and j we have

$$\frac{d b_i}{d c_j} - \frac{d b_j}{d c_i} = 0 \tag{72}$$

The general proof of this theorem is found in my paper on the General Integrals of Planetary Motion, so that it is not necessary to repeat it here.

It appears that we may omit the term $t\frac{db}{dt}$ in the value of $\frac{d\lambda}{dt}$ in the disturbed orbit; provided that in taking the derivatives with respect to c, we suppose the values of b under the signs sin and cos, to be invariable, and vary only the coefficients k. We then have in the disturbed orbit

$$\frac{d\lambda_i}{dt} = \frac{d l_i}{dt} + b_i$$

$$\lambda_i = l_i + \int b_i\, dt$$

This result may be expressed in the elegant form adopted by DELAUNAY in his theory of the Moon. Taking λ_1, λ_2, etc., as the variables instead of l_1, l_2, etc., the above equations reduce the forms (68) to the following:—

$$\frac{d c_i}{dt} = \frac{d\mathrm{R}}{d\lambda_i} \qquad\qquad \frac{d\lambda_i}{dt} = -\frac{d\mathrm{R}}{d c_i} + b_i \tag{73}$$

Consider the differential function

$$b_1\, d c_1 + b_2\, d c_2 + \quad . \quad . \quad . \quad + b_6\, d c_6$$

The equations (72) show that this expression is an exact differential of a certain function of c_1, c_2, etc., which we may represent by Θ. If, then, we find θ by integrating this differential, we have

$$b_i = \frac{d\Theta}{d c_i}$$

Substituting this value of b_i in the above expressions for $\frac{d\lambda_i}{dt}$, and putting

$$\mathrm{R}' = \mathrm{R} - \Theta$$

the equations which give c_i and λ_i become

$$\frac{d c_i}{dt} = \frac{d\mathrm{R}'}{d\lambda_i} \qquad\qquad \frac{d\lambda_i}{dt} = -\frac{d\mathrm{R}'}{d c_i} \tag{74}$$

The fact that the mean motions of the Sun and Moon and the secular variation of the nodes and apsides of the solar and lunar orbits under the influence of the mutual

attractions of the three bodies can all be expressed as partial derivatives of a single function lends an interest to the form and properties of this function. As a preliminary lemma in its investigation we remark that b_1, b_2, etc , k and θ may all be expressed as homogeneous functions of $c_1, c_2, c_3, c_4, c_5, c_6$. If we express the coordinates x', y', z', x, y, z, of the Sun and Moon by the general expression

$$x,\ y,\ z,\ x',\ y',\ z' = \Sigma\, k \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(l + b\,t,) \tag{75}$$

the quantity k will necessarily be a homogeneous function of the first degree in a and a', the mean distances, and therefore in the language of physics, of dimensions L, while, since $b\,t$ is an angle, and so a pure number, b itself is of dimensions $T^{-1} = M^{\frac{1}{2}} L^{-\frac{3}{2}}$ and therefore homogeneous and of the order $-\frac{3}{2}$ in a and a' when unit mass is defined as that which attracts with unit force at unit distance. Now, c_i is of the general form (62), that is, we have

$$\begin{aligned} c_1 &= \frac{1}{2}\ \Sigma\ \mu\, i_1\, b\, k^2 \\ c_2 &= \frac{1}{2}\ \Sigma\ \mu\, i_2\, b\, k^2 \\ &\cdot \qquad \cdot \\ &\cdot \qquad \cdot \\ c_6 &= \frac{1}{2}\ \Sigma\ \mu\, i_6\, b\, k^2 \end{aligned} \tag{76}$$

$i_1, i_2, \quad . \quad . \quad . \quad i_6$ being the coefficients of $\lambda_1\ \lambda_2, \quad . \quad . \quad . \quad \lambda_6$ respectively in the angle $l + b\,t$, and μ being the functions of the masses employed in (62). Hence c_i considered as a function of

$$m_1,\ m_2,\ m_3,\ a,\ e,\ \gamma,\ a',\ e',\ \gamma',$$

will be homogeneous and of the degree $\frac{1}{2}$ in a and a'. Hence, conversely when we express these elements as functions of c_1, c_2, etc., these functions will be homogeneous and of the second degree in c_1, c_2, etc. Hence,

a, a', and k are functions of the degree 2 in c_1, c_2, etc.,
$b_1, b_2, \ . \ . \ b_6$ " " " -3 " "

and, since

$$d\,\Theta = b_1\, d\,c_1 + b_2\, d\,c_2 + \quad . \quad . \quad . \quad . \quad + b_6\, d\,c_6$$

it follows that Θ is homogeneous and of the order -2 in c_1, c_2, etc.

The same reasoning may be expressed in another way. Let us suppose the linear dimensions of the system increased in the ratio $\nu{:}1$, the masses, inclinations, and eccentricities all remaining unaltered. Then the analytical forms of the integrals of motion will remain unaltered, while the following changes will result in the values of the principal constants:—

All the values of k will be increased in the ratio $\nu{:}1$
. . . . b . . . . $\nu^{-\frac{3}{2}}{:}1$
. . . . c . . . . $\nu^{\frac{1}{2}}{:}1$

These conditions can be fulfilled without changing the form of b and k only when b is homogeneous and of the order -3 and k homogeneous and of the order 2 in c. It follows that each value of k, b, and Θ must satisfy the partial differential equations

$$c_1\frac{d\,k}{d\,c_1}+c_2\frac{d\,k}{d\,c_2}+c_3\frac{d\,k}{d\,c_3}+c_4\frac{d\,k}{d\,c_4}+c_5\frac{d\,k}{d\,c_5}+c_6\frac{d\,k}{d\,c_6}-2\,k=0$$

$$c_1\frac{d\,b_i}{d\,c_1}+c_2\frac{d\,b_i}{d\,c_2}+c_3\frac{d\,b_i}{d\,c_3}+c_4\frac{d\,b_i}{d\,c_4}+c_5\frac{d\,b_i}{d\,b_5}+c_6\frac{d\,b_i}{d\,c_6}+3\,b_i=0$$

$$c_1\frac{d\,\Theta}{d\,c_1}+c_2\frac{d\,\Theta}{d\,c_2}+c_3\frac{d\,\Theta}{d\,c_3}+c_4\frac{d\,\Theta}{d\,c_4}+c_5\frac{d\,\Theta}{d\,c_5}+c_6\frac{d\,\Theta}{d\,c_6}+2\,\Theta=0$$

Since

$$b_1=\frac{d\,\Theta}{d\,c_1} \qquad b_2=\frac{d\,\Theta}{d\,c_2}\ \text{etc.}$$

we have, from the last equation

$$\Theta=-\frac{1}{2}\,(b_1\,c_1+b_2\,c_2+b_3\,c_3+b_4\,c_4+b_5\,c_5+b_6\,c_6)$$

If we substitute for c_1, c_2, etc., their values (76) this equation becomes

$$\Theta=-\frac{1}{4}\,\Sigma\,\mu\,(i_1\,b_1+i_2\,b_2+i_3\,b_3+i_4\,b_4+i_5\,b_5+i_6\,b_6)\,b\,k^2$$

or since

$$b=i_1\,b_1+i_2\,b_2+\text{etc.}$$

$$\Theta=-\frac{1}{4}\,\Sigma\,\mu\,b^2\,k^2$$

From this formula it can be shown that Θ is the negative of the function which CLAUSIUS has named the *viriel* of the system, that is, the constant term in the expression for the living force of the system of three bodies. Let us represent this living force by T. Then

$$2\,\mathrm{T}=m_1\,(\xi'^2+\eta'^2_1+\zeta'^2_1)+m_2\,(\xi'^2_2+\eta'^2_2+\zeta'^2_2)+m_3\,(\xi'^2_3+\eta'^2_3+\zeta'^2_3)$$

the accents indicating derivatives as to t. If we differentiate the first three equations (5) with respect to the time, with ξ_0 a constant, and substitute for ξ, η, ζ their values in x, y, etc., we have

$$2\,\mathrm{T}=\mu_1\left(\frac{d\,x'^2}{d\,t^2}+\frac{d\,y'^2}{d\,t^2}+\frac{d\,z'^2}{d\,t^2}\right)+\mu_2\left(\frac{d\,x^2}{d\,t^2}+\frac{d\,y^2}{d\,t^2}+\frac{d\,z^2}{d\,t^2}\right)$$

Differentiating the expression (75) we have

$$\frac{d}{d\,t}\left(x,\ y,\ z,\ x',\ y',\ \text{or}\ z'\right)=\Sigma\pm b\,k\,{\cos \atop \sin}\,(f+b\,t)$$

If we square each equation of this form, each term under the sign Σ will give in the square the constant term

$$\frac{1}{2}\,b^2\,k^2$$

Hence the sum of all the constants in 2 T, which we shall represent by 2 T_0, will be

$$\frac{1}{2} \Sigma \mu b^2 k^2$$

the sign Σ being extended so as to include all the terms in the six coordinates x', y', z', x, y, z. Comparing this with the above value of Θ we have

$$\Theta = -T_0$$

T_0 being the viriel of CLAUSIUS.

This result may be expressed as follows: Taking for the elements c_1, c_2, etc., the functions k_ϵ, k_π, k_θ, etc., and expressing the constant of the living force of the system composed of the Sun, Earth, and Moon as a function of k_ϵ, k_π, etc., the mean motions of the Sun and Moon, and the secular motions of their nodes and apsides will be given by the equations

$$\left.\begin{aligned}
\frac{d\,\epsilon}{d\,t} + \frac{d\,T_0}{d\,k_\epsilon} &= 0 \\
\frac{d\,\pi}{d\,t} + \frac{d\,T_0}{d\,k_\pi} &= 0 \\
\frac{d\,\theta}{d\,t} + \frac{d\,T_0}{d\,k_\theta} &= 0 \\
\frac{d\,\epsilon'}{d\,t} + \frac{d\,T_0}{d\,k'_\epsilon} &= 0 \\
\frac{d\,\pi'}{d\,t} + \frac{d\,T_0}{d\,k'_\pi} &= 0 \\
\frac{d\,\theta'}{d\,t} + \frac{d\,T_0}{d\,k'_\theta} &= 0
\end{aligned}\right\} \tag{77}$$

These equations can not be directly used to obtain the values of these secular motions of ϵ, π, etc., because we can not form T_0 until we have solved the equations of motion. But, the solutions of these equations being once obtained, they may form a valuable check on their accuracy.

CHAPTER II.

SPECIAL CASE OF THE MOON'S MOTION.

§ 11.

COMPUTATION OF THE ANALYTIC EXPRESSIONS k_e, k_π, ETC., EMPLOYED IN THE PRECEDING THEORY.

In the preceding chapters we have shown, in a general way, how we may deduce expressions for the simultaneous variations of the elements of the orbits of the Earth and of the Moon which are due to the action of a planet.

We now proceed to the actual calculation of these expressions. The coordinates of the Moon relatively to the Earth, and of the Sun relatively to the center of gravity of the Earth and Moon, are supposed to be given in advance, in terms of the time and of arbitrary constants. As already indicated, we shall take these data from the theory of Delaunay. In this work, however, the parallax is carried only to terms of the fifth order, while, for the present purpose, it is desired to carry the approximation to terms of a higher order.

During my visit to Paris in April, 1871, Mr. Delaunay was so good as to calculate the constant term of the parallax to terms of the sixth order, and afterward Professor J. C. Adams kindly communicated his results for this same term to the seventh and higher orders. I am thus enabled to present the actual analytic expressions for k_e, k_π, etc., to terms of the order m^7 and $e'^2 m^5$, and to terms of the sixth order in e, γ, e' and m.

The notation of the constants is as follows:

The elements of the orbit of the moon, a, n, e, γ, have the same signification as in the theory of Delaunay, n being the actual mean motion of the Moon, and a the mean distance, derived from n by the equation

$$a^3 n^2 = m_2 + m_3$$

e is the eccentricity of the lunar orbit, and γ the sine of half inclination to the ecliptic. The coordinates of the moon relatively to the ecliptic are expressed in terms of g, g', λ, and λ', which represent respectively the mean anomalies of the Sun and Moon, and the mean angular distances of these bodies from the node of the Moon's orbit on the ecliptic.

These quantities are used for convenience instead of the l, l', F, and D of DELAUNAY, with which they are connected by the relations

$$l = g$$
$$l' = g'$$
$$F = \lambda$$
$$D = \lambda - \lambda'$$

The required formulæ are presented in the form I have found most convenient for manipulation in the following pages Instead of writing in full each expression such as

$$C \frac{\cos}{\sin} \left(i\,g + i'\,g' + j\,\lambda + j'\,\lambda' \right)$$

I use the plan of the German astronomers, simply writing the indices i, i', j, j', opposite the coefficient C of the term.

We commence with the value of sin P or $\frac{a}{r}$ given by DELAUNAY. If we put

$$\frac{a}{r} = 1 + \delta$$

we have

$$\frac{r}{a} = 1 - \delta + \delta^2 - \delta^3 + \text{etc.} \quad . \quad . \quad .$$

The several terms of $\frac{a}{r}$, so far as is necessary for our present purpose, are given in the first of the following series of tables. Only those terms are given which are necessary for determining k_ϵ, k_π, and k_θ as high as m^7 and $e'^2 m^5$, and as high as the sixth dimension in all the other small quantities, as well as their combinations. Now, the mode of forming the quantities sought is such that it is not necessary to carry any coefficient to a greater degree of approximation than is necessary for obtaining its square to the necessary order of accuracy, with the single exception of the coefficient of $\cos g'$ in $\frac{a}{r}$. All the coefficients exceeding the third order are therefore omitted, and those retained are only given to such a degree of approximation that the sum of the lowest and highest orders of the terms of each separate coefficient shall be 6 or 7, according to circumstances. This remark applies to all the following expressions for the Moon's coordinates.

The several powers of $\frac{a}{r} - 1$ to the sixth, inclusive, being formed, they are combined according to the preceding formulæ to obtain the value of $\frac{r}{a}$, the coefficients for which are given in the next table.

The value of β, or the Moon's latitude, as given by DELAUNAY, follows. The several powers of β being formed, sin β and cos β are obtained by the formulæ

$$\sin \beta = \beta - \frac{1}{6}\,\beta^3 + \frac{1}{120}\,\beta^5,$$

$$\cos \beta = 1 - \frac{1}{2}\,\beta^2 + \frac{1}{24}\,\beta^4 - \frac{1}{720}\,\beta^6,$$

for which the expressions are given.

These quantities are then multiplied by $\frac{r}{a}$, and the products are given in the next two tables. The expressions for $\cos \delta v$ and $\sin \delta v$ are formed in the same way from δv, and their products by $\frac{r}{a} \cos \beta$ are given. We thus have the coefficients of $\cos l$ and $\sin l$ in the expressions (48) for ξ and η, the rectangular coordinates of the Moon referred to the ecliptic.

cos g,	g',	λ,	λ'	$\frac{a}{r} = 1 +$
0	0	0	0	$\frac{1}{6} m^2 + \frac{1}{4} e'^2 m^2 - \frac{179}{288} m^4 - \frac{97}{48} m^5 + \frac{1}{16} e^4 m^2 + \frac{5}{16} e'^4 m^2 + 2 e^2 \gamma^2 m^2 - \gamma^4 m^2 + \frac{3}{16} \alpha^2 m^2 - \frac{799}{192} e'^2 m^4 - \frac{757}{162} m^6 - \frac{873}{32} e'^2 m^5 - \frac{4039}{432} m^7$
1	0	0	0	$e - \frac{1}{8} e^3 - \frac{7}{12} e m^2 - \frac{285}{64} e m^3 + \frac{5}{2} e \gamma^4 - \frac{5}{4} e^3 \gamma^2 + \frac{1}{192} e^5 + \frac{19}{32} e \gamma^2 m^2 + \frac{19}{96} e^3 m^2 - \frac{7}{8} e e'^2 m^2 - \frac{45091}{2304} e m^4$
2	0	0	0	$e^2 - \frac{1}{3} e^4 - \frac{5}{6} e^2 m^2$
3	0	0	0	$\frac{9}{8} e^3$
−1	1	0	0	$\frac{21}{8} e e' m$
0	1	0	0	$-\frac{3}{2} e' m^2 + \frac{449}{16} e' m^4$
1	1	0	0	$-\frac{21}{8} e e' m$
−1	0	2	0	$-\frac{5}{2} e \gamma^2$
0	0	2	0	$-5 e^2 \gamma^2$
−2	0	2	−2	$-\frac{15}{14} e^2 m^2$
−1	0	2	−2	$\frac{15}{8} e m + \frac{187}{32} e m^2 - \frac{15}{4} e \gamma^2 m - \frac{75}{16} e e'^2 m + \frac{29513}{1536} e m^3$
0	0	2	−2	$m^2 + \frac{15}{4} e^2 m + \frac{19}{6} m^3 + \frac{189}{16} e^2 m^2 - \frac{5}{2} e'^2 m^2 - 2 \gamma^2 m^2 + \frac{131}{18} m^4 - \frac{239}{12} e'^2 m^3 + \frac{383}{27} m^5$

cos g, g', λ, λ'	$\frac{a}{r}=1+$
1 0 2 −2	$\frac{33}{16} e m^2$
−1 −1 2 −2	$\frac{35}{8} e e' m$
0 −1 2 −2	$\frac{7}{2} e' m^2 + \frac{157}{8} e' m^3$
−1 1 2 −2	$-\frac{15}{8} e e' m$
0 1 2 −2	$-\frac{1}{2} e' m^2 - \frac{91}{24} e' m^3$
0 1 1 −1	$\frac{5}{4} e' \alpha$
0 0 1 −1	$-\frac{15}{16} \alpha m$

From this we obtain

cos g, g', λ, λ'	$\frac{r}{a}=1+$
0 0 0 0	$\frac{1}{2} e^2 - \frac{1}{6} m^2 + \frac{355}{384} e^2 m^2 - \frac{1}{4} e'^2 m^2 + \frac{331}{288} m^4 + \frac{1905}{256} e^2 m^3 + \frac{83}{16} m^5 - \frac{5}{4} e^4 \gamma^2$ $+ \frac{45}{8} e^2 \gamma^4 + \frac{25}{32} e'^2 \alpha^2 + \frac{225}{128} e^4 m^2 - \frac{5}{16} e'^4 m^2 + \gamma^4 m^2$ $+ \frac{1047}{128} e^2 e'^2 m^2 - \frac{135}{16} e^2 \gamma^2 m^2 + \frac{129}{512} \alpha^2 m^2 + \frac{1\,499\,551}{36\,864} e^2 m^4$ $+ \frac{1751}{192} e'^2 m^4 - 2 \gamma^2 m^4 + \frac{42\,775}{2592} m^6 + \frac{2241}{32} e'^2 m^5 + \frac{4787}{108} m^7$
1 0 0 0	$-e + \frac{3}{8} e^3 + \frac{11}{12} e m^2 + \frac{405}{64} e m^3 - \frac{5}{192} e^5 + \frac{5}{2} e \gamma^4 + \frac{5}{4} e^3 \gamma^2 + \frac{181}{128} e^3 m^2$ $+ \frac{11}{8} e e'^2 m^2 - \frac{19}{32} e \gamma^2 m^2 + \frac{70\,027}{2304} e m^4$
2 0 0 0	$-\frac{1}{2} e^2 + \frac{1}{3} e^4 + \frac{1}{3} e^2 m^2$
3 0 0 0	$-\frac{3}{8} e^3$

cos g, g', λ, λ'	$\frac{r}{a}$ (continued).
$-1\ \ 1\ \ 0\ \ 0$	$-\frac{21}{8}ee'm$
$0\ \ 1\ \ 0\ \ 0$	$\frac{3}{2}e'm^2-\frac{409}{16}e'm^4$
$1\ \ 1\ \ 0\ \ 0$	$\frac{21}{8}ee'm$
$-1\ \ 0\ \ 2\ \ 0$	$\frac{5}{2}e\gamma^2$
$0\ \ 0\ \ 2\ \ 0$	$\frac{5}{2}e^2\gamma^2$
$-2\ \ 0\ \ 2\ -2$	$\frac{15}{8}e^2m$
$-1\ \ 0\ \ 2\ -2$	$-\frac{15}{8}em-\frac{155}{32}em^2+\frac{15}{16}e^3m+\frac{75}{16}ee'^2m+\frac{15}{4}e\gamma^2m-\frac{23\,689}{1536}em^3$
$0\ \ 0\ \ 2\ -2$	$-m^2-\frac{15}{8}e^2m-\frac{19}{6}m^3-\frac{173}{32}e^2m^2+\frac{5}{2}e'^2m^2+2\gamma^2m^2-\frac{125}{18}m^4$ $+\frac{239}{12}e'^2m^3-\frac{709}{54}m^5$
$1\ \ 0\ \ 2\ -2$	$-\frac{17}{16}em^2$
$-1\ -1\ \ 2\ -2$	$-\frac{35}{8}ee'm$
$0\ -1\ \ 2\ -2$	$-\frac{7}{2}e'm^2-\frac{157}{8}e'm^3$
$-1\ \ 1\ \ 2\ -2$	$\frac{15}{8}ee'm$
$0\ \ 1\ \ 2\ -2$	$\frac{1}{2}e'm^2+\frac{91}{24}e'm^3$
$0\ \ 0\ \ 1\ -1$	$\frac{15}{16}\alpha m$
$0\ \ 1\ \ 1\ -1$	$-\frac{5}{4}e'\alpha$

We have from DELAUNAY

sin $g,\ g',\ \lambda,\ \lambda'$	$\beta = \gamma \times$
$-2\quad 0\quad 1\quad 0$	$-\frac{3}{2}e^2$
$-1\quad 0\quad 1\quad 0$	$-2e+\frac{5}{4}e^3-5e\gamma^2+\frac{189}{32}em^2$
$0\quad 0\quad 1\quad 0$	$2-2e^2+\frac{7}{32}e^4-\frac{1}{4}\gamma^4$
$1\quad 0\quad 1\quad 0$	$2e-\frac{5}{2}e^3-\frac{1}{2}em^2$
$2\quad 0\quad 1\quad 0$	$\frac{9}{4}e^2$
$0\ -1\quad 1\quad 0$	$\frac{3}{4}e'm$
$0\quad 1\quad 1\quad 0$	$-\frac{3}{4}e'm$
$0\quad 0\quad 3\quad 0$	$-\frac{1}{3}\gamma^2$
$-1\quad 0\quad 3\ -2$	$\frac{15}{4}em$
$0\quad 0\quad 3\ -2$	$\frac{11}{8}m^2$
$0\ -1\quad 1\ -2$	$\frac{7}{4}e'm$
$-1\quad 0\quad 1\ -2$	$3em$
$0\quad 0\quad 1\ -2$	$\frac{3}{4}m+\frac{25}{16}m^2+\frac{27}{16}e^2m-\frac{15}{8}e'^2m+\frac{9}{8}\gamma^2m+\frac{29\text{[illegible]}7}{768}m^3$
$1\quad 0\quad 1\ -2$	$\frac{3}{4}em$
$0\quad 1\quad 1\ -2$	$-\frac{3}{4}e'm$

From this we obtain

sin g	g'	λ	λ'	$\sin\beta =$
-2	0	1	0	$-\frac{3}{2}e^2\gamma$
-1	0	1	0	$-2e\gamma-4e\gamma^3+\frac{5}{4}e^3\gamma+\frac{189}{32}e\gamma m^2$
0	0	1	0	$2\gamma-2e^2\gamma-\gamma^3+\frac{7}{32}e^4\gamma+e^2\gamma^3-\frac{1}{4}\gamma^5-\frac{9}{32}\gamma^3m^2$
1	0	1	0	$2e\gamma-\frac{5}{2}e^3\gamma-e\gamma^3-\frac{1}{2}e\gamma m^2$
2	0	1	0	$\frac{9}{4}e^2\gamma$
0	-1	1	0	$\frac{3}{4}e'\gamma m$
0	1	1	0	$-\frac{3}{4}e'\gamma m$
-1	0	3	-2	$\frac{15}{4}e\gamma m$
0	0	3	-2	$\frac{11}{8}\gamma m^2$
0	-1	1	-2	$\frac{7}{4}e'\gamma m$
-1	0	1	-2	$3e\gamma m$
0	0	1	-2	$\frac{3}{4}\gamma m+\frac{25}{16}\gamma m^2+\frac{27}{16}e^2\gamma m-\frac{15}{8}e'^2\gamma m+\frac{3}{8}\gamma^3 m+\frac{2957}{768}\gamma m^3$
1	0	1	-2	$\frac{3}{4}e\gamma m$
0	1	1	-2	$-\frac{3}{4}e'\gamma m$

cos g	g'	λ	λ'	$\cos\beta = 1+$
0	0	0	0	$-\gamma^2+\frac{1}{4}\gamma^4-\frac{9}{64}\gamma^2m^2-\frac{75}{128}\gamma^2m^3+\frac{45}{64}e\ \gamma^2-5e^2\gamma^4+\frac{1}{4}\gamma^6-\frac{17}{128}e^2\gamma^2m^2-\frac{31}{64}e'^2\gamma^2m^2-\frac{9}{32}\gamma^4m^2-\frac{5175}{2048}\gamma^2m^4$
1	0	0	0	$-\frac{5}{2}e^3\gamma^2+5e\gamma^4-\frac{109}{16}e\gamma^2m^2$

cos β (continued).

cos g	g'	λ	λ'	
2	0	0	0	$\frac{5}{4}e^2\gamma^2$
0	1	0	0	0
−2	0	2	0	$-\frac{1}{2}e^2\gamma^2$
−1	0	2	0	$-2e\gamma^2$
0	0	2	0	$\gamma^2-4e^2\gamma^2$
1	0	2	0	$2e\gamma^2$
2	0	2	0	$\frac{13}{4}e^2\gamma^2$
−1	0	2	−2	$-\frac{3}{2}e\gamma^2 m$
0	0	2	−2	$\frac{3}{4}\gamma^2 m+\frac{3}{16}\gamma^2 m^2$
1	0	2	−2	$\frac{3}{2}e\gamma^2 m$

$$\frac{r}{a}\sin\beta=$$

sin g	g'	λ	λ'	
−2	0	1	0	$-e^2\gamma$
−1	0	1	0	$-3e\gamma-6e\gamma^3+\frac{15}{8}e^3\gamma+\frac{503}{64}e\gamma m^2$
0	0	1	0	$2\gamma-e^2\gamma-\gamma^3-\frac{1}{3}\gamma m^2+\frac{3}{8}\gamma m^3-\frac{11}{32}e^4\gamma+3e^2\gamma^3-\frac{1}{4}\gamma^5-\frac{25}{48}e^2\gamma m^2$ $-\frac{1}{2}e'^2\gamma m^2-\frac{11}{96}\gamma^3 m^2+\frac{1031}{288}\gamma m^4$
1	0	1	0	$e\gamma-\frac{3}{4}e^3\gamma-\frac{1}{2}e\gamma^3+\frac{1}{12}e\gamma m^2$
2	0	1	0	$\frac{3}{4}e^2\gamma$
0	−1	1	0	$\frac{3}{4}e'\gamma m$

$\frac{r}{a}\sin\beta$ (continued).

sin g,	g',	λ	λ'	
0	1	1	0	$-\frac{3}{4}e'\gamma m$
−1	0	3	−2	$\frac{15}{8}e\gamma m$
0	0	3	−2	$\frac{3}{8}\gamma m^2$
0	−1	1	−2	$\frac{7}{4}e'\gamma m$
−1	0	1	−2	$\frac{9}{2}e\gamma m$
0	0	1	−2	$\frac{3}{4}\gamma m+\frac{41}{16}\gamma m^2+\frac{3}{8}\gamma^3 m+\frac{3}{16}e^2\gamma m-\frac{15}{8}e'^2\gamma m+\frac{5293}{768}\gamma m^3$
1	0	1	−2	$\frac{3}{8}e\gamma m$
0	1	1	−2	$-\frac{3}{4}e'\gamma m$

$\frac{r}{a}\cos\beta=1+$

cos g	g'	λ	λ'	
0	0	0	0	$\frac{1}{2}e^2-\gamma^2-\frac{1}{6}m^2-\frac{1}{2}e^2\gamma^2+\frac{1}{4}\gamma^4+\frac{355}{384}e^2m^2+\frac{5}{192}\gamma^2m^2-\frac{1}{4}e'^2m^2$ $+\frac{331}{288}m^4+\frac{1905}{256}e^2m^3-\frac{123}{128}\gamma^2m^3+\frac{83}{16}m^5+\frac{25}{64}e^4\gamma^2$ $-3e^2\gamma^4+\frac{1}{4}\gamma^6-\frac{2095}{384}e^2\gamma^2m^2+\frac{1047}{128}e^2e'^2m^2-\frac{15}{64}e'^2\gamma^2m^2$ $+\frac{225}{18}e^4m^2+\frac{65}{96}\gamma^4m^2-\frac{5}{16}e'^4m^2+\frac{1\,499\,551}{36\,864}e^2m^4$ $-\frac{127\,807}{18\,432}\gamma^2m^4+\frac{1751}{192}e'^2m^4+\frac{42\,775}{2592}m^6+\frac{2241}{32}e'^2m^5$ $+\frac{4787}{108}m^7+\frac{25}{32}e'^2\alpha^2+\frac{129}{512}\alpha^2m^2$
1	0	0	0	$-e+\frac{3}{8}e^3+e\gamma^2+\frac{11}{12}em^2+\frac{405}{64}em^3-\frac{5}{192}e^5-\frac{9}{4}e^3\gamma^2+\frac{7}{2}e\gamma^4$ $+\frac{181}{128}e^3m^2-\frac{853}{96}e\gamma^2m^2+\frac{11}{8}ee'^2m^2+\frac{70\,027}{2304}em^4$

cos $g,\ g',\ \lambda,\ \lambda'$	$\frac{r}{a}\cos\beta = 1 +$
2 0 0 0	$-\frac{1}{2}e^2+\frac{1}{3}e^4+\frac{7}{4}e^2\gamma^2+\frac{1}{3}e^2m^2$
3 0 0 0	$-\frac{3}{8}e^3$
−1 1 0 0	$-\frac{21}{8}ee'm$
0 1 0 0	$\frac{3}{2}e'm^2-\frac{409}{16}e'm^4$
1 1 0 0	$\frac{21}{8}ee'm$
−1 0 2 0	0
0 0 2 0	$\gamma^2-e^2\gamma^2-\frac{1}{6}\gamma^2m^2$
1 0 2 0	$\frac{3}{2}e\gamma^2$
−2 0 2 −2	$\frac{15}{8}e^2m$
−1 0 2 −2	$-\frac{15}{8}em-\frac{155}{32}em^2$
0 0 2 −2	$-m^2-\frac{15}{8}e^2m+\frac{3}{4}\gamma^2m-\frac{19}{6}m^3-\frac{173}{32}e^2m^2+\frac{51}{16}\gamma^2m^2+\frac{5}{2}e'^2m^2$ $-\frac{125}{18}m^4+\frac{239}{12}e'^2m^3-\frac{709}{54}m^5$
1 0 2 −2	$-\frac{17}{16}em^2$
−1 −1 2 −2	$-\frac{35}{8}ee'm$
0 −1 2 −2	$-\frac{7}{2}e'm^2-\frac{157}{8}e'm^3$
−1 1 2 −2	$\frac{15}{8}ee'm$
0 1 2 −2	$\frac{1}{2}e'm^2+\frac{91}{24}e'm^3$

cos g,	g',	λ,	λ'	$\frac{r}{a}\cos\beta$ (continued).
0	0	1	−1	$\frac{15}{16}\alpha m$
0	1	1	−1	$-\frac{5}{4}e'\alpha$
0	0	0	2	$-\frac{3}{4}\gamma^2 m$

We have also from Delaunay

sin g,	g',	λ,	λ'	$\delta v =$
1	0	0	0	$2e - \frac{1}{4}e^3 + \frac{5}{96}e^5$
2	0	0	0	$\frac{5}{4}e^2 - \frac{11}{24}e^4 - \frac{5}{4}e^2\gamma^2 - \frac{7}{16}e^2 m^2$
3	0	0	0	$\frac{13}{12}e^3$
−1	1	0	0	$-\frac{21}{4}e e' m$
0	1	0	0	$-3e'm - \frac{27}{8}e^2 e' m + \frac{27}{2}\gamma^2 e' m - \frac{27}{8}e'^3 m + \frac{735}{16}e' m^3 + \frac{1261}{4}e' m^4$
1	1	0	0	$-\frac{21}{4}e e' m$
0	2	0	0	$-\frac{9}{4}e'^2 m$
−1	0	2	0	$-3e\gamma^2$
0	0	2	0	$\gamma^2 - \frac{9}{4}e^2\gamma^2 - \gamma^4 + \frac{11}{4}\gamma^2 m^2$
1	0	2	0	$-2e\gamma^2$
−2	0	2	−2	$\frac{45}{16}e^2 m$
−1	0	2	−2	$\frac{15}{4}e m + \frac{263}{16}e m^2 - 6e\gamma^2 m - \frac{75}{8}e e'^2 m + \frac{48217}{768}e m^3$

sin g, g', λ, λ'	δv (continued).
0 0 2—2	$\frac{11}{8}m^2+\frac{75}{16}e^2 m-\frac{3}{4}\gamma^2 m+\frac{59}{12}m^3+\frac{1101}{64}e^2 m^2-\frac{47}{16}\gamma^2 m^2-\frac{55}{16}e'^2 m^2 +\frac{893}{72}m^4-\frac{691}{24}e'^2 m^3+\frac{2855}{108}m^5$
1 0 2—2	$\frac{17}{8}e m^2$
—1—1 2—2	$\frac{35}{4}e e' m$
0—1 2—2	$\frac{77}{16}e' m^2+\frac{479}{16}e' m^3$
—1 1 2—2	$-\frac{15}{4}e e' m$
0 1 2—2	$-\frac{11}{16}e' m^2-\frac{257}{48}e' m^3$
0 0 0 2	$-\frac{9}{4}\gamma^2 m$
0 1 1—1	$\frac{5}{2}e' \alpha$
0 0 1—1	$-\frac{15}{8}\alpha m$

From this we derive

cos g, g, λ, λ'	$\cos\delta v=1+$
0 0 0 0	$-e^2+\frac{7}{64}e^4-\frac{1}{4}\gamma^4-\frac{225}{64}e^2 m^2-\frac{9}{4}e'^2 m^2-\frac{121}{256}m^4-\frac{8715}{256}e^2 m^3+\frac{33}{64}\gamma^2 m -\frac{649}{192}m^5-\frac{5}{288}e^6+\frac{25}{32}e^4\gamma^2-\frac{33}{8}e^2\gamma^4-\frac{1}{2}\gamma^6-\frac{1885}{512}e^4 m^2 -\frac{1387}{64}e^2 e'^2 m^2-\frac{405}{64}e'^4 m^2+\frac{1665}{128}e^2\gamma^2 m^2+\frac{81}{4}e'^2\gamma^2 m^2 -\frac{1}{32}\gamma^4 m^2-\frac{428\,589}{2048}e^2 m^4+\frac{33\,465}{512}e'^2 m^4+\frac{989}{256}\gamma^2 m^4-\frac{1865}{128}m^6 -\frac{25}{16}e'^2\alpha^2-\frac{225}{256}\alpha^2 m^2+\frac{54\,687}{128}e'^2 m^5-\frac{21\,023}{432}m^7$

cos $g,\ g,\ \lambda,\ \lambda'$	$\cos\delta v$ (continued).
1 0 0 0	$-\frac{5}{4}e^3-\frac{165}{64}em^3+\frac{17}{48}e^5+\frac{5}{4}e^3\gamma^2-\frac{5}{2}e\gamma^4-\frac{109}{8}e^3m^2-\frac{63}{4}ee'^2m^2$ $+\frac{45}{32}e\gamma^2m^2-\frac{5627}{256}em^4$
2 0 0 0	$e^2-\frac{5}{3}e^4$
3 0 0 0	$\frac{5}{4}e^3$
−1 1 0 0	$3ee'm$
0 1 0 0	$-\frac{363}{128}e'm^4$
1 1 0 0	$-3ee'm$
−1 0 2 0	$e\gamma^2$
0 0 2 0	$-e^2\gamma^2$
1 0 2 0	$-e\gamma^2$
−2 0 2 −2	$-\frac{15}{4}e^2m$
−1 0 2 −2	$-\frac{11}{8}em^2-\frac{15}{8}e^3m-\frac{59}{12}em^3+\frac{3}{4}e\gamma^2m$
0 0 2 −2	$\frac{15}{4}e^2m+\frac{229}{16}e^2m^2-\frac{33}{4}e'^2m^3$
1 0 2 −2	$\frac{11}{8}em^2$
0 −1 2 −2	$\frac{35}{4}e^2e'm+\frac{33}{16}e'm^3$
0 1 2 −2	$-\frac{15}{4}e^2e'm-\frac{33}{16}e'm^3$
sin $g,\ g',\ \lambda,\ \lambda',$	$\sin\delta v=$
1 0 0 0	$2e-\frac{5}{4}e^3+\frac{17}{48}e^5-\frac{1}{2}e\gamma^4-\frac{225}{32}e^3m^2-\frac{9}{2}ee'^2m^2-\frac{121}{128}em^4$

sin g,	g',	λ,	λ'	$\sin \delta v =$
2	0	0	0	$\frac{5}{4}e^2 - \frac{41}{24}e^4 - \frac{5}{4}e^2\gamma^2 - \frac{7}{16}e^2m^2$
3	0	0	0	$\frac{17}{12}e^3$
-1	1	0	0	$-\frac{21}{4}ee'm$
0	1	0	0	$-3e'm - \frac{3}{8}e^2e'm + \frac{27}{2}\gamma^2e'm - \frac{27}{8}e'^3m - \frac{735}{16}e'm^3 + \frac{1261}{4}e'm^4$
1	1	0	0	$-\frac{21}{4}ee'm$
0	2	0	0	$-\frac{9}{4}e'^2m$
-1	0	2	0	$-3e\gamma^2$
0	0	2	0	$-\gamma^2 - \frac{5}{4}e^2\gamma^2 - \gamma^4 + \frac{11}{4}\gamma^2m^2$
1	0	2	0	$-2e\gamma^2$
-2	0	2	-2	$\frac{45}{16}e^2m$
-1	0	2	-2	$\frac{15}{4}em + \frac{263}{16}em^2 - \frac{15}{4}e^3m - 6e\gamma^2m - \frac{75}{8}ee'^2m + \frac{48217}{768}em^3$
0	0	2	-2	$\frac{11}{8}m^2 + \frac{75}{16}e^2m - \frac{3}{4}\gamma^2m + \frac{59}{12}m^3 + \frac{1013}{64}e^2m^2 - \frac{47}{16}\gamma^2m^2 - \frac{55}{16}e'^2m^2$ $+ \frac{893}{72}m^4 - \frac{691}{24}e'^2m^3 + \frac{2855}{108}m^5$
1	0	2	-2	$\frac{17}{8}em^2$
-1	-1	2	-2	$\frac{35}{4}ee'm$
0	-1	2	-2	$\frac{77}{16}e'm^2 + \frac{479}{16}e'm^3$
-1	1	2	-2	$-\frac{15}{4}ee'm$

sin g, g', λ, λ'	$\sin \delta v =$
0 1 2 −2	$-\frac{11}{16} e' m^2 - \frac{257}{48} e' m^3$
0 0 0 2	$-\frac{9}{4} \gamma^2 m$
0 1 1 −1	$\frac{5}{2} e' \alpha$
0 0 1 −1	$-\frac{15}{8} \alpha m$

Multiplying by $\frac{r}{a} \cos \beta$ we find

cos g, g', λ, λ'	$\frac{r}{a} \cos \beta \cos \delta v = 1 +$
0 0 0 0	$-\frac{1}{2} e^2 - \gamma^2 - \frac{1}{6} m^2 - \frac{1}{64} e^4 + \frac{1}{2} e^2 \gamma^2 - \frac{931}{384} e^2 m^2 + \frac{5}{192} \gamma^2 m^2 - \frac{5}{2} e'^2 m^2$ $+ \frac{1559}{2304} m^4 - \frac{3315}{128} e^2 m^3 - \frac{57}{128} \gamma^2 m^3 + \frac{347}{192} m^5 - \frac{29}{1152} e^6$ $+ \frac{19}{16} e^4 \gamma^2 - \frac{15}{2} e^2 \gamma^4 - \frac{5363}{1536} e^4 m^2 + \frac{265}{24} e^2 \gamma^2 m^2 - \frac{425}{64} e'^4 m^2$ $- \frac{1839}{128} e^2 e'^2 m^2 + \frac{11}{16} \gamma^2 m^2 + \frac{1425}{64} e'^2 \gamma^2 m^2 - \frac{6\,056\,323}{36\,864} e^2 m^4$ $- \frac{47\,887}{18\,432} \gamma^2 m^4 + \frac{114\,979}{1536} e'^2 m^4 + \frac{83\,407}{41\,472} m^6 - \frac{25}{32} e'^2 \alpha^2 - \frac{321}{512} \alpha^2 m^2$ $+ \frac{63\,651}{128} e'^2 m^5 - \frac{4351}{1152} m^7$
1 0 0 0	$-e - \frac{3}{8} e^3 + e\gamma^2 + \frac{11}{12} e m^2 + \frac{15}{4} e m^3 + \frac{5}{96} e^5 - \frac{1}{4} e^3 \gamma^2 + \frac{5}{4} e \gamma^4$ $- \frac{1145}{128} e^3 m^2 - \frac{359}{48} e \gamma^2 m^2 - \frac{97}{8} e e'^2 m^2 + \frac{20\,473}{2304} e m^4$
2 0 0 0	$\frac{1}{2} e^2 - \frac{1}{3} e^4 + \frac{3}{4} e^2 \gamma^2 + \frac{1}{6} e^2 m^2$
3 0 0 0	$\frac{3}{8} e^3$
−1 1 0 0	$\frac{3}{8} e e' m$
0 1 0 0	$\frac{3}{2} e' m^2 - \frac{3635}{128} e' m^4$

cos g, g', λ, λ'	$\frac{r}{a}\cos\beta\cos\delta v = 1 +$
1 1 0 0	$-\frac{3}{8}ee'm$
−1 0 2 0	$e\gamma^2$
0 0 2 0	$\gamma^2 - 3e^2\gamma^2 - \frac{1}{6}\gamma^2 m^2$
1 0 2 0	$\frac{1}{2}e\gamma^2$
−2 0 2 −2	$-\frac{15}{8}e^2 m$
−1 0 2 −2	$-\frac{15}{8}em - \frac{199}{32}em^2 + \frac{9}{2}e\gamma^2 m + \frac{75}{16}ee'^2 m + \frac{15}{16}e^3 m - \frac{31241}{1536}em^3$
0 0 2 −2	$-m^2 + \frac{15}{8}e^2 m + \frac{3}{4}\gamma^2 m - \frac{19}{6}m^3 + \frac{317}{32}e^2 m^2 + \frac{51}{16}\gamma^2 m^2 + \frac{5}{2}e'^2 m^2$ $-\frac{125}{18}m^4 + \frac{35}{3}e'^2 m^3 - \frac{709}{54}m^5$
1 0 2 −2	$\frac{5}{16}em^2$
−1 −1 2 −2	$-\frac{35}{8}ee'm$
0 −1 2 −2	$-\frac{7}{2}e'm^2 - \frac{281}{16}e'm^3$
−1 1 2 −2	$\frac{15}{8}ee'm$
0 1 2 −2	$\frac{1}{2}e'm^2 + \frac{83}{48}e'm^3$
0 0 0 2	$-\frac{3}{4}\gamma^2 m$
0 0 1 −1	$\frac{15}{16}\alpha m$
0 1 1 −1	$-\frac{5}{4}e'\alpha$

sin g	g'	λ	λ'	$\frac{r_*}{a}\cos\beta\sin\delta v=$
1	0	0	0	$2e-\frac{3}{8}e^3-2e\gamma^2-\frac{1}{3}em^2+\frac{75}{128}em^3+\frac{5}{96}e^5-\frac{1}{4}e^3\gamma^2+\frac{5}{4}e\gamma^4$ $+\frac{97}{128}e^3m^2-\frac{125}{192}e\gamma^2m^2+\frac{23}{8}ee'^2m^2+\frac{33\,353}{4608}em^4$
2	0	0	0	$\frac{1}{4}e^2-\frac{15}{12}e^4-\frac{3}{2}e^2\gamma^2+\frac{13}{48}e^2m^2$
3	0	0	0	$\frac{7}{24}e^3$
−1	1	0	0	$-\frac{15}{4}ee'm$
0	1	0	0	$-3e'm-\frac{15}{8}e^2e'm+\frac{33}{2}\gamma^2e'm-\frac{27}{8}e'^3m+\frac{743}{16}e'm^3+\frac{1261}{4}e'm^4$
1	1	0	0	$-\frac{15}{4}ee'm$
0	2	0	0	$-\frac{9}{4}e'^2m$
−1	0	2	0	$-\frac{7}{2}e\gamma^2$
0	0	2	0	$-\gamma^2-\frac{3}{4}e^2\gamma^2+\frac{35}{12}\gamma^2m^2$
1	0	2	0	$-\frac{1}{2}e\gamma^2$
−2	0	2	−2	$\frac{45}{16}e^2m$
−1	0	2	−2	$\frac{15}{4}em+\frac{67}{4}em^2-\frac{15}{8}e^3m-\frac{81}{8}e\gamma^2m-\frac{75}{8}ee'^2m+\frac{48\,281}{768}em^3$
0	0	2	−2	$\frac{11}{8}m^2+\frac{15}{16}e^2m-\frac{3}{4}\gamma^2m+\frac{59}{12}m^3+\frac{221}{64}e^2m^2-\frac{69}{16}\gamma^2m^2-\frac{55}{16}e'^2m^2$ $+\frac{1753}{144}m^4-\frac{547}{24}e'^2m^3+\frac{5533}{216}m^5$
1	0	2	−2	$\frac{7}{16}em^2$
−1	−1	2	−2	$\frac{35}{4}ee'm$

sin g, g', λ, λ'	$\frac{r}{a}\cos\beta\sin\delta v$ (continued).
0 —1 2 —2	$\frac{77}{16}e'm^2+\frac{455}{16}e'm^3$
—1 1 2 —2	$-\frac{15}{4}ee'm$
0 1 2 —2	$-\frac{11}{16}e'm^2-\frac{185}{48}e'm^3$
0 0 0 2	$-\frac{9}{4}\gamma^2 m$
0 1 1 —1	$\frac{5}{2}e'\alpha$
0 0 1 —1	$-\frac{15}{8}\alpha m$

The values of k_e, k_π, etc., may be formed from these expressions without going through the process of forming the rectangular coordinates x, y, z, in full. Let us consider any term in the value of $\frac{r}{a}\cos\beta\cos\delta v$, the corresponding term in the value of $\frac{r}{a}\cos\beta\sin\delta v$, and any term whatever in $\frac{r}{a}\sin\beta$. Let these terms be

$$\frac{r}{a}\cos\beta\cos\delta v=k\cos \mathrm{N}$$

$$\frac{r}{a}\cos\beta\sin\delta v=k'\sin \mathrm{N}$$

$$\frac{r}{a}\sin\beta \qquad =c\sin \mathrm{N}'$$

k and k' representing the several coefficients given on the preceding pages. If we substitute these values in the second members of (48) p. 129, we find

$$\frac{\xi}{a}=\frac{1}{2}(k+k')\cos(l+\mathrm{N})+\frac{1}{2}(k-k')\cos(l-\mathrm{N})$$

$$\frac{\eta}{a}=\frac{1}{2}(k+k')\sin(l+\mathrm{N})+\frac{1}{2}(k-k')\sin(l-\mathrm{N})$$

$$\frac{\zeta}{a}=c\sin\mathrm{N}'$$

The general expressions for the rectangular coordinates of the Moon, referred to any system of axes having their origin at the center of the Earth are

$$x = \mathrm{a}\,\xi + \mathrm{b}\,\eta + \mathrm{c}\,\zeta$$
$$y = \mathrm{a}'\,\xi + \mathrm{b}'\,\eta + \mathrm{c}'\,\zeta$$
$$z = \mathrm{a}''\,\xi + \mathrm{b}''\,\eta + \mathrm{c}''\,\zeta$$

the direction cosines a, b, . . . having the values given in (51) p. 131. If we put

$$\left.\begin{aligned} k_1 &= \frac{1}{2}(k+k') \\ k_2 &= \frac{1}{2}(k-k') \end{aligned}\right\} \qquad (78)$$

we find

$$\begin{aligned} \frac{x}{a} &= k_1(1-\gamma'^2)\cos(\theta'-\tau+l+\mathrm{N}) \\ &+k_2(1-\gamma'^2)\cos(\theta'-\tau+l-\mathrm{N}) \\ &+k_1\gamma'^2\cos(\theta'+\tau-l-\mathrm{N}) \\ &+k_2\gamma'^2\cos(\theta'+\tau-l+\mathrm{N}) \\ &-c\gamma'\sqrt{1-\gamma'^2}\cos(\theta'+\mathrm{N}') \\ &+c\gamma'\sqrt{1-\gamma'^2}\cos(\theta'-\mathrm{N}') \end{aligned} \qquad (79)$$

$$\begin{aligned} \frac{y}{a} &= k_1(1-\gamma'^2)\sin(\theta'-\tau+l+\mathrm{N}) \\ &+k_2(1-\gamma'^2)\sin(\theta'-\tau+l-\mathrm{N}) \\ &+k_1\gamma'^2\sin(\theta'+\tau-l-\mathrm{N}) \\ &+k_2\gamma'^2\sin(\theta'+\tau-l+\mathrm{N}) \\ &-c\gamma'\sqrt{1-\gamma'^2}\sin(\theta'+\mathrm{N}') \\ &+c\gamma'\sqrt{1-\gamma'^2}\sin(\theta'-\mathrm{N}') \end{aligned}$$

$$\begin{aligned} \frac{z}{a} &= 2\,k_1\gamma'\sqrt{1-\gamma'^2}\sin(l-\tau+\mathrm{N}) \\ &+2\,k_2\gamma'\sqrt{1-\gamma'^2}\sin(l-\tau-\mathrm{N}) \\ &+c(1-2\,\gamma'^2)\sin\mathrm{N}' \end{aligned}$$

Representing the coefficients of g, g', λ, λ' in N by i, i', j, j', respectively, we have

$$\left.\begin{aligned} \mathrm{N} &= i\,g + i'\,g' + j\,\lambda + j'\,\lambda' \\ \mathrm{N}' &= i_1\,g + i'_1\,g' + j_1\,\lambda + j'_1\,\lambda' \end{aligned}\right\} \qquad (80)$$

Substituting these values of N and N′, after effecting the substitutions (53) to reduce the expressions to the form (54), remembering that

$$g = \lambda - \omega = \varepsilon - \pi$$
$$g' = \lambda' - \omega' = \varepsilon' - \pi'$$

the values of x and z become

$$
\begin{aligned}
\frac{x}{a}=&k_1(1-\gamma'^2)\cos\{(1+i+j)\,\varepsilon-i\,\pi-(j+j')\,\theta+(i'+j')\,\varepsilon'-i'\,\pi'\}\\
&+k_2(1-\gamma'^2)\cos\{(1-i-j)\,\varepsilon+i\,\pi+(j+j')\,\theta-(i'+j')\,\varepsilon'+i'\,\pi'\}\\
&+k_1\gamma'^2\cos\{-(1+i+j)\,\varepsilon+i\,\pi+(j+j')\,\theta-(i'+j')\,\varepsilon'+i'\,\pi'+2\,\theta'\}\\
&+k_2\gamma'^2\cos\{-(1-i-j)\,\varepsilon-i\,\pi-(j+j')\,\theta+(i'+j')\,\varepsilon'-i'\,\pi'+2\,\theta'\}\\
&-c\,\gamma'\sqrt{1-\gamma'^2}\cos\{(i_1+j_1)\,\varepsilon-i_1\,\pi-(j_1+j'_1)\,\theta+(i'_1+j'_1)\,\varepsilon'-i'_1\,\pi'+\theta'\}\\
&+c\,\gamma'\sqrt{1-\gamma'^2}\cos\{-(i_1+j_1)\,\varepsilon+i_1\,\pi+(j_1+j'_1)\,\theta-(i'_1+j'_1)\,\varepsilon'+i'_1\,\pi'+\theta'\}\\
\frac{z}{a}=&2\,k_1\gamma'\sqrt{1-\gamma'^2}\sin\{(1+i+j)\,\varepsilon-i\,\pi-(j+j')\,\theta+(i'+j')\,\varepsilon'-i'\,\pi'-\theta'\}\\
&+2\,k_2\gamma'\sqrt{1-\gamma'^2}\sin\{(1-i-j)\,\varepsilon+i\,\pi+(j+j')\,\theta-(i'+j')\,\varepsilon'+i'\,\pi'-\theta'\}\\
&+c\,(1-2\,\gamma'^2)\sin\{(i_1+j_1)\,\varepsilon-i_1\,\pi-(j_1+j'_1)\,\theta+(i'_1+j'_1)\,\varepsilon'-i'_1\,\pi'\}
\end{aligned}
\tag{81}
$$

It is unnecessary to write the value of y, as it is the same with that of x, except that cos is replaced by sin. The entire values of x, y, and z will consist of the sum of an infinite series of terms like the above, formed by the successive substitution of each of the different values of N in the general expressions (79), each term of N giving rise to four terms in x, and each term of N$'$ to two. Should the argument be the same in any two of the terms thus originating they must be combined into one. But it is easy to see that this can not be the case. For, as already shown (§ 8 p. 137), we must then have the coefficients of ε, π, θ, ε', π', and θ' the same in both terms. Now, let us represent any one system of values of i, i', j, j' by

$$i,\ i',\ j,\ j'=s_1,\ s_2,\ s_3,\ s_4,$$

and any other system by

$$i,\ i',\ j,\ j'=t_1,\ t_2,\ t_3,\ t_4.$$

In order that the substitution of these two systems of values in (81) should give the same coefficients of ε, π, etc., in any two terms we must have

$$
\begin{gathered}
s_1+s_3=\pm(t_1+t_3)\\
s_1=\pm t_1\\
s_3+s_4=\pm(t_3+t_4)\\
s_2=\pm t_2
\end{gathered}
$$

the upper signs alone or the lower signs alone being used.

But these equations give

$$s_1=\pm t_1;\quad s_2=\pm t_2;\quad s_3=\pm t_3;\quad s_4=\pm t_4$$

and consequently

$$\mathrm{N}_1=\pm\mathrm{N}_2$$

But, all the pairs of terms in which the values of N were the same, with opposite signs only, have been combined into one as we went along, so that there are none fulfilling this last condition, as we may easily see by simple inspection of the indices corresponding to the preceding values of k and k'.

Let us represent that part of k_ϵ, k_π, etc., formed from the expressions of the coordinates of the Moon by

$$a\mu_2 h_\epsilon; \quad a\mu_2 h_\pi \quad \text{etc.} \tag{82}$$

Let us also put

p, the coefficient of the time in $i\,g+i'\,g'+j\,\lambda+j'\,\lambda'$, or in N

p_1 that in N′

so that the coefficients of the time in the first four terms of x and y, formerly represented by b, are now $\pm n \pm p$. Referring to the formulæ (62) we see that each value of N and of N′ in the preceding tables gives rise to the following terms in h_ϵ

$$\begin{aligned}
&(1+i+j)\,(n+p)\,k_1^2\,(1-\gamma'^2)^2\\
&+(1-i-j)\,(n-p)\,k_2^2\,(1-\gamma'^2)^2\\
&+(1+i+j)\,(n+p)\,k_1^2\,\gamma'^4\\
&+(1-i-j)\,(n-p)\,k_2^2\,\gamma'^4\\
&+2\,(i_1+j_1)\,p_1\,c^2\,\gamma'^2\,(1-\gamma'^2)\\
&+2\,(1+i+j)\,(n+p)\,k_1^2\,\gamma'^2\,(1-\gamma'^2)\\
&+2\,(1-i-j)\,(n-p)\,k_2^2\,\gamma'^2\,(1-\gamma'^2)\\
&+\frac{1}{2}\,(i_1+j_1)\,p_1\,c^2\,(1-2\,\gamma'^2)^2
\end{aligned}$$

This expression reduces to

$$(1+i+j)\,(n+p)\,k_1^2+(1-i-j)\,(n-p)\,k_2^2+\frac{1}{2}\,(i_1+j_1)\,p_1\,c^2$$

The values of h_π, h_θ, etc., are formed in the same way by replacing $1+i+j$, $1-i-j$, etc., by the corresponding coefficients of π, θ, etc.

Completing the expressions by the sign Σ, which refers to the different systems of values of i, i', j, j', we have

$$\begin{aligned}
h_\epsilon &= \Sigma\left\{(1+i+j)\,(n+p)\,k_1^2+(1-i-j)\,(n-p)\,k_2^2+\frac{1}{2}\,(i_1+j_1)\,p_1\,c^2\right\}\\
h_\pi &= \Sigma\left\{-i\,(n+p)\,k_1^2+i\,(n-p)\,k_2^2-\frac{1}{2}\,i_1\,p_1\,c^2\right\}\\
h_\theta &= \Sigma\left\{-(j+j')\,(n+p)\,k_1^2+(j+j')\,(n-p)\,k_2^2-\frac{1}{2}\,(j_1+j'_1)\,p_1\,c^2\right\}\\
h'_\epsilon &= \Sigma\left\{(i'+j')\,(n+p)\,k_1^2-(i'+j')\,(n-p)\,k_2^2+\frac{1}{2}\,(i'_1+j'_1)\,p_1 c^2\right\}\\
h'_\pi &= \Sigma\left\{-i'\,(n+p)\,k_1^2-i'\,(n-p)\,k_2^2-\frac{1}{2}\,i'_1\,p_1\,c^2\right\}\\
h'_\theta &= \Sigma\left\{-2\,(n+p)\,k_1^2-2\,(n-p)\,k_2^2\right\}\gamma'^2
\end{aligned} \tag{82}$$

We now give the expressions of k_1^2, k_2^2, c^2, and p, corresponding to each term of the expressions for $\frac{r}{a} \cos \beta \cos \delta v$, $\frac{r}{a} \cos \beta \sin \delta v$, and $r \sin \beta$, except the first, or constant term of $\frac{r}{a} \cos \beta \cos \delta v$, to which the above expression does not apply. If we consider this term alone, and call it k_0 we shall have

$$h_\epsilon = n\, k_0^2$$

while all the other values of h will vanish. The formation of k_0^2, k_1^2, k_2^2, is too simple to render any explanation necessary. To form the values of p, it is necessary to have the expressions for the secular motion of the perigee and node of the Moon, or of its mean anomaly and argument of latitude, under the action of the Sun, in terms of the elements of the Sun and Moon. These are obtained from Tome II, p. 237, of DELAUNAY's Theory, by making the change of elements indicated on p. 800.

We have

$$\frac{d\,g}{d\,t} = n \left\{ 1 - \frac{3}{4} m^2 - \frac{225}{32} m^3 + \frac{3}{8} e^2 m^2 - \frac{9}{8} e'^2 m^2 + 6 \gamma^2 m^2 - \frac{4071}{128} m^4 \right\}$$

$$\frac{d\,g'}{d\,t} = m n \tag{83}$$

$$\frac{d\,\lambda}{d\,t} = n \left\{ 1 + \frac{3}{4} m^2 - \frac{9}{32} m^3 + \frac{3}{2} e^2 m^2 + \frac{9}{8} e'^2 m^2 - \frac{3}{2} \gamma^2 m^2 - \frac{273}{128} m^4 \right\}$$

$$\frac{d\,\lambda'}{d\,t} = n\,(m - 1) + \frac{d\,\lambda}{d\,t}$$

Thus arise the following values of the quantities which enter into the equations (82).

i, i', j, j'	
0 0 0 0	$k_0^2 = 1 - e^2 - 2\gamma^2 - \frac{1}{3} m^2 + \frac{7}{32} e^4 + 2 e^2 \gamma^2 + \gamma^4 - \frac{899}{192} e^2 m^2 + \frac{37}{96} \gamma^2 m^2$ $- 5 e'^2 m^2 + \frac{1591}{1152} m^4 - \frac{3315}{64} e^2 m^3 - \frac{57}{64} \gamma^2 m^3 + \frac{347}{96} m^5 - \frac{5}{144} e^6$ $+ \frac{61}{32} e^4 \gamma^2 - 16 e^2 \gamma^4 - \frac{3497}{768} e^4 m^2 + \frac{127}{96} \gamma^4 m^2 - \frac{425}{32} e'^4 m^2$ $+ \frac{2567}{96} e^2 \gamma^2 m^2 - \frac{1679}{64} e^2 e'^2 m^2 + \frac{1585}{32} e'^2 \gamma^2 m^2 - \frac{6\,053\,899}{18\,432} e^2 m^4$ $- \frac{60\,439}{9216} \gamma^2 m^4 + \frac{115\,619}{768} e'^2 m^4 + \frac{39\,365}{10\,368} m^6 - \frac{25}{16} e'^2 \alpha^2$ $- \frac{321}{256} \alpha^2 m^2 + \frac{63\,651}{64} e'^2 m^5 - \frac{261}{32} m^7$
	$p = 0$

$i,$	$i',$	$j,$	j'	
1	0	0	0	$k_1^2=\frac{1}{4}e^2-\frac{3}{8}e^4-\frac{1}{2}e^2\gamma^2+\frac{7}{24}e^2m^2+\frac{555}{256}e^2m^3+\frac{37}{192}e^6+\frac{1}{8}e^4\gamma^2+\frac{3}{2}e^2\gamma^4-\frac{69}{16}e^4m^2-\frac{1673}{384}e^2\gamma^2m^2-\frac{37}{8}e^2e'^2m^2+\frac{75\,083}{9216}e^2m^4$
				$k_2^2=\frac{9}{4}e^2-\frac{9}{2}e^2\gamma^2-\frac{15}{8}e^2m^2-\frac{1215}{256}e^2m^3+\frac{9}{4}e^2\gamma^4+\frac{1863}{128}e^4m^2+\frac{1551}{128}e^2\gamma^2m^2+\frac{45}{2}e^2e'^2m^2-\frac{2131}{1024}e^2m^4$
				$p=n\left(1-\frac{3}{4}m^2-\frac{225}{32}m^3+\frac{3}{8}e^2m^2-\frac{9}{8}e'^2m^2+6\gamma^2m^2-\frac{4071}{128}m^4\right)$
2	0	0	0	$k_1^2=\frac{9}{64}e^4-\frac{9}{32}e^6-\frac{9}{32}e^4\gamma^2+\frac{21}{128}e^4m^2$
				$k_2^2=\frac{1}{64}e^4+\frac{1}{96}e^6+\frac{9}{32}e^4\gamma^2-\frac{5}{384}e^4m^2$
				$p=n\left(2-\frac{3}{2}m^2\right)$
3	0	0	0	$k_1^2=\frac{1}{9}e^6$
				$k_2^2=\frac{1}{576}e^6$
				$p=3n$
−1	1	0	0	$k_1^2=\frac{729}{256}e^2e'^2m^2$
				$k_2^2=\frac{1089}{256}e^2e'^2m^2$
				$p=-n$
0	1	0	0	$k_1^2=\frac{9}{4}e'^2m^2-\frac{9}{4}e'^2m^3+\frac{45}{16}e^2e'^2m^2-\frac{99}{4}e'^2\gamma^2m^2+\frac{81}{16}e'^4m^2-\frac{2211}{32}e'^2m^4-\frac{101\,235}{256}e'^2m^5$
				$k_2^2=\frac{9}{4}e'^2m^2+\frac{9}{4}e'^2m^3+\frac{45}{16}e^2e'^2m^2-\frac{99}{4}e'^2\gamma^2m^2+\frac{81}{16}e'^4m^2-\frac{2211}{32}e'^2m^4-\frac{140\,877}{256}e'^2m^5$
				$p=mn$

$i,$	$i',$	$j,$	j'	
1	1	0	0	$k_1^2 = \frac{1089}{256} e^2 e'^2 m^2$
				$k_2^2 = \frac{729}{256} e^2 e'^2 m^2$
				$p = n$
0	2	0	0	$k_1^2 = \frac{81}{64} e'^4 m^2$
				$k_2^2 = \frac{81}{64} e'^4 m^2$
				$p = 2\,m\,n$
-1	0	2	0	$k_1^2 = \frac{25}{16} e^2 \gamma^4$
				$k_2^2 = \frac{81}{16} e^2 \gamma^4$
				$p = n$
0	0	2	0	$k_1^2 = 0$
				$k_2^2 = \gamma^4 - \frac{9}{4} e^2 \gamma^4 - \frac{37}{12} \gamma^4 m^2$
				$p = n\left(2 + \frac{3}{2} m^2\right)$
1	0	2	0	$k_1^2 = 0$
				$k_2^2 = \frac{1}{4} e^2 \gamma^4$
				$p = 3\,n$
-2	0	2	-2	$k_1^2 = \frac{225}{1024} e^4 m^2$
				$k_2^2 = \frac{5625}{1024} e^4 m^2$
				$p = 0$
-1	0	2	-2	$k_1^2 = \frac{225}{256} e^2 m^2 + \frac{5055}{512} e^2 m^3 - \frac{225}{256} e^4 m^2 - \frac{675}{128} e^2 \gamma^2 m^2 - \frac{1125}{256} e^2 e'^2 m^2 + \frac{553743}{8192} e^2 m^4$

i	i'	j	j'	
-1	0	2	-2	$k_2^2=\frac{2025}{256}e^2m^2+\frac{33\,075}{512}e^2m^3-\frac{2025}{256}e^4m^2-\frac{5265}{128}e^2\gamma^2m^2$
				$-\frac{10\,125}{256}e^2e'^2m^2+\frac{2\,997\,495}{8192}e^2m^4$
				$p=n\left(1-2m+\frac{3}{4}m^2\right)$
0	0	2	-2	$k_1^2=\frac{9}{256}m^4+\frac{135}{256}e^2m^3+\frac{21}{64}m^5+\frac{2025}{1024}e^4m^2+\frac{5085}{1024}e^2m^4-\frac{27}{128}\gamma^2m^4$
				$-\frac{45}{256}e'^2m^4+\frac{447}{256}m^6-\frac{93}{32}e'^2m^5+\frac{83}{12}m^7$
				$k_2^2=\frac{361}{256}m^4-\frac{285}{256}e^2m^3-\frac{57}{32}\gamma^2m^3+\frac{1843}{192}m^5-\frac{11\,727}{1024}e^2m^4-\frac{479}{32}\gamma^2m^4$
				$-\frac{1805}{256}e'^2m^4+\frac{225}{1024}e^4m^2+\frac{45}{64}e^2\gamma^2m^2+\frac{9}{16}\gamma^4m^2+\frac{29\,981}{768}m^6$
				$-\frac{779}{12}e'^2m^5+\frac{106\,513}{864}m^7$
				$p=n(2-2m)$
1	0	2	-2	$k_1^2=\frac{9}{64}e^2m^4$
				$k_2^2=\frac{1}{256}e^2m^4$
				$p=3n$
-1	-1	2	-2	$k_1^2=\frac{1225}{256}e^2e'^2m^2$
				$k_2^2=\frac{11\,025}{256}e^2e'^2m^2$
				$p=n$
0	-1	2	-2	$k_1^2=\frac{441}{1024}e'^2m^4+\frac{1827}{256}e'^2m^5$
				$k_2^2=\frac{17\,689}{1024}e'^2m^4+\frac{3059}{16}e'^2m^5$
				$p=n(2-3m)$
-1	1	2	-2	$k_1^2=\frac{225}{256}e^2e'^2m^2$
				$k_2^2=\frac{2025}{256}e^2e'^2m^2$
				$p=n$

i	i'	j	j'	
0	1	2	−2	$k_1^2 = \frac{9}{1024} e'^2 m^4 + \frac{51}{256} e'^2 m^5$
				$k_2^2 = \frac{361}{1024} e'^2 m^4 + \frac{1273}{384} e'^2 m^5$
				$p = n(2 - m)$
0	0	0	2	$k_1^2 = \frac{9}{4} \gamma^4 m^2$
				$k_2^2 = \frac{9}{16} \gamma^4 m^2$
				$p = 0$
0	0	1	−1	$k_1^2 = \frac{225}{1024} \alpha^2 m^2$
				$k_2^2 = \frac{2025}{1024} \alpha^2 m^2$
				$p = n$
0	1	1	−1	$k_1^2 = \frac{25}{64} e'^2 \alpha^2$
				$k_2^2 = \frac{225}{64} e'^2 \alpha^2$
				$p = n$
−2	0	1	0	$c^2 = e^4 \gamma^2$
				$p_1 = -n$
−1	0	1	0	$c^2 = 9 e^2 \gamma^2 + 36 e^2 \gamma^4 - \frac{45}{4} e^4 \gamma^2 - \frac{1509}{32} e^2 \gamma^2 m^2$
				$p_1 = \frac{3}{2} m^2 n$
0	0	1	0	$c^2 = 4\gamma^2 - 4 e^2 \gamma^2 - 4\gamma^4 - \frac{4}{3} \gamma^2 m^2 + \frac{3}{2} \gamma^2 m^3 - \frac{3}{8} e^4 \gamma^2 + 14 e^2 \gamma^4 + \frac{5}{24} \gamma^4 m^2 - \frac{17}{12} e^2 \gamma^2 m^2 - 2 e'^2 \gamma^2 m^2 + \frac{1039}{72} \gamma^2 m^4$
				$p_1 = n \left(1 + \frac{3}{4} m^2 - \frac{9}{32} m^3 + \frac{3}{2} e^2 m^2 + \frac{9}{8} e'^2 m^2 - \frac{3}{2} \gamma^2 m^2 - \frac{273}{128} m^4\right)$

$i,$	$i',$	$j,$	j'	
1	0	1	0	$c^2 = e^2\gamma^2 - \frac{3}{2}e^4\gamma^2 - e^2\gamma^4 + \frac{1}{6}e^2\gamma^2 m^2$
				$p_1 = 2n$
2	0	1	0	$c^2 = \frac{9}{16}e^4\gamma^2$
				$p_1 = 3n$
0	−1	1	0	$c^2 = \frac{9}{16}e'^2\gamma^2 m^2$
				$p_1 = n$
0	1	1	0	$c^2 = \frac{9}{16}e'^2\gamma^2 m^2$
				$p_1 = n$
−1	0	3	−2	$c^2 = \frac{225}{64}e^2\gamma^2 m^2$
				$p_1 = 2n$
0	−1	1	−2	$c^2 = \frac{49}{16}e'^2\gamma^2 m^2$
				$p_1 = n$
−1	0	1	−2	$c^2 = \frac{81}{4}e^2\gamma^2 m^2$
				$p_1 = 0$
0	0	1	−2	$c^2 = \frac{9}{16}\gamma^2 m^2 + \frac{123}{32}\gamma^2 m^3 + \frac{9}{16}\gamma^4 m^2 + \frac{9}{32}e^2\gamma^2 m^2 - \frac{45}{16}e'^2\gamma^2 m^2$
				$+ \frac{8655}{512}\gamma^2 m^4$
				$p_1 = n\left(1 - 2m - \frac{3}{4}m^2\right)$
1	0	1	−2	$c^2 = \frac{9}{64}e^2\gamma^2 m^2$
				$p_1 = 2n$
0	1	1	−2	$c^2 = \frac{9}{16}e'^2\gamma^2 m^2$
				$p_1 = n$

From these expressions we form the values of $\frac{n+p}{n}k_1^2$ and $\frac{n-p}{n}k_2^2$, which, when multiplied by n enter into the expressions (82) for h_ϵ, h_π, etc. The results are shown in the following table. On the left margin are given the indices i, j, i', j', as heretofore; then follow the corresponding expressions for $\frac{n+p}{n}k_1^2$ and $\frac{n-p}{n}k_2^2$ which are given in the middle of the page. For brevity the former is represented by the symbol (1) and the latter by (2). On the right margin are given the coefficients by which these quantities are multiplied in the expressions for h_ϵ, h_π, h_θ, etc. To save space, each set of multipliers is arranged in two lines, and the several sets are separated by a dash.

Each term being multiplied by the corresponding coefficient, and the sums taken, and, in the case of h_ϵ, the value of k_0^2 being also included, we have the values of $\frac{h_\epsilon}{n}$, $\frac{h_\pi}{n}$, etc.

Terms arising from the Longitude.

$i,\ i',\ j,\ j'$		Coefficients for $\epsilon,\ \pi,\ \theta,$ $\epsilon',\ \pi',\ \theta'$
1 0 0 0	$(1)=\frac{1}{2}e^2-\frac{3}{4}e^4-e^2\gamma^2+\frac{19}{48}e^2m^2+\frac{165}{64}e^2m^3+\frac{37}{96}e^6$ $+\frac{1}{4}e^4\gamma^2+3e^2\gamma^4-\frac{33}{4}e^4m^2-\frac{1313}{192}e^2\gamma^2m^2$ $-\frac{305}{32}e^2e'^2m^2+\frac{9359}{1152}e^2m^4$	+2 −1 0; 0 0 −2
	$(2)=\frac{27}{16}e^2m^2-\frac{27}{32}e^4m^2+\frac{2025}{128}e^2m^3-\frac{135}{8}e^2\gamma^2m^2$ $+\frac{81}{32}e^2e'^2m^2+\frac{35\,919}{512}e^2m^4$	0 +1 0; 0 0 −2
2 0 0 0	$(1)=\frac{27}{64}e^4-\frac{27}{32}e^6-\frac{27}{32}e^4\gamma^2+\frac{9}{32}e^4m^2$	+3 −2 0; 0 0 −2
	$(2)=-\frac{1}{64}e^4-\frac{1}{96}e^6-\frac{9}{32}e^4\gamma^2+\frac{7}{192}e^4m^2$	−1 +2 0; 0 0 −2
3 0 0 0	$(1)=\frac{4}{9}e^6$	+4 −3 0; 0 0 −2
	$(2)=-\frac{1}{288}e^6$	−2 +3 0; 0 0 −2
−1 1 0 0	$(1)=0$	
	$(2)=\frac{1089}{128}e^2e'^2m^2$	+2 −1 0; −1 +1 −2

Terms arising from the Longitude—Continued.

$i,\ i',\ j,\ j'$		Coefficients for $\varepsilon,\ \pi,\ \theta,$ $\varepsilon',\ \pi',\ \theta'$
0 1 0 0	$(1)=\frac{9}{4}e'^2m^2+\frac{45}{16}e^2e'^2m^2-\frac{99}{4}e'^2\gamma^2m^2+\frac{81}{16}e'^4m^2$ $-\frac{2283}{32}e'^2m^4-\frac{118\,923}{256}e'^2m^5$	$+1\ \ 0\ \ 0$ $+1-1-2$
	$(2)=\frac{9}{4}e'^2m^2+\frac{45}{16}e^2e'^2m^2-\frac{99}{4}e'^2\gamma^2m^2+\frac{81}{16}e'^4m^2$ $-\frac{2283}{32}e'^2m^4-\frac{123\,189}{256}e'^2m^5$	$+1\ \ 0\ \ 0$ $-1+1-2$
1 1 0 0	$(1)=\frac{1089}{128}e^2e'^2m^2$	$+2-1\ \ 0$ $+1-1-2$
	$(2)=0$	
0 2 0 0	$(1)=\frac{81}{64}e'^4m^2$	$+1\ \ 0\ \ 0$ $+2-2-2$
	$(2)=\frac{81}{64}e'^4m^2$	$+1\ \ 0\ \ 0$ $-2+2-2$
−1 0 2 0	$(1)=\frac{25}{8}e^2\gamma^4$	$+2+1-2$ $0\ \ 0-2$
	$(2)=0$	
0 0 2 0	$(1)=0$	
	$(2)=-\gamma^4+\frac{9}{4}e^2\gamma^4+\frac{19}{12}\gamma^4m^2$	$-1\ \ 0+2$ $0\ \ 0-2$
1 0 2 0	$(1)=0$	
	$(2)=-\frac{1}{2}e^2\gamma^4$	$-2+1+2$ $0\ \ 0-2$
−2 0 2 −2	$(1)=\frac{225}{1024}e^4m^2$	$+1+2\ \ 0$ $-2\ \ 0-2$
	$(2)=\frac{5625}{1024}e^4m^2$	$+1-2\ \ 0$ $+2\ \ 0-2$

Terms arising from the Longitude—Continued.

$i,\ i',\ j,\ j'$		Coefficients for $\varepsilon,\ \pi,\ \theta,$ $\varepsilon',\ \pi',\ \theta'$
$-1\quad 0\quad 2-2$	$(1)=\frac{225}{128}e^2m^2+\frac{4605}{256}e^2m^3-\frac{225}{128}e^4m^2-\frac{675}{64}e^2\gamma^2m^2$ $-\frac{1125}{128}e^2e'^2m^2+\frac{475\,563}{4096}e^2m^4$	$+2+1\quad 0$ $-2\quad 0-2$
	$(2)=\frac{2025}{128}e^2m^3+\frac{126\,225}{1024}e^2m^4$	$0-1\quad 0$ $+2\quad 0-2$
$0\quad 0\quad 2-2$	$(1)=\frac{27}{256}m^4+\frac{405}{256}e^2m^3+\frac{117}{128}m^5+\frac{6075}{1024}e^4m^2$ $+\frac{14\,175}{1024}e^2m^4-\frac{81}{128}\gamma^2m^4-\frac{135}{256}e'^2m^4+\frac{1173}{256}m^6$ $-\frac{1071}{128}e'^2m^5+\frac{6627}{384}m^7$	$+3\quad 0\quad 0$ $-2\quad 0-2$
	$(2)=-\frac{361}{256}m^4+\frac{285}{256}e^2m^3+\frac{57}{32}\gamma^2m^3-\frac{2603}{384}m^5$ $-\frac{225}{1024}e^4m^2-\frac{45}{64}e^2\gamma^2m^2-\frac{9}{16}\gamma^4m^2+\frac{9447}{1024}e^2m^4$ $+\frac{365}{32}\gamma^2m^4+\frac{1805}{256}e'^2m^4-\frac{15\,237}{768}m^6$ $+\frac{19\,513}{384}e'^2m^5-\frac{156\,223}{3456}m^7$	$-1\quad 0\quad 0$ $+2\quad 0-2$
$1\quad 0\quad 2-2$	$(1)=\frac{9}{16}e^2m^4$	$+4-1\quad 0$ $-2\quad 0-2$
	$(2)=-\frac{1}{128}e^2m^4$	$-2+1\quad 0$ $+2\quad 0-2$
$-1-1\quad 2-2$	$(1)=\frac{1225}{128}e^2e'^2m^2$	$+2+1\quad 0$ $-3+1-2$
	$(2)=0$	
$0-1\quad 2-2$	$(1)=\frac{1323}{1024}e'^2m^4+\frac{20\,601}{1024}e'^2m^5$	$+3\quad 0\quad 0$ $-3+1-2$
	$(2)=-\frac{17\,689}{1024}e'^2m^4-\frac{142\,709}{1024}e'^2m^5$	$-1\quad 0\quad 0$ $+3-1-2$

Terms arising from the Longitude—Continued.

$i,\ i',\ j,\ j'$		Coefficients for $\varepsilon,\ \pi,\ \theta,$ $\varepsilon',\ \pi',\ \theta'$
$-1\quad 1\quad 2-2$	$(1)=\frac{225}{128}e^2e'^2m^2$	$+2+1\quad 0$ $-1-1-2$
	$(2)=0$	
$0\quad 1\quad 2-2$	$(1)=\frac{27}{1024}e'^2m^4+\frac{603}{1024}e'^2m^5$	$+3\quad 0\quad 0$ $-1-1-2$
	$(2)=-\frac{361}{1024}e'^2m^4-\frac{9101}{3072}e'^2m^5$	$-1\quad 0\quad 0$ $+1+1-2$
$0\quad 0\quad 0\quad 2$	$(1)=\frac{9}{4}\gamma^4m^2$	$+1\quad 0-2$ $+2\quad 0-2$
	$(2)=\frac{9}{16}\gamma^4m^2$	$+1\quad 0+2$ $-2\quad 0-2$
$0\quad 0\quad 1-1$	$(1)=\frac{225}{512}\alpha^2m^2$	$+2\quad 0\quad 0$ $-1\quad 0-2$
	$(2)=0$	
$0\quad 1\quad 1-1$	$(1)=\frac{25}{32}e'^2\alpha^2$	$+2\quad 0\quad 0$ $0-1-2$
	$(2)=0$	

Terms arising from the Latitude,

or values of $\frac{1}{2}\frac{p}{n}c^2$

$i,\ i',\ j,\ j'$		Coefficients for $\varepsilon,\ \pi,\ \theta$ $\varepsilon',\ \pi',\ \theta'$
$-2\quad 0\quad 1\quad 0$	$-\frac{1}{2}e^4\gamma^2$	$-1+2-1$ $0\quad 0\quad 0$
$-1\quad 0\quad 1\quad 0$	$\frac{27}{4}e^2\gamma^2m^2$	$0+1-1$ $0\quad 0\quad 0$

Terms arising from the Latitude,

or values of $\frac{1}{2}\frac{p}{n}c^2$—Continued.

$i,\ i',\ j,\ j'$		Coefficients for $\epsilon,\ \pi,\ \theta$ $\epsilon',\ \pi',\ \theta'$
0 0 1 0	$2\gamma^2 - 2e^2\gamma^2 - 2\gamma^4 + \frac{5}{6}\gamma^2 m^2 + \frac{3}{16}\gamma^2 m^3 - \frac{3}{16}e^4\gamma^2$ $+ 7e^2\gamma^4 + \frac{19}{24}e^2\gamma^2 m^2 + \frac{5}{4}e'^2\gamma^2 m^2 - \frac{211}{48}\gamma^4 m^2$ $+ \frac{1411}{576}\gamma^2 m^4$	+1 0 −1 0 0 0
1 0 1 0	$e^2\gamma^2 - \frac{3}{2}e^4\gamma^2 - e^2\gamma^4 + \frac{1}{6}e^2\gamma^2 m^2$	+2 −1 −1 0 0 0
2 0 1 0	$\frac{27}{32}e^4\gamma^2$	+3 −2 −1 0 0 0
0 −1 1 0	$\frac{9}{32}e'^2\gamma^2 m^2$	+1 0 −1 −1 +1 0
0 1 1 0	$\frac{9}{32}e'^2\gamma^2 m^2$	+1 0 −1 +1 −1 0
−1 0 3 −2	$\frac{225}{64}e^2\gamma^2 m^2$	+2 +1 −1 −2 0 0
0 0 3 −2	$\frac{27}{128}\gamma^2 m^4$	+3 0 −1 −2 0 0
0 −1 1 −2	$\frac{49}{32}e'^2\gamma^2 m^2$	+1 0 +1 −3 +1 0
−1 0 1 −2	0	0 +1 +1 −2 0 0
0 0 1 −2	$\frac{9}{32}\gamma^2 m^2 + \frac{87}{64}\gamma^2 m^3 + \frac{9}{64}e^2\gamma^2 m^2 - \frac{45}{32}e'^2\gamma^2 m^2$ $+ \frac{9}{32}\gamma^4 m^2 + \frac{4503}{1024}\gamma^2 m^4$	+1 0 +1 −2 0 0
1 0 1 −2	$\frac{9}{64}e^2\gamma^2 m^2$	+2 −1 +1 −2 0 0
0 1 1 −2	$\frac{9}{32}e'^2\gamma^2 m^2$	+1 0 +1 −1 −1 0

We have now all the data for forming, by means of the equations (82), the final values of h_e, h_π, etc., so far as they depend on the coordinates x, y, z of the Moon, which parts are represented below by adding the symbol ☽. We shall, however, for convenience in our subsequent work, change the form of the common factor $\mu_2 a^2 n$ by which all these quantities are multiplied in order to obtain k_e, k_π, etc. We recall that, in DELAUNAY's theory, the quantities a and n are connected by the relation

$$a^3 n^2 = m_2 + m_3$$

Then, since we have put

$$\mu_2 = \frac{m_2 m_3}{m_2 + m_3}$$

we have

$$\mu_2 a^2 n = \frac{m_2 m_3}{a n}$$

which we shall adopt for the common factor in question.

We now find from the preceding tables for the terms in question.

$$\begin{aligned}
k_{e☽} = \frac{m_2 m_3}{a n}\Big\{ 1 &+ \Big(-\frac{1}{3} - \frac{3}{8}e^2 - \frac{1}{2}e'^2 + \frac{3}{2}\gamma^2 - \frac{1}{32}e^4 - \frac{9}{16}e^2 e'^2 - \frac{5}{8}e'^4 + \frac{5}{4}e^2\gamma^2 + \frac{9}{4}e'^2\gamma^2 \\
&\quad - \gamma^4 - \frac{3}{8}\alpha^2\Big) m^2 \\
&+ \Big(-\frac{225}{32}e^2 - \frac{9}{8}\gamma^2\Big) m^3 \\
&+ \Big(\frac{895}{288} - \frac{23\,255}{512}e^2 + \frac{3995}{192}e'^2 - \frac{1585}{128}\gamma^2\Big) m^4 \\
&+ \Big(\frac{1261}{96} + \frac{11\,349}{64}e'^2\Big) m^5 + \frac{3028}{81}m^6 + \frac{76\,741}{864}m^7 \Big\} \\
k_{\pi☽} = \frac{m_2 m_3}{a n}\Big\{ &-\frac{1}{2}e^2 - \frac{1}{8}e^4 + \frac{5}{8}e^2\gamma^4 - \frac{5}{16}e^4\gamma^2 - \frac{1}{16}e^6 \\
&+ \Big(\frac{1171}{384}e^2 - \frac{2069}{384}e^4 - \frac{309}{128}e^2 e'^2 - \frac{85}{8}e^2\gamma^2\Big) m^2 \\
&+ \frac{3945}{256}e^2 m^3 + \frac{2\,001\,623}{36\,864}e^2 m^4 \Big\} \\
k_{\theta☽} = \frac{m_2 m_3}{a n}\Big\{ &-2\gamma^2 + e^2\gamma^2 - \frac{35}{4}e^2\gamma^4 + \frac{43}{32}e^4\gamma^2 \\
&+ \Big(-\frac{53}{96}\gamma^2 - \frac{2101}{192}e^2\gamma^2 - \frac{45}{32}e'^2\gamma^2 + \frac{143}{32}\gamma^4\Big) m^2 \\
&+ \frac{75}{64}\gamma^2 m^3 + \frac{16\,007}{9216}\gamma^2 m^4 \Big\}
\end{aligned} \qquad (84)$$

$$k_{\epsilon'\mathfrak{D}} = -\frac{m_2 m_3}{a n}\left\{\left(\frac{225}{64} e^2 + \frac{9}{16}\gamma^2\right) m^2 - \frac{97}{32} m^4 + \text{etc.}\right\}$$

$$k_{\pi'\mathfrak{D}} = +\frac{m_2 m_3}{a n}\left\{\left(\frac{125}{16} e^2 e'^2 + \frac{5}{4} e'^2 \gamma^2\right) m^2 + \frac{291}{16} e'^2 m^4 \text{etc.}\right\} \tag{85}$$

$$k_{\theta'\mathfrak{D}} = +\frac{m_2 m_3}{a n}\left\{-2\gamma'^2 + 4\gamma^2\gamma'^2 + e^2\gamma'^2 + \frac{2}{3}\gamma'^2 m^2 + \text{etc.}\right\}$$

We have next to consider the corresponding quantities formed from the solar coordinates, x', y', z'. Owing to the minutenesss of the perturbations of these coordinates in the problem of three bodies, the largest terms of this kind will be those arising from the values of x', y', and z' in the theory of elliptic motion. Into this theory the lunar elements g, π, and θ do not enter; it will not therefore add any terms to k_ϵ, k_π or k_θ. However from the form of the canonical equations (64) it appears that k_ϵ', k_π', and k_θ' are the conjugate elements of ϵ', π', and θ', that is to say, functions of ϵ', π', and θ' satisfying the conditions

$$[\epsilon', k_\epsilon'] = 1; \quad [\pi', k_\pi'] = 1; \quad [\theta', k_\theta'] = 1$$

while all other combinations of the same kind vanish. Now in the theory of elliptic motion the values of these conjugate functions are known to be

$$k_\epsilon'{}_\odot = m_1 \frac{(m_2 + m_3)}{a' n'}$$

$$k_\pi'{}_\odot = k_\epsilon (1 - e'^2)^{\frac{1}{2}} \tag{86}$$

$$k_\theta'{}_\odot = k_\pi' \cos i' = k_\pi' (1 - 2\gamma'^2)$$

which are therefore the principal terms in question.

We have next to consider the order of magnitude of the terms which will be introduced into the values of x', y', and z' by the perturbations of these quantities due to the separation of the Earth and Moon. From the process of integration developed in §§ 3, 5, and 6 when Ω_1 is taken account of, it follows that any term of x' in (60) which contains either ϵ, π, or θ in its argument will contain the factor

$$k = \mu \frac{a^2}{a'^2}$$

and will be of the form

$$x' = k m s a' \cos (\mathrm{I}\,\epsilon + \mathrm{I}^1 \pi + \ \ . \ \ . \ \ .)$$

s being a numerical factor of the order of magnitude unity or less. The principal term of k_ϵ thus arising will be by (62)

$$k_{\epsilon\odot} = \mu_1 \mathrm{I}^2 n k^2 m^2 s^2 a'^2$$

and the complete expression for $k_{\epsilon\odot}$ will contain the factor $\mu_1 n k^2 m^2 a'^2$ the ratio of which to the common factor $\mu_2 a^2 n$ of $k_{\epsilon\mathfrak{D}}$ is of the evanescent order of magnitude $m^2 k$. These terms are, therefore, entirely unimportant, and the complete values of k_ϵ, k_π, and k_θ may be taken as equal to $k_{\epsilon\mathfrak{D}}$, etc.

The terms of x' which depend on the arguments ϵ', π', θ' exclusively will receive increments of the order of magnitude of the terms in ϵ, π, and θ; that is to say, the terms in question will be of the form

$$x' = a'(s' + k\,m\,s) \cos(i_4\,\varepsilon' + i_5\,\pi' + i_6\,\theta')$$

s' and s being numerical factors. Each term of this kind will give rise in $k'_{\epsilon\odot}$ to the term

$$i_4^2\,a'^2\,n'\,(s' + k\,m\,s)^2$$

The principal term $i_4^2\,a'^2\,n'\,s'^2$ of this expression will be contained in the value of $k'_{\epsilon\odot}$ already given in (86). The principal part of the remaining terms will have as a factor the factor of the former terms, $\mu_1\,a'^2\,n'$, multiplied by $m\,k$. This gives rise to the common factor

$$\frac{m_2\,m_3}{a\,n}m^2$$

in the terms of k_ϵ', k_π', and k_θ'. The terms introduced by the coordinates x, y, and, z, as found in (85), contain this same factor. We conclude, therefore, that this portion of k_ϵ' etc., must remain undetermined until the solution of the problem of three bodies is complete.

§ 12.

FORMATION OF THE DIFFERENTIAL EQUATIONS.

We have now to differentiate the preceding values of k_ϵ, k_θ, etc., with respect to the elements of which they are functions. To find the derivatives of the powers of m with respect to the logarithm of a, which I call ν, we remark that its value is

$$m = \frac{n'}{n} = \left\{\frac{m_1 + m_2 + m_3}{m_2 + m_3}\,\frac{a^3}{a'^3}\right\}^{\frac{1}{2}}$$

From this we derive the general rule that if

$$u = a^i\,\{c_0 + c_1\,m + c_2\,m^2 + c_3\,m^3 + \text{etc.}\}$$

then

$$\begin{aligned}\frac{d\,u}{d\,\nu} &= a^i\left\{i\,c_0 + \left(i + \frac{3}{2}\right)c_1\,m + \left(i + \frac{6}{2}\right)c_2\,m^2 + \text{etc.}\right\}\\ \frac{d\,u}{d\,\nu'} &= a^i\left\{-\frac{3}{2}c_1\,m - \frac{6}{2}c_2\,m^2 - \frac{9}{2}c_3\,m^3 - \text{etc.}\right\}\end{aligned} \qquad (87)$$

We thus find, dropping the symbols ☽ and ⊙ in (84) and (86),

$$\begin{aligned}\frac{d\,k_\epsilon}{d\,\nu} = \frac{m_2\,m_3}{a\,n}\Big\{&\frac{1}{2} + \Big(-\frac{7}{6} - \frac{21}{16}e^2 + \frac{21}{4}\gamma^2 - \frac{7}{4}e'^2 - \frac{7}{64}e^4 + \frac{35}{8}e^2\gamma^2 - \frac{7}{2}\gamma^4 - \frac{63}{32}e^2e'^2 + \frac{63}{8}e'^2\gamma^2\\ &- \frac{35}{16}e'^4 - \frac{33}{16}\alpha^2\Big)m^2 + \Big(-\frac{1125}{32}e^2 - \frac{45}{8}\gamma^2\Big)m^3 + \Big(\frac{11\,635}{576} - \frac{302\,315}{1024}e^2\\ &- \frac{20\,605}{256}\gamma^2 + \frac{51\,935}{384}e'^2\Big)m^4 + \Big(\frac{1261}{12} + \frac{11\,349}{8}e'^2\Big)m^5 + \frac{28\,766}{81}m^6\\ &+ \frac{844\,151}{864}m^7\Big\}\end{aligned}$$

$$\frac{d\,k_\epsilon}{d\,e}=\frac{m_2 m_3}{a\,n}\left\{\left(-\frac{3}{4}e-\frac{1}{8}e^3+\frac{5}{2}e\,\gamma^2-\frac{9}{8}e\,e'^2\right)m^2-\frac{225}{16}e\,m^3-\frac{23\,255}{256}e\,m^4\right\}$$

$$\frac{d\,k_\epsilon}{d\,\gamma}=\frac{m_2 m_3}{a\,n}\left\{\left(3\,\gamma+\frac{5}{2}e^2\gamma-4\,\gamma^3+\frac{9}{2}e'^2\gamma\right)m^2-\frac{9}{4}\gamma\,m^3-\frac{1585}{64}\gamma\,m^4\right\}$$

$$\frac{d\,k_\epsilon}{d\,e'}=\frac{m_2 m_3}{a\,n}\left\{\left(-e'-\frac{9}{8}e^2e'+\frac{9}{2}\gamma^2e'-\frac{5}{2}e'^3\right)m^2+\frac{3995}{96}e'\,m^4+\frac{11\,349}{32}e'\,m^5\right\}$$

$$\frac{d\,k_\pi}{d\,v}=\frac{m_2 m_3}{a\,n}\left\{-\frac{1}{4}e^2-\frac{1}{16}e^4-\frac{1}{32}e^6-\frac{5}{32}e^4\gamma^2+\frac{5}{16}e^2\gamma^4+\left(\frac{8197}{768}e^2-\frac{14\,483}{768}e^4-\frac{595}{16}e^2\gamma^2\right.\right.$$
$$\left.\left.-\frac{2163}{256}e^2e'^2\right)m^2+\frac{19\,725}{256}e^2m^3+\frac{26\,021\,099}{73\,728}e^2m^4\right\}$$

$$\frac{d\,k_\pi}{d\,e}=\frac{m_2 m_3}{a\,n}\left\{-e-\frac{1}{2}e^3+\frac{5}{4}e\,\gamma^4-\frac{5}{4}e^3\gamma^2-\frac{3}{8}e^5+\left(\frac{1171}{192}e-\frac{2069}{96}e^3-\frac{85}{4}e\,\gamma^2-\frac{309}{64}e\,e'^2\right)m^2\right.$$
$$\left.+\frac{3945}{128}e\,m^3+\frac{2\,001\,623}{18\,432}e\,m^4\right\}$$

$$\frac{d\,k_\pi}{d\,\gamma}=\frac{m_2 m_3}{a\,n}\left\{\frac{5}{2}e^2\gamma^3-\frac{5}{8}e^4\gamma-\frac{85}{4}e^2\gamma\,m^2\right\}$$

$$\frac{d\,k_\pi}{d\,e'}=\frac{m_2 m_3}{a\,n}\left\{-\frac{309}{64}e^2e'\,m^2\right\}$$

$$\frac{d\,k_\theta}{d\,v}=\frac{m_2 m_3}{a\,n}\left\{-\gamma^2+\frac{1}{2}e^2\gamma^2-\frac{35}{8}e^2\gamma^4-\frac{43}{64}e^4\gamma^2+\left(-\frac{371}{192}\gamma^2-\frac{14\,707}{384}e^2\gamma^2+\frac{1001}{64}\gamma^4\right.\right.$$
$$\left.\left.-\frac{315}{64}e'^2\gamma^2\right)m^2+\frac{375}{64}\gamma^2m^3+\frac{208\,091}{18\,432}\gamma^2m^4\right\}$$

$$\frac{d\,k_\theta}{d\,e}=\frac{m_2 m_3}{a\,n}\left\{2\,e\,\gamma^2-\frac{35}{2}e\,\gamma^4+\frac{43}{8}e^3\gamma^2-\frac{2101}{96}e\,\gamma^2m^2\right\}$$

$$\frac{d\,k_\theta}{d\,\gamma}=\frac{m_2 m_3}{a\,n}\left\{-4\,\gamma+2\,e^2\gamma-35\,e^2\gamma^3+\frac{43}{16}e^4\gamma+\left(-\frac{53}{48}\gamma-\frac{2101}{96}e^2\,\gamma+\frac{143}{8}\gamma^3\right.\right.$$
$$\left.\left.-\frac{45}{16}e'^2\gamma\right)m^2+\frac{75}{32}\gamma\,m^3+\frac{16\,007}{4608}\gamma\,m^4\right.$$

$$\frac{d\,k_\theta}{d\,e'}=\frac{m_2 m_3}{a\,n}\left\{-\frac{45}{16}\gamma^2\,e'\,m^2\right\}$$

The derivatives with respect to v' may be found by the condition

$$\frac{d\,k}{d\,v}+\frac{d\,k}{d\,v'}=\frac{1}{2}k$$

A rigorous solution of the problem will now be obtained by taking the derivatives of k_ϵ', k_π', and k_θ', with respect to the six elements which they contain, substituting all the derivatives in general equations of the form (64), and then solving these equations with respect to $\frac{d\,v}{d\,t}$, $\frac{d\,e}{d\,t}$, etc. But we have seen that in the values of k_ϵ', k_θ', etc.,

the lunar elements are multiplied by factors so minute that they may be wholly neglected. The perturbations of the solar elements may therefore be obtained from the values of k_ϵ', k_π' and k_θ', and the results substituted in the expressions for the lunar elements. The process will be as follows:

From the equations

$$\begin{aligned} \frac{d\,\nu'}{d\,t} &= \frac{d\,\nu'}{d k_\epsilon'}\frac{d\,\mathrm{R}}{d\,\epsilon'} + \frac{d\,\nu'}{d k_\pi'}\frac{d\,\mathrm{R}}{d\,\pi'} + \frac{d\,\nu'}{d\,k'_\theta}\frac{d\,\mathrm{R}}{d\,\theta'} \\ \frac{d\,e'}{d\,t} &= \frac{d\,e'}{d k'_\epsilon}\frac{d\,\mathrm{R}}{d\,\epsilon'} + \frac{d\,e'}{d\,k'_\pi}\frac{d\,\mathrm{R}}{d\,\pi'} + \frac{d\,e'}{d\,k'_\theta}\frac{d\,\mathrm{R}}{d\,\theta'} \end{aligned} \tag{88}$$

we are to derive the perturbations of ν' and e'. Those of γ' are at present omitted, because this quantity does not enter into k_ϵ, k_π, or k_θ, so far as we have written them.

From the equations

$$\begin{aligned} \frac{d\,\epsilon'}{d\,t} &= n' - \frac{d\,\nu'}{d\,k_\epsilon'}\frac{d\,\mathrm{R}}{d\,\nu'} - \frac{d\,e'}{d k_\epsilon'}\frac{d\,\mathrm{R}}{d\,e'} - \frac{d\,\gamma'}{d\,k_\epsilon'}\frac{d\,\mathrm{R}}{d\,\gamma'} \\ \frac{d\,\pi'}{d\,t} &= \quad - \frac{d\,\nu'}{d\,k_\pi'}\frac{d\,\mathrm{R}}{d\nu'} - \frac{d\,e'}{d k_\pi'}\frac{d\,\mathrm{R}}{d\,e'} - \frac{d\,\gamma'}{d k_\epsilon'}\frac{d\,\mathrm{R}'}{d\,\gamma'} \end{aligned}$$

where

$$n' = \{m_1 + m_2 + m_3\}^{\frac{1}{2}}\, a'^{-\frac{3}{2}} = (m_1 + m_2 + m_3)^{\frac{1}{2}}\, c^{-\frac{3}{2}\nu'}$$

c being the Neperien base, we are to derive ϵ' and π', or their perturbations.

Putting the first three of the equations (64) in the form

$$\begin{aligned} \frac{d\,k_\epsilon}{d\,\nu}\frac{d\,\nu}{d\,t} + \frac{d\,k_\epsilon}{d\,e}\frac{d\,e}{d\,t} + \frac{d\,k_\epsilon}{d\,\gamma}\frac{d\,\gamma}{d\,t} &= \frac{d\,\mathrm{R}}{d\,\epsilon} - \frac{d\,k_\epsilon}{d\,\nu'}\frac{d\,\nu'}{d\,t} - \frac{d\,k_\epsilon}{d\,e'}\frac{d\,e'}{d\,t} \\ \frac{d\,k_\pi}{d\,\nu}\frac{d\,\nu}{d\,t} + \frac{d\,k_\pi}{d\,e}\frac{d\,e}{d\,t} + \frac{d\,k_\pi}{d\,\gamma}\frac{d\,\gamma}{d\,t} &= \frac{d\,\mathrm{R}}{d\,\pi} - \frac{d\,k_\pi}{d\,\nu'}\frac{d\,\nu'}{d\,t} - \frac{d\,k_\pi}{d\,e'}\frac{d\,e'}{d\,t} \\ \frac{d\,k_\theta}{d\,\nu}\frac{d\,\nu}{d\,t} + \frac{d\,k_\theta}{d\,e}\frac{d\,e}{d\,t} + \frac{d\,k_\theta}{d\,\gamma}\frac{d\,\gamma}{d\,t} &= \frac{d\,\mathrm{R}}{d\,\theta} - \frac{d\,k_\theta}{d\,\nu'}\frac{d\,\nu'}{d\,t} - \frac{d\,k_\theta}{d\,e'}\frac{d\,e'}{d\,t} \end{aligned} \tag{89}$$

We are to solve these equations with respect to $\frac{d\,\nu}{d\,t}, \frac{d\,e}{d\,t}$, and $\frac{d\,\gamma}{d\,t}$. Let us put

t', the time introduced through the varying solar elements, so that

$$\frac{d\,k_\epsilon}{d t'} = \frac{d\,k_\epsilon}{d\,\nu'}\frac{d\,\nu'}{d\,t} + \frac{d\,k_\epsilon}{d\,e'}\frac{d\,e'}{d\,t}$$

etc. etc. etc.

We shall then have

$$
\begin{aligned}
\frac{d\nu}{dt} &= \left(\frac{d\nu}{dk_\epsilon}\right) \cdot \left(\frac{dR}{d\varepsilon} - \frac{dk_\epsilon}{dt'}\right) \\
&+ \left(\frac{d\nu}{dk_\pi}\right) \cdot \left(\frac{dR}{d\pi} - \frac{dk_\pi}{dt'}\right) \\
&+ \left(\frac{d\nu}{dk_\theta}\right) \cdot \left(\frac{dR}{d\theta} - \frac{dk_\theta}{dt'}\right) \\
\frac{de}{dt} &= \left(\frac{de}{dk_\epsilon}\right) \cdot \left(\frac{dR}{d\varepsilon} - \frac{dk_\epsilon}{dt'}\right) \\
&+ \left(\frac{de}{dk_\pi}\right) \cdot \left(\frac{dR}{d\pi} - \frac{dk_\pi}{dt'}\right) \\
&+ \left(\frac{de}{dk_\theta}\right) \cdot \left(\frac{dR}{d\theta} - \frac{dk_\theta}{dt'}\right) \\
\frac{d\gamma}{dt} &= \left(\frac{d\gamma}{dk_\epsilon}\right) \cdot \left(\frac{dR}{d\varepsilon} - \frac{dk_\epsilon}{dt'}\right) \\
&+ \left(\frac{d\gamma}{dk_\pi}\right) \cdot \left(\frac{dR}{d\pi} - \frac{dk_\pi}{dt}\right) \\
&+ \left(\frac{d\gamma}{dk_\theta}\right) \cdot \left(\frac{dR}{d\theta} - \frac{dk_\pi}{dt'}\right)
\end{aligned}
\qquad (90)
$$

The derivatives $\left(\frac{d\nu}{dk_\epsilon}\right)$, etc., are here inclosed in parentheses to indicate that they are to be taken subject to the condition that ν' and e' shall remain constant; in other words, the derivatives with respect to k_ϵ are to be formed by solving the equations

$$
\begin{aligned}
\frac{dk_\epsilon}{d\nu}\left(\frac{d\nu}{dk_\epsilon}\right) + \frac{dk_\epsilon}{de}\left(\frac{de}{dk_\epsilon}\right) + \frac{dk_\epsilon}{d\gamma}\left(\frac{d\gamma}{dk_\epsilon}\right) &= 1 \\
\frac{dk_\pi}{d\nu}\left(\frac{d\nu}{dk_\epsilon}\right) + \frac{dk_\pi}{de}\left(\frac{de}{dk_\epsilon}\right) + \frac{dk_\pi}{d\gamma}\left(\frac{d\gamma}{dk_\epsilon}\right) &= 0 \\
\frac{dk_\theta}{d\nu}\left(\frac{d\nu}{dk_\epsilon}\right) + \frac{dk_\theta}{de}\left(\frac{de}{dk_\epsilon}\right) + \frac{dk_\theta}{d\gamma}\left(\frac{d\gamma}{dk_\epsilon}\right) &= 0
\end{aligned}
\qquad (91)
$$

and those with respect to k_π and k_θ by solving similar sets in which those quantities enter. Having found the perturbations of ν, e, and γ, those of ε, π, and θ are to be obtained from the equations

$$\frac{d k_\epsilon}{d x}\frac{d \epsilon_0}{d t}+\frac{d k_\pi}{d x}\frac{d \pi_0}{d t}+\frac{d k_\theta}{d x}\frac{d \theta_0}{d t}=-\frac{d \mathrm{R}}{d x}$$

$$\frac{d k_\epsilon}{d e}\frac{d \epsilon_0}{d t}+\frac{d k_\pi}{d e}\frac{d \pi_0}{d t}+\frac{d k_\theta}{d e}\frac{d \theta_0}{d t}=-\frac{d \mathrm{R}}{d e}$$

$$\frac{d k_\epsilon}{d \gamma}\frac{d \epsilon_0}{d t}+\frac{d k_\pi}{d \gamma}\frac{d \pi_0}{d t}+\frac{d k_\theta}{d \gamma}\frac{d \theta_0}{d t}=-\frac{d \mathrm{R}}{d \gamma}$$

which give

$$\begin{aligned}
\frac{d \epsilon}{d t}&=\frac{d \epsilon_0}{d t}+n=n-\left(\frac{d x}{d k_\epsilon}\right)\frac{d \mathrm{R}}{d x}-\left(\frac{d e}{d k_\epsilon}\right)\frac{d \mathrm{R}}{d e}-\left(\frac{d \gamma}{d k_\epsilon}\right)\frac{d \mathrm{R}}{d \gamma}\\
\frac{d \pi}{d t}&=\frac{d \pi_0}{d t}+\pi_1=\pi_1-\left(\frac{d x}{d k_\pi}\right)\frac{d \mathrm{R}}{d x}-\left(\frac{d e}{d k_\pi}\right)\frac{d \mathrm{R}}{d e}-\left(\frac{d \gamma}{d k_\pi}\right)\frac{d \mathrm{R}}{d \gamma}\\
\frac{d \theta}{d t}&=\frac{d \theta_0}{d t}+\theta_1=\theta_1-\left(\frac{d x}{d k_\theta}\right)\frac{d \mathrm{R}}{d x}-\left(\frac{d e}{d k_\theta}\right)\frac{d \mathrm{R}}{d e}-\left(\frac{d \gamma}{d k_\theta}\right)\frac{d \mathrm{R}}{d \gamma}
\end{aligned}\tag{92}$$

The values of n, π_1, and θ_1 are the expressions for the mean motion and the secular motions of the perigee and the node in the theory of three bodies, which last are taken from DELAUNAY (*Comptes Rendus*, 1872, I, Vol. 74, p. 19), as follows:

$$n=(m_2+m_3)^{\frac{1}{2}}c^{-\frac{3}{2}}$$

$$\begin{aligned}
\pi_1=m^2n\Big\{&\frac{3}{4}-\frac{3}{8}e^2-6\gamma^2+\frac{9}{8}e'^2-\frac{3}{32}e^4+\frac{69}{8}e^2\gamma^2-\frac{45}{4}\gamma^4-9\gamma^2e'^2-\frac{9}{16}e^2e'^2+\frac{45}{32}e'^4\\
&+\left(\frac{225}{32}-\frac{675}{64}e^2-\frac{189}{8}\gamma^2+\frac{825}{32}e'^2+\frac{81}{32}e^2\gamma^2+\frac{1107}{16}\gamma^4-\frac{349}{4}\gamma^2e'^2-\frac{2475}{64}e^2e'^2\right)m\\
&+\left(\frac{4071}{128}-\frac{31\,605}{512}e^2-\frac{3963}{32}\gamma^2+\frac{61\,179}{256}e'^2\right)m^2\\
&+\left(\frac{265\,493}{2048}-\frac{1\,483\,665}{4096}e^2-\frac{335\,403}{512}\gamma^2+\frac{1\,767\,849}{1024}e'^2\right)m^3\\
&+\frac{12\,822\,631}{24\,576}m^4+\frac{1\,273\,925\,965}{589\,824}m^5+\frac{71\,028\,685\,589}{7\,077\,888}m^6\Big\}
\end{aligned}$$

$$\begin{aligned}
\theta_1=m^2n\Big\{&-\frac{3}{4}-\frac{3}{2}e^2+\frac{3}{2}\gamma^2-\frac{9}{8}e'^2+\frac{21}{64}e^4-\frac{51}{8}e^2\gamma^2+\frac{9}{4}e'^2\gamma^2-\frac{9}{4}e^2e'^2-\frac{45}{32}e'^4\\
&+\left(\frac{9}{32}-\frac{189}{32}e^2-\frac{27}{16}\gamma^2+\frac{23}{32}e'^2-\frac{675}{256}e^4+\frac{567}{16}e^2\gamma^2+\frac{27}{16}\gamma^4-\frac{349}{16}e^2e'^2-\frac{99}{16}\gamma^2e'^2\right)m\\
&+\left(\frac{273}{128}-\frac{2739}{128}e^2-\frac{843}{128}\gamma^2+\frac{3261}{256}e'^2\right)m^2\\
&+\left(\frac{9797}{2048}-\frac{165\,411}{2048}e^2-\frac{7185}{1024}\gamma^2+\frac{73\,423}{1024}e'^2\right)m^3+\frac{199\,273}{24\,576}m^4+\frac{6\,657\,733}{589\,824}m^5\Big\}
\end{aligned}$$

The perturbations of ϖ, e, and γ, which are of the first order as to the disturbing forces, and found by integrating (90) as if the elements were constant in the second members, may be divided into two classes: (1) Those which arise from the derivatives of R with respect to ϵ, π, and θ, and (2) those which arise from $\frac{d\,k_\epsilon}{d\,t'}$, etc. Let us consider the latter. They are given by the integration of the equations

$$
\begin{aligned}
\frac{d\varpi}{dt} &= -\left(\frac{d\,\varpi}{d\,k_\epsilon}\right)\frac{d\,k_\epsilon}{d\,t'} - \left(\frac{d\,\varpi}{dk_\pi}\right)\frac{d\,k_\pi}{d\,t'} - \left(\frac{d\,\varpi}{d\,k_\theta}\right)\frac{d\,k_\theta}{d\,t'} \\
\frac{d\,e}{dt} &= -\left(\frac{d\,e}{d\,k_\epsilon}\right)\frac{d\,k_\epsilon}{d\,t'} - \left(\frac{d\,e}{d\,k_\pi}\right)\frac{d\,k_\pi}{d\,t'} - \left(\frac{d\,e}{d\,k_\theta}\right)\frac{d\,k_\theta}{d\,t'} \qquad (93) \\
\frac{d\gamma}{dt} &= -\left(\frac{d\gamma}{d\,k_\epsilon}\right)\frac{d\,k_\epsilon}{d\,t'} - \left(\frac{d\,\gamma}{d\,k_\pi}\right)\frac{d\,k_\pi}{d\,t'} - \left(\frac{d\,\gamma}{dk_\theta}\right)\frac{d\,k_\theta}{d\,t'}
\end{aligned}
$$

The corresponding perturbations, $\delta\,\varpi$, $\delta\,e$, and $\delta\,\gamma$, to quantities of the first order will be given by solving the equations

$$
\begin{aligned}
&\frac{d\,k_\epsilon}{d\varpi}\delta\,\varpi + \frac{d\,k_\epsilon}{de}\delta\,e + \frac{d\,k_\epsilon}{d\gamma}\delta\,\gamma + \frac{d\,k_\epsilon}{d\varpi'}\delta\,\varpi' + \frac{d\,k_\epsilon}{de'}\delta\,e' = 0 \\
&\frac{d\,k_\pi}{d\,\varpi}\delta\,\varpi + \frac{d\,k_\pi}{d\,e}\delta\,e + \frac{d\,k_\pi}{d\,\gamma}\delta\,\gamma + \frac{d\,k_\pi}{d\varpi'}\delta\,\varpi' + \frac{d\,k_\pi}{d\,e'}\delta\,e' = 0 \qquad (94) \\
&\frac{d\,k_\theta}{d\,\varpi}\delta\,\varpi + \frac{d\,k_\theta}{d\,e}\delta\,e + \frac{d\,k_\theta}{d\,\gamma}\delta\,\gamma + \frac{d\,k_\theta}{d\,\varpi'}\delta\,\varpi' + \frac{d\,k_\theta}{d\,e'}\delta\,e' = 0
\end{aligned}
$$

The values of $\delta\ \varpi'$ and $\delta\ e'$ are regarded as known in advance. This curious theorem may embody some principle applicable to the disturbed motion of three bodies, which has not yet been fully mastered. The quantities I have called k_ϵ, k_π, and k_θ, are closely related to DELAUNAY's L, G, and H, as is shown in the next section. The theorem shows that the secular acceleration of the mean motion and of the motion of the perigee and nodes may be found by the equations

$$\delta\,\mathrm{L}=0; \quad \delta\,\mathrm{G}=0; \quad \delta\,\mathrm{H}=0;$$

by making the transformation on p. 800 of DELAUNAY's *Theorie*, Vol. II. This result I have verified so far as the mean motion is concerned.

The numerical values of the derivatives of k_e, k_π, and k_θ from the formulæ of pp. 186–87 are next to be computed. To exhibit the uncertainty to which many of the derivatives are subject, owing to the approximation in powers of m, γ, e, and e' not being carried far enough, the results are arranged by powers of m, the quantity whose powers converge least rapidly. From the last two or three terms the probable law of formation is roughly guessed at by induction, and a correction for the sum of the higher terms thus concluded. This correction must always be regarded as uncertain by a not inconsiderable fraction of its amount, and several of the results are therefore quite uncertain.

Formation of $\dfrac{a\,n}{m_2\,m_3}\dfrac{d\,k_e}{d\,x}$

Terms independent of m	+ .500 000
multiplied by m^2	− .006 492
m^3	− .000 050
m^4	+ .000 602
m^5	+ .000 246
m^6	+ .000 063
m^7	+ .000 012
Higher terms by induction	2
Sum	+0.494 383

Formation of $\dfrac{a\,n}{m_2\,m_3}\dfrac{1}{e}\dfrac{d\,k_e}{d\,e}$

Terms multiplied by m^2	−.0042
m^3	−.0058
m^4	−.0028
Higher terms by induction	−.0010
Sum	−.0138

Formation of $\dfrac{a\,n}{m_2\,m_3}\dfrac{1}{\gamma}\dfrac{d\,k_e}{d\,\gamma}$

Terms multiplied by m^2	+.0168
m^3	−.0009
m^4	−.0008
Higher terms by induction	−.0004
Sum	+.0147

Formation of $\frac{a\,n}{m_2\,m_3}\frac{d\,k_\iota}{d\,\varpi'}$

Terms multiplied by m^2	+.005 565
m^3	+.000 045
m^4	−.000 556
m^5	−.000 231
m^6	−.000 059
m^7	−.000 012
Higher terms by induction	−.000 003
Sum	+.004 749

Formation of $\frac{a\,n}{m_2\,m_3}\frac{1}{e'}\frac{d\,k_\iota}{d\,e'}$

Terms multiplied by m^2	−.00 560
m^4	+.00 131
m^5	+.00 085
Higher terms by induction	+.00 044
Sum	−.00 300

From the magnitude of the term concluded from induction we may conclude that this quantity is uncertain by $\frac{1}{20}$ of its entire amount, in consequence of the slowness with which the series converges.

Formation of $\frac{a\,n}{m_2\,m_3}\frac{1}{e^2}\frac{d\,k_\pi}{d\,\varpi}$

Terms independent of m	−.2502
multiplied by m^2	+.0590
m^3	+.0320
m^4	+.0111
Higher terms by induction	+.0030
Sum	−.1451

Formation of $\frac{a\,n}{m_2\,m_3}\frac{1}{e}\frac{d\,k_\pi}{d\,e}$

Terms independent of m	−1.0016
multiplied by m^2	+ .0335
m^3	+ .0129
m^4	+ .0034
Higher terms by induction	+ .0008
Sum	− .9510

Formation of $\dfrac{a\,n}{m_2\,m_3}\,\dfrac{1}{e^2\,\gamma}\,\dfrac{d\,k_\pi}{d\,\gamma}$

Terms independent of m	$+.003$
multiplied by m^2	$-.119$
Sum	$-.116$

Formation of $\dfrac{a\,n}{m_2\,m_3}\,\dfrac{1}{e^2}\,\dfrac{d\,k_\pi}{d\,\varpi'}$

Terms multiplied by m^2	$-.0506$
m^3	$-.0290$
m^4	$-.0102$
Higher terms by induction	$-$ 0034
Sum	$-.0932$

Formation of $\dfrac{a\,n}{m_2\,m_3}\,\dfrac{1}{e^2\,e'}\,\dfrac{d\,k_\pi}{d\,e'}$

Terms in m^2	-0.027

Formation of $\dfrac{a\,n}{m_2\,m_3}\,\dfrac{1}{\gamma^2}\,\dfrac{d\,k_\theta}{d\,\varpi}$

Terms independent of m	$-$ .99 852
multiplied by m^2	$-$ .01 128
m^3	$+$ .00 245
m^4	$+$ .00 035
Sum	$-$ 1.00 700

Formation of $\dfrac{a\,n}{m_2\,m_3}\,\dfrac{1}{e\,\gamma^2}\,\dfrac{d\,k_\theta}{d\,e}$

Terms independent of m	$+1.98$
multiplied by m^2	-0.12
Sum	$+1.86$

Formation of $\dfrac{a\,n}{m_2\,m_3}\,\dfrac{1}{\gamma}\,\dfrac{d\,k_\theta}{d\,\gamma}$

Terms independent of m	-3.9942
multiplied by m^2	$-$ .0064
m^3	$+$ .0010
m^4	$+$ .0001
Sum	-3.9995

Formation of $\frac{a\,n}{m_2\,m_3}\,\frac{1}{\gamma^2}\,\frac{d\,k_\theta}{d\,\varpi'}$

Terms multiplied by m^2	. . . .	$-.00\,968$
m^3	. . . .	$+.00\,220$
m^4	. . . .	$+.00\,033$
Sum	. . . .	$-.00\,715$

We therefore have the following numerical values of the derivatives, in which, however, the last figure is nearly always to be regarded as quite uncertain.

$$\frac{d\,k_\epsilon}{d\,\varpi} = +.494\,383\,\frac{m_2\,m_3}{a\,n}$$

$$\frac{d\,k_\epsilon}{d\,e} = -.0138\,e\,\frac{m_2\,m_3}{a\,n}$$

$$\frac{d\,k_\epsilon}{d\gamma} = +.0147\,\gamma\,\frac{m_2\,m_3}{a\,n}$$

$$\frac{d\,k_\epsilon}{d\varpi'} = +.004\,749\,\frac{m_2\,m_3}{a\,n}$$

$$\frac{d\,k_\epsilon}{d\,e'} = -.00\,300\,e'\,\frac{m_2\,m_3}{a\,n}$$

$$\frac{d\,k_\pi}{d\,\varpi} = -.000\,437\,\frac{m_2\,m_3}{a\,n}$$

$$\frac{d\,k_\pi}{d\,e} = -.9510\,e\,\frac{m_2\,m_3}{a\,n} \qquad (95)$$

$$\frac{d\,k_\pi}{d\,\gamma} = -.000\,35\,\gamma\,\frac{m_2\,m_3}{a\,n}$$

$$\frac{d\,k_\pi}{d\,\varpi'} = -.000\,281\,\frac{m_2\,m_3}{a\,n}$$

$$\frac{d\,k_\pi}{d\,e'} = -.000\,08\,e'\,\frac{m_2\,m_3}{a\,n}$$

$$\frac{d\,k_\theta}{d\,\varpi} = -.002\,028\,\frac{m_2\,m_3}{a\,n}$$

$$\frac{d\,k_\theta}{d\,e} = +.003\,75\,e\,\frac{m_2\,m_3}{a\,n}$$

$$\frac{d\,k_\theta}{d\,\gamma} = -3.9995\,\gamma\,\frac{m_2\,m_3}{a\,n}$$

$$\frac{d\,k_\theta}{d\,v'} = -.000\,014\,\frac{m_2\,m_3}{a\,n}$$

$$\frac{d\,k_\theta}{d\,e'} = -.000\,032\,e'\,\frac{m_2\,m_3}{a\,n} \tag{95}$$

The values of $\frac{d\,v}{d\,k_\epsilon}$, etc., may be obtained either by solving the equations (89), when they will be the coefficients of $\frac{d\mathrm{R}}{d\,\varepsilon}$, etc., or by solving the equations (91). We thus find

$$\left(\frac{d\,v}{d\,k_\epsilon}\right) = 2.0225\,\frac{a\,n}{m_2\,m_3}$$

$$\left(\frac{d\,v}{d\,k_\pi}\right) = -.0293\,\frac{a\,n}{m_2\,m_3}$$

$$\left(\frac{d\,v}{d\,k_\theta}\right) = +.0075\,\frac{a\,n}{m_2\,m_3}$$

$$\left(\frac{d\,e}{d\,k_\epsilon}\right) = -.000\,928\,\frac{a\,n}{e\,m_2\,m_3}$$

$$\left(\frac{d\,e}{d\,k_\pi}\right) = -1.0514\,\frac{a\,n}{e\,m_2\,m_3}$$

$$\left(\frac{d\,e}{d\,k_\theta}\right) = +.000\,09\,\frac{a\,n}{e\,m_2\,m_3}$$

$$\left(\frac{d\,\gamma}{d\,k_\epsilon}\right) = -.001\,045\,\frac{a\,n}{\gamma\,m_2\,m_3}$$

$$\left(\frac{d\,\gamma}{d\,k_\pi}\right) = -.000\,97\,\frac{a\,n}{\gamma\,m_2\,m_3}$$

$$\left(\frac{d\,\gamma}{d\,k_\theta}\right) = -0.250\,04\,\frac{a\,n}{\gamma\,m_2\,m_3}$$

Now, resuming the equations (90), let us begin with those terms $\frac{d\,v}{d\,t}$, etc., which are dependent only on the variations of the solar elements, and which are given in (93). We have put

$$\frac{d\,k_\epsilon}{d\,t'} = \frac{d\,k_\epsilon}{d\,v'}\,\frac{d\,v'}{d\,t} + \frac{d\,k_\epsilon}{d\,e'}\,\frac{d\,e'}{d\,t}$$

$$\frac{d\,k_\pi}{d\,t'} = \frac{d\,k_\pi}{d\,v'}\,\frac{d\,v'}{d\,t} + \frac{d\,k_\pi}{d\,e'}\,\frac{d\,e'}{d\,t}$$

$$\frac{d\,k_\theta}{d\,t'} = \frac{d\,k_\theta}{d\,v'}\,\frac{d\,v'}{d\,t} + \frac{d\,k_\theta}{d\,e'}\,\frac{d\,e'}{d\,t}$$

If we here substitute the numerical values of $\frac{d\,k}{d\,\varpi''}$, etc., just found, and then substitute in (93) both these values and the values of $\frac{d\,\varpi}{d\,k_{\epsilon}}$, etc., in (96), we have

$$\frac{d\,\varpi}{d\,t} = -.0096\,\frac{d\,\varpi'}{d\,t} + .006\,07\,e'\,\frac{d\,e'}{d\,t}$$

$$e\,\frac{d\,e}{d\,t} = -.000\,29\,\frac{d\,\varpi'}{d\,t} - .000\,09\,e'\,\frac{d\,e'}{d\,t}$$

$$\gamma\,\frac{d\,\gamma}{d\,t} = .000\,001\,\frac{d\,\varpi'}{d\,t} - .000\,011\,e'\,\frac{d\,e'}{d\,t}$$

and, by integration,

$$\delta\,\varpi = -.0096\;\delta\,\varpi' + .003\,04\;\delta.e'^2$$

$$\delta.e^2 = -.000\,58\;\delta\,\varpi' - .000\,09\;\delta.e'^2 \qquad (96)$$

$$\delta.\gamma^2 = \quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad - .000\,011\;\delta.e'^2$$

These quantities are to be substituted in $n\,\pi_1$, and θ_1, to obtain the corresponding variations of ϵ, π, and θ. We have first

$$\delta\,n = -\frac{3}{2}\,n\,\delta\,\varpi$$

$$= +\frac{3}{2}\,n\left\{.0096\;\delta\,\varpi' - .003\,04\;\delta.e'^2\right\}$$

Taking the partial derivatives of π_1 and θ_1 (p. 190) with respect to ϖ, ϖ', and e' we find the following terms of

$$\frac{1}{n}\,\frac{d\,\pi_1}{d\,\varpi}$$

Terms in m^2	.00 619
m^3	873
m^4	443
m^5	179
m^6	68
m^7	25
m^8	10
m^9	4
Higher terms by induction	2
Sum	.02 223

Formation of $\frac{1}{n}\frac{d\,\pi_1}{d\,x'}$

Terms in m_2	$-.0123$
m_3	$-.0131$
m_4	$-.0059$
m_5	$-.0022$
m_6	$-.0008$
m_7	$-.0003$
m_8 and upwards	$-.0002$
Sum	$-.0348$

Formation of $\frac{1}{n}\frac{2}{e'}\frac{d\,\pi_1}{d\,e'}$

Terms in m_2	$+.0063$
m_3	$+.0108$
m_4	$+.0075$
m_5	$+.0040$
Higher terms by induction	$+.0032$
Sum	$+.0318$

The derivatives of θ_1 are much more convergent. We find

$$\frac{1}{n}\frac{d\,\theta_1}{d\,x}=-.005\,61$$

$$\frac{1}{n}\frac{d\,\theta_1}{d\,x'}=+.011\,64$$

$$\frac{1}{n}\frac{2}{e'}\frac{d\,\theta_1}{d\,e'}=-.005\,37$$

The terms of ε, π, and θ, which are independent of the derivatives of R with respect to the lunar elements, are, by (92)

$$\left.\begin{aligned}\delta\,\varepsilon&=\int\delta\,n\,dt\\ \delta\,\pi&=\int\delta\,\pi_1\,dt\\ \delta\,\theta&=\int\delta\,\theta_1\,dt\end{aligned}\right\}\qquad(97)$$

We have, from what precedes

$$\delta n = -\frac{3}{2} n \, \delta w$$

$$\delta \pi_1 = n \left\{ +.022\,23\, \delta w - .0348\, \delta w' + .0318\, \delta.e'^2 \right\}$$

$$\delta \theta_1 = n \left\{ -.005\,61\, \delta w + .011\,64\, \delta w' - .005\,37\, \delta.e'^2 \right\}$$

Substituting for $\delta\, \alpha$ its value in $\delta\, w'$ and $\delta.e'^2$ just found, these values become

$$\begin{aligned} \delta n &= n \{ \quad .0144\, \delta w' - .004\,56\, \delta.e'^2 \} \\ \delta \pi_1 &= n \{ -.0350\, \delta w' + .0319\, \delta.e'^2 \} \\ \delta \theta_1 &= n \{ +.01\,169\, \delta w' - .005\,39\, \delta.e'^2 \} \end{aligned} \tag{98}$$

The forms of the integrals deduced from these values will be different according as we consider the secular or periodic terms. There are, however, no secular terms in $\delta\, w'$. Corresponding to the adopted values of the planetary masses are the following secular perturbations of e'

$$\delta.e' = -8''.73\, \mathrm{T} - 0''.028\, \mathrm{T}^2$$

T being the number of centuries from 1800. From this and the adopted constant term of e' we derive

$$\delta.e'^2 = -0''.2926\, \mathrm{T} - 0''.000\,56\, \mathrm{T}^2$$

and hence

$$\begin{aligned} \delta\, \varepsilon &= \quad 5''.60\, \mathrm{T}^2 + 0''.007\, \mathrm{T}^3 \\ \delta\, \pi &= -39''.3 \quad \mathrm{T}^2 - 0''.050\, \mathrm{T}^3 \\ \delta\, \theta &= \quad 6''.64\, \mathrm{T}^2 + 0''.008\, \mathrm{T}^3 \end{aligned} \tag{99}$$

From the manner in which these quantities are obtained it may be shown that the coefficient of T^2 in $\delta\, \varepsilon$ depends mainly on the value of $\frac{d\, k_{\epsilon}}{d\, e'}$, and the value of $\delta\, \pi$ mainly on that of $\frac{d\, \pi_1}{d\, e'}$. We may, therefore, conclude from the defect of convergence in the series which express these quantities that the first quantity, the secular acceleration of the mean longitude, is uncertain by more than $\frac{1}{20}$, and the second by $\frac{1}{50}$ of its entire amount, while the third is nearly exact. Although all the observations hitherto made on the Moon would not enable us to detect errors of this magnitude, a more accurate determination would be desirable from a theoretical point of view. But this can not be made without further investigations into the problem of three bodies, which it is not the object of the present paper to undertake.

Let us now take up the periodic terms of $\delta\,x'$ and $\delta\,e'$. If we put

$$\delta\,x' = a_c \cos \mathrm{N} + a_s \sin \mathrm{N}$$

$$\delta\,e' = e_c \cos \mathrm{N} + e_s \sin \mathrm{N}$$

where N is an angle increasing uniformly with the time and therefore of the form $\mathrm{N}_0 + bt$, and if we substitute these values in the equations for $\delta\,n$, $\delta\,\pi_1$, and $\delta\,\theta_1$, and integrate, we find

$$\delta\,\varepsilon = + \quad .0144\,\nu\{a_c \sin \mathrm{N} - a_s \cos \mathrm{N}\}$$
$$- .000\,155\,\nu\{e_c \sin \mathrm{N} - e_s \cos \mathrm{N}\}$$

or,

$$\delta\,\varepsilon = (.0144\,a_c - .000\,155\,e_c)\,\nu \sin \mathrm{N} - (.0144\,a_s - .000\,155\,e_s)\,\nu \cos \mathrm{N} \qquad (100)$$

where we put

$$\nu = \frac{n}{b}$$

It will be seen that ν represents the quotient of the period of the inequality divided by the period of revolution of the Moon.

Applying the same process to π and θ, we have

$$\begin{aligned} \delta\,\pi &= (-.0350\,a_c + .00\,106\,e_c)\,\nu \sin \mathrm{N} \\ &\qquad + (0350\,a_s - .00\,106\,e_s)\,\nu \cos \mathrm{N} \\ \delta\,\theta &= (.0117\,a_c - .00\,018\,e_c)\,\nu \sin \mathrm{N} + (-\,0117\,a_s + 00\,018\,e_s)\,\nu \cos \mathrm{N} \end{aligned} \qquad (101)$$

The derivation of the numerical results of these formulæ requires the determination of the periodic inequalities of the solar elements produced by the action of each planet. The results will be found in Chapter III.

§ 13.

REMARKS ON THE RELATION OF THE PRECEDING THEORY TO THAT OF DELAUNAY.

The relation of the preceding theory to that of DELAUNAY is so remarkable as to deserve special attention. DELAUNAY continually reduced the differential equations to the canonical form

$$\frac{d\,\mathrm{L}}{d\,t} = \frac{d\,\mathrm{R}}{d\,l} \qquad \frac{d\,l}{d\,t} = -\frac{d\,\mathrm{R}}{d\,\mathrm{L}}$$

$$\frac{d\,\mathrm{G}}{d\,t} = \frac{d\,\mathrm{R}}{d\,g} \qquad \frac{d\,g}{d\,t} = -\frac{d\,\mathrm{R}}{d\,\mathrm{G}}$$

$$\frac{d\,\mathrm{H}}{d\,t} = \frac{d\,\mathrm{R}}{d\,h} \qquad \frac{d\,h}{d\,t} = -\frac{d\,\mathrm{R}}{d\,\mathrm{H}}$$

where l, g, and h correspond as follows to the variable angles in the theory in the present paper:

$$l=\varepsilon-\pi$$

$$g=\pi-\theta$$

$$h=\theta$$

Here R may be regarded as the potential of any disturbing force whatever tending to change the lunar elements. Comparing DELAUNAY's R with the equations (14) of the present paper, and the preceding equations with (65), it will be seen that those parts of our k_ϵ, k_π, and k_θ which are derived from the coordinates of the Moon alone, which parts are the only ones yet considered, should be related to the constants L, G, H, of DELAUNAY in the following way:

$$k_\epsilon=\mu_2\,\mathrm{L}$$

$$k_\pi=\mu_2\,(\mathrm{G}-\mathrm{L})$$

$$k_\theta=\mu_2\,(\mathrm{H}-\mathrm{G})$$

The final values of these constants, in terms of a, e, and γ, are given by DELAUNAY in Vol. II of his work, pp. 235–236. We could, therefore, immediately compare these expressions with the values of our k_ϵ, etc., finally obtained, but for the circumstance that the elements a, e, and γ here used are not identical with those in which the coordinates of the Moon are finally expressed at the end of the work. To reduce the one to the other, certain substitutions must be made, which are given on page 800 The identity of the expressions which would be derived by making these substitutions in L, G, H, with the values which I have derived from the Moon's coordinates would afford so remarkable confirmation of the numerical correctness of DELAUNAY's results, that I have deemed it worth while to make the transformation in question. The results are as follows:

$$\mathrm{L}=\sqrt{a\,\mu}\left\{\begin{array}{l}1+\left(-\frac{1}{3}+\frac{3}{2}\gamma^2-\gamma^4-\frac{1}{2}e'^2-\frac{5}{8}e'^4+\frac{9}{4}\gamma^2e'^2-\frac{3}{8}e^2+\frac{5}{4}\gamma^2e^2-\frac{9}{16}e'^2e^2\right.\\ \qquad\left.-\frac{1}{32}e^4\right)m^2\\ \quad+\left(-\frac{225}{32}e^2-\frac{9}{8}\gamma^2-\frac{825}{32}e'^2e^2\right)m^3+\left(+\frac{895}{288}-\frac{1585}{128}\gamma^2+\frac{3995}{192}e'^2\right.\\ \qquad\left.-\frac{23\,255}{512}e^2\right)m^4\\ \quad+\left(+\frac{1261}{96}+\frac{11\,349}{64}e'^2\right)m^5+\frac{3028}{81}m^6+\frac{76\,741}{864}m^7\end{array}\right\}$$

$$G=\sqrt{a\,\mu}\left\{\begin{array}{l}
1-\frac{1}{2}e^2+\frac{5}{8}\gamma^4 e^2-\frac{1}{8}e^4-\frac{5}{16}\gamma^2 e^4-\frac{1}{16}e^6 \\
+\left(-\frac{1}{3}+\frac{3}{2}\gamma^2-\gamma^4-\frac{1}{2}e'^2-\frac{5}{8}e'^4+\frac{9}{4}\gamma^2 e'^2+\frac{1027}{384}e^2-\frac{75}{8}\gamma^2 e^2\right. \\
\qquad\left.-\frac{381}{128}e'^2 e^2-\frac{2081}{384}e^4\right)m^2 \\
+\left(+\frac{2145}{256}e^2-\frac{9}{8}\gamma^2-\frac{25\,495}{512}e'^2 e^2\right)m^3+\left(+\frac{895}{288}-\frac{1585}{128}\gamma^2\right. \\
\qquad\left.+\frac{3995}{192}e'^2+\frac{327\,263}{36\,864}e^2\right)m^4 \\
+\left(+\frac{1261}{96}+\frac{11\,349}{64}e'^2\right)m^5+\frac{3028}{81}m^6+\frac{76\,741}{864}m^7
\end{array}\right\}$$

$$H=\sqrt{a\,\mu}\left\{\begin{array}{l}
1-2\gamma^2-\frac{1}{2}e^2+\gamma^2 e^2-\frac{65}{8}\gamma^4 e^2-\frac{1}{8}e^4+\frac{33}{32}\gamma^2 e^4-\frac{1}{16}e^6 \\
+\left(-\frac{1}{3}+\frac{91}{96}\gamma^2+\frac{111}{32}\gamma^4-\frac{1}{2}e'^2-\frac{5}{8}e'^4+\frac{1027}{384}e^2-\frac{3901}{192}\gamma^2 e^2\right. \\
\qquad\left.-\frac{381}{128}e'^2 e^2-\frac{2081}{384}e^4+\frac{27}{32}e'^2\gamma^2\right)m^2 \\
+\left(+\frac{2145}{256}e^2+\frac{3}{64}\gamma^2-\frac{25\,495}{512}e'^2 e^2\right)m^3+\left(+\frac{895}{288}-\frac{98\,113}{9216}\gamma^2\right. \\
\qquad\left.+\frac{3995}{192}e'^2+\frac{327\,263}{36\,864}e^2\right)m^4 \\
+\left(+\frac{1261}{96}+\frac{11\,349}{64}e'^2\right)m^5+\frac{3028}{81}m^6+\frac{76\,741}{864}m^7
\end{array}\right\}$$

The above relation affords a remarkable method of determining the secular variations of the Moon's mean motion, and of the motions of its perigee and node. From the equations (94) of the preceding section it appears that these variations are immediately obtainable from the three equations

$$\frac{d\,\mathrm{L}}{d\,a}\frac{d\,a}{d\,t}+\frac{d\,\mathrm{L}}{d\,e}\frac{d\,e}{d\,t}+\frac{d\,\mathrm{L}}{d\,\gamma}\frac{d\,\gamma}{d\,t}+\frac{d\,\mathrm{L}}{d\,e'}\frac{d\,e'}{d\,t}=0$$

$$\frac{d\,\mathrm{G}}{d\,a}\frac{d\,a}{d\,t}+\frac{d\,\mathrm{G}}{d\,e}\frac{d\,e}{d\,t}+\frac{d\,\mathrm{G}}{d\,\gamma}\frac{d\,\gamma}{d\,t}+\frac{d\,\mathrm{G}}{d\,e'}\frac{d\,e'}{d\,t}=0$$

$$\frac{d\,\mathrm{H}}{d\,a}\frac{d\,a}{d\,t}+\frac{d\,\mathrm{H}}{d\,e}\frac{d\,e}{d\,t}+\frac{d\,\mathrm{H}}{d\,\gamma}\frac{d\,\gamma}{d\,t}+\frac{d\,\mathrm{H}}{d\,e'}\frac{d\,e'}{d\,t}=0$$

The value of the secular variation of e' is to be inserted in the last terms, and the equations are to be solved for

$$\frac{d\,a}{d\,t}, \frac{d\,e}{d\,t}, \text{ and } \frac{d\,\gamma}{d\,t}$$

We then have

$$\frac{d^2\,\varepsilon}{d\,t^2}=\frac{d\,n}{d\,a}\frac{d\,a}{d\,t}$$

$$\frac{d^2\,\pi}{d\,t^2}=\frac{d\,\pi_1}{d\,a}\frac{d\,a}{d\,t}+\frac{d\,\pi_1}{d\,e}\frac{d\,e}{d\,t}+\frac{d\,\pi_1}{d\,\gamma}\frac{d\,\gamma}{d\,t}+\frac{d\,\pi_1}{d\,e'}\frac{d\,e'}{d\,t}$$

$$\frac{d^2\,\theta}{d\,t^2}=\frac{d\,\theta_1}{d\,a}\frac{d\,a}{d\,t}+\frac{d\,\theta_1}{d\,e}\frac{d\,e}{d\,t}+\frac{d\,\theta_1}{d\,\gamma}\frac{d\,\gamma}{d\,t}+\frac{d\,\theta_1}{d\,e'}\frac{d\,e'}{d\,t}$$

π_1 and θ_1 being the secular motions of π and θ, as given in the preceding section p. 190.

The analytic expression of these variations in this way offers no difficulty as far as the terms of the seventh order. Above this, however, we meet with the difficulty that the transformations given by DELAUNAY, on page 800, do not extend in any case beyond terms of this order in e, γ, and m. Moreover, DELAUNAY's supplementary expression for R, on page 742, does not seem to afford the means of carrying the transformation further. Were it required to write the value of L, or of k_{ϵ}, with a higher degree of precision than this, it might be possible to do so inductively from DELAUNAY's expression for the secular variation of the Moon's mean motion given in the *Comptes Rendus*, Vol. 72, p. 496, which extends to terms in m^9. This course does not, however, seem necessary, as the coefficient in question, when carried to the fifth power of m, is sufficiently accurate for all purposes except the computation of the secular acceleration itself.

It might be remarked, in this connection, that DELAUNAY never published any details of the process by which he effected the transformation of the values of e, γ, and a, beyond terms of the seventh order in m. It is understood that these were reserved for his intended third volume, which was never prepared for the press.

§ 14.

TRANSFORMATION OF THE PERTURBATIVE FUNCTION R AND OF ITS DERIVATIVES.

We have in the first chapter reduced the function R to the following form:

$$\text{R}=m_4\,\mu_1\left(\frac{1}{\rho}+\frac{x'\,x_4+y'\,y_4+z'\,z_4}{r_4^3}\right)$$

$$+\frac{1}{2}m_4\,\mu_2\left(\frac{3\,\Delta^2}{\rho^5}-\frac{r^2}{\rho^3}\right)$$

where

$$\Delta=x(x_4+x')+y(y_4+y')+z(z_4+z')$$

In this form, however, the function R is subject to a serious inconvenience, in containing the square of Δ, which is a function of the elements of the Sun, Moon, and planet. This makes it very difficult to select all the terms corresponding to any particular combination of the varying angles. We shall, therefore, express R in the form of a series of products, in which that portion of its derivatives which contains the disturbed elements of the Sun and Moon shall be separated from that portion which depends upon the elements of the planet.

We have from the preceding expression for Δ

$$\begin{aligned}\Delta^2 = & (x'+x_4)^2 x^2 + (y'+y_4)^2 y^2 + (z'+z_4)^2 z^2 \\ & + 2\,(x'+x_4)\ (y'+y_4)\, x\, y \\ & + 2\,(y'+y_4)\ (z'+z_4)\, y\, z \\ & + 2\,(z'+z_4)\ (x'+x_4)\, z\, x\,\cdot\end{aligned}$$

We have now by differentiating the preceding expression for R with respect to any element a

$$\begin{aligned}\frac{d\,\mathrm{R}}{d\,a} = & m_4\,\mu_1 \Big\{ \Big(\frac{x_4}{r_4^{\,3}} - \frac{x'}{\rho^3} - \frac{x_4}{\rho^3}\Big)\frac{d\,x'}{d\,a} \\ & + \Big(\frac{y_4}{r_4^{\,3}} - \frac{y'}{\rho^3} - \frac{y_4}{\rho^3}\Big)\frac{d\,y'}{d\,a} \\ & + \Big(\frac{z_4}{r_4^{\,3}} - \frac{z'}{\rho^3} - \frac{z_4}{\rho^3}\Big)\frac{d\,z'}{d\,a}\Big\} \\ & + m_4\,\mu_2 \Big\{ \Big(3\,\frac{(x'+x_4)^2}{\rho^5} - \frac{1}{\rho^3}\Big)\, x\,\frac{d\,x}{d\,a} \\ & + \Big(3\,\frac{(y'+y_4)^2}{\rho^5} - \frac{1}{\rho^3}\Big)\, y\,\frac{d\,y}{d\,a} \\ & + \Big(3\,\frac{(z'+z_4)^2}{\rho^5} - \frac{1}{\rho^3}\Big)\, z\,\frac{d\,z}{d\,a} \\ & + \frac{3\,(x'+x_4)\,(y'+y_4)}{\rho^5}\Big(x\,\frac{d\,y}{d\,a} + y\,\frac{d\,x}{d\,a}\Big) \\ & + \frac{3\,(y'+y_4)\,(z'+z_4)}{\rho^5}\Big(y\,\frac{d\,z}{d\,a} + z\,\frac{d\,y}{d\,a}\Big) \\ & + \frac{3\,(z'+z_4)\,(x'+x_4)}{\rho^5}\Big(z\,\frac{d\,x}{d\,a} + x\,\frac{d\,z}{d\,a}\Big)\Big\}\end{aligned} \qquad (102)$$

Putting the latter part of this expression in this form affords us a great advantage in the selection of the terms which may become sensible. We see that in each of the

last six products the first factor contains only the coordinates of the Sun and planet, and not those of the Moon. Therefore, when we omit, as we must in a first approximation, the squares and products of the masses of the planets, the factors in question admit of being developed as a function of the mean anomalies of the Sun and planet. Each term of this development is of the form

$$K \frac{\sin}{\cos} (i g' + j g_4) = K \frac{\sin}{\cos} U$$

g' being the mean anomaly of the Sun, g_4 that of the planet, and K a function of the eccentricities, mean distances, and inclinations. As it is necessary to carry the development to high multiples of i and j, the terms, in the case of a single planet, may amount to several hundred in number.

The second factor of each product is a function only of the coordinates of the Moon relatively to the Earth, which we have represented by x, y, z, or, rather, it is a function of the elements of the motion of the Moon around the Earth. But these necessarily include the elements of the Sun, multiplied by its disturbing force. As we have developed x, y, and z, each term of the second factor will be of the form

$$\varkappa \frac{\sin}{\cos} (i_1 l + i_2 \pi + i_3 \theta + i_4 l' + i_5 \pi' + i_6 \theta')$$

In the first approximation we may regard π' and θ', or the solar node and perigee as constant, and the terms in question may be put in the form

$$\varkappa \frac{\sin}{\cos} (i_1 l + i_2 \pi + i_3 \theta + i_4 g') = \varkappa \frac{\sin}{\cos} N$$

Although we have here four variable angles, all of whose multiples are to be included, yet they converge so rapidly that the terms we have retained in the lunar coordinates include all that can be expected to give any appreciable result.

Having thus developed each factor, if we form their products, we shall find each term of $\frac{d R}{d a}$, which contains the lunar elements, to be of the form

$$\frac{1}{2} m_4 \mu_2 K \frac{\sin}{\cos} (N \pm U)$$

The derivatives of each of the elements contain these terms multiplied by a factor which is constant in the approximation. If, then, we represent by s the coefficient of the time in $N \pm n$, we see that the integrated values of all the elements of the Moon contain terms of the form

$$\frac{c}{s} \frac{\sin}{\cos} (N \pm U)$$

and ε, π, and θ contain in addition terms of the form

$$\frac{c'}{s^2} \frac{\sin}{\cos} (N \pm U)$$

We hence derive the two following theorems of the motion of the Moon disturbed by a planet:

1. *The elements of the orbit of the Moon disturbed by a planet contain an infinity of terms of the form*

$$c \begin{matrix}\sin\\ \cos\end{matrix} (i_1 l + i_2 \pi + i_3 \theta + i_4 g' + i_5 g_4)$$

2. *If any such term is of very long period it may become sensible owing to the smallness of the divisor, the first and second powers of which enter into the coefficient* c.

§ 15.

FORM OF THE PERTURBATIONS ARISING FROM THE SECOND TERM OF R.

Let us take up the second part of the expression (102) which we shall call $\frac{d\,R_2}{d\,a}$, and examine more fully the perturbations to which it may give rise. We have just shown that the coefficients which depend on the coordinates of the Sun and planetmay be developed in multiples of their mean anomalies. Let us then put

$$\begin{aligned}
\frac{3}{2}\frac{(x'+x_4)^2}{\rho^5} - \frac{1}{2}\frac{1}{\rho^3} &= K_1 \cos U + K'_1 \sin U \\
\frac{3}{2}\frac{(y'+y_4)^2}{\rho^5} - \frac{1}{2}\frac{1}{\rho^3} &= K_2 \cos U + K'_2 \sin U \\
\frac{3}{2}\frac{(z'+z_4)^2}{\rho^5} - \frac{1}{2}\frac{1}{\rho^3} &= K_3 \cos U + K'_3 \sin U \\
\frac{3\,(x'+x_4)\,(y'+y_4)}{\rho^5} &= K_4 \cos U + K'_4 \sin U \\
\frac{3\,(y'+y_4)\,(z'+z_4)}{\rho^5} &= K_5 \cos U + K'_5 \sin U \\
\frac{3\,(z'+z_4)\,(x'+x_4)}{\rho^5} &= K_6 \cos U + K'_6 \sin U
\end{aligned} \qquad (103)$$

Here K and K′ are numerical coefficients, calculated in the manner to be hereafter described, and U an angle of the form

$$i'\,g' + i_4\,g_4$$

Let us also put

$$\begin{aligned}
\frac{x^2}{a^2} &= \varkappa_1 \cos N \\
\frac{y^2}{a^2} &= \varkappa_2 \cos N
\end{aligned} \qquad (104)$$

$$\begin{aligned}
\frac{z^2}{a^2} &= \varkappa_3 \cos \mathrm{N} \\
\frac{xy}{a^2} &= \varkappa_4 \sin \mathrm{N} \\
\frac{yz}{a^2} &= \varkappa_5 \sin \mathrm{N} \\
\frac{zx}{a^2} &= \varkappa_6 \sin \mathrm{N}
\end{aligned} \tag{104}$$

Each N is of the form

$$i\,l + i'\,\pi + i''\,\theta + i'''g'$$

and each K a function of a, e, γ, a', e' and γ'.

We therefore have for the derivatives relatively to ε,

$$\begin{aligned}
2x\frac{dx}{d\varepsilon} &= -i a^2 \varkappa_1 \sin \mathrm{N} \\
2y\frac{dy}{d\varepsilon} &= -i a^2 \varkappa_2 \sin \mathrm{N} \\
2z\frac{dz}{d\varepsilon} &= -i a^2 \varkappa_3 \sin \mathrm{N} \\
x\frac{dy}{d\varepsilon} + y\frac{dx}{d\varepsilon} &= i a^2 \varkappa_4 \cos \mathrm{N} \\
y\frac{dz}{d\varepsilon} + z\frac{dy}{d\varepsilon} &= i a^2 \varkappa_5 \cos \mathrm{N} \\
z\frac{dx}{d\varepsilon} + x\frac{dz}{d\varepsilon} &= i a^2 \varkappa_6 \cos \mathrm{N}
\end{aligned} \tag{105}$$

while the derivatives with respect to π and θ are of the same form, except that i' and i'', respectively, are to be substituted for i. For the derivatives relatively to ν, e, and γ, we have

$$\begin{aligned}
2x\frac{dx}{d\nu} &= a^2\left(2\varkappa_1 + \frac{d\varkappa_1}{d\nu}\right) \cos \mathrm{N} \\
2y\frac{dy}{d\nu} &= a^2\left(2\varkappa_2 + \frac{d\varkappa_2}{d\nu}\right) \cos \mathrm{N} \\
2z\frac{dz}{d\nu} &= a^2\left(2\varkappa_3 + \frac{d\varkappa_3}{d\nu}\right) \cos \mathrm{N}
\end{aligned} \tag{106}$$

$$x\frac{d\,y}{d\,\varpi}+y\frac{d\,x}{d\,\varpi}=a^2\left(2\,\varkappa_4+\frac{d\,\varkappa_4}{d\,\varpi}\right)\sin \mathrm{N}$$

etc., etc., etc.

$$2\,x\frac{d\,x}{d\,e}=a^2\frac{d\,\varkappa_1}{d\,e}\cos \mathrm{N}$$

. . . .

$$x\frac{d\,y}{de}+y\frac{d\,x}{de}=a^2\frac{d\,\varkappa_4}{d\,e}\sin \mathrm{N}$$

etc., etc.,

while the derivatives with respect to γ are of the same form.

Substituting in the general form (102) the expressions in (103), (105) and (106) we find for the result of any combination of terms

$$\frac{d\,\mathrm{R}_2}{d\,\varepsilon}=\frac{1}{2}\,i\,m_4\,\mu_2(\pm K'_1\varkappa_1\pm K'_2\varkappa_2\pm K'_3\varkappa_3+K_4\varkappa_4+K_5\varkappa_5+K_6\varkappa_6)\,a^2\cos(\mathrm{N}\pm\mathrm{U})$$
$$+\frac{1}{2}\,i\,m_4\,\mu_2(-K_1\varkappa_1-K_2\varkappa_2-K_3\varkappa_3\pm K'_4\varkappa_4\pm K'_5\varkappa_4\pm K'_6\varkappa_6)\,a^2\sin(\mathrm{N}\pm\mathrm{U})$$

$$\frac{d\,\mathrm{R}_2}{d\,\varpi}=\frac{1}{2}\,m_4\,\mu_2\Big(2\,K_1\varkappa_1+\ \ldots\ +K_1\frac{d\,\varkappa_1}{d\,\varpi}+K_2\frac{d\,\varkappa_2}{d\,\varpi}+K_3\frac{d\,\varkappa_3}{d\,\varpi}\mp K'_4\frac{d\,\varkappa_4}{d\,\varpi}\mp K'_5\frac{d\,\varkappa_5}{d\,\varpi}$$
$$\mp K'_6\frac{d\,\varkappa_6}{d\,\varpi}\Big)\,a^2\cos(\mathrm{N}\pm\mathrm{U})$$
$$+\frac{1}{2}\,m_4\,\mu_2\Big(\pm 2\,K'_1\varkappa_1\pm\ \ldots\ \pm K'_1\frac{d\,\varkappa_1}{d\,\varpi}\pm K'_2\frac{d\,\varkappa_2}{d\,\varpi}\pm K'_3\frac{d\,\varkappa_3}{d\,\varpi}+K_4\frac{d\,\varkappa_4}{d\,\varpi}$$
$$+K_5\frac{d\,\varkappa_5}{d\,\varpi}+K_6\frac{d\,\varkappa_6}{d\,\varpi}\Big)\,a^2\sin(\mathrm{N}\pm\mathrm{U})$$

If we put for brevity

$$\begin{aligned}K&=\pm K'_1\varkappa_1\pm K'_2\varkappa_2\pm K'_3\varkappa_3+K_4\varkappa_4+K_5\varkappa_5+K_6\varkappa_6\\ K'&=\quad K_1\varkappa_1+K_2\varkappa_2+K_3\varkappa_3\mp K'_4\varkappa_4\mp K'_5\varkappa_5\mp K'_6\varkappa_6\end{aligned}\qquad(107)$$

we shall have

$$\begin{aligned}\frac{d\,\mathrm{R}_2}{d\,\varepsilon}&=\frac{1}{2}\,i\,m_4\,\mu_2\,a^2\Big\{K\cos(\mathrm{N}\pm\mathrm{U})-K'\sin(\mathrm{N}\pm\mathrm{U})\Big\}\\ \frac{d\,\mathrm{R}_2}{d\,\pi}&=\frac{1}{2}\,i'\,m_4\,\mu_2\,a^2\Big\{K\cos(\mathrm{N}\pm\mathrm{U})-K'\sin(\mathrm{N}\pm\mathrm{U})\Big\}\\ \frac{d\,\mathrm{R}_2}{d\,\theta}&=\frac{1}{2}\,i''\,m_4\,\mu_2\,a^2\Big\{K\cos(\mathrm{N}\pm\mathrm{U})-K'\sin(\mathrm{N}\pm\mathrm{U})\Big\}\end{aligned}\qquad(108)$$

$$\frac{d\,\mathrm{R}_2}{d\,\varpi}=\frac{1}{2}\,m_4\,\mu_2\,a^2\Big\{\Big(2\,K+\frac{d\,K}{d\,\varpi}\Big)\sin(\mathrm{N}\pm\mathrm{U})+\Big(2\,K'+\frac{d\,K_1}{d\,\varpi}\Big)\cos(\mathrm{N}\pm\mathrm{U})\Big\}$$

$$\frac{d\,\mathrm{R}_2}{d\,e}=\frac{1}{2}m_4\,\mu_2\,a^2\left\{\frac{d\,K}{d\,e}\sin(\mathrm{N}\pm\mathrm{U})+\frac{d\,K'}{d\,e}\cos(\mathrm{N}\pm\mathrm{U})\right\}$$

$$\frac{d\,\mathrm{R}_2}{d\,\gamma}=\frac{1}{2}m_4\,\mu_2\,a^2\left\{\frac{d\,K}{d\,\gamma}\sin(\mathrm{N}\pm\mathrm{U})+\frac{d\,K'}{d\,\gamma}\cos(\mathrm{N}\pm\mathrm{U})\right\}$$

These values of the derivatives of R being substituted in the equations (89) and (90), will give the values of the derivatives of the elements with respect to the time.

§ 16.

FINAL FORM OF THE DIFFERENTIAL EQUATIONS FOR THE VARIATIONS OF THE LUNAR ELEMENTS.

Returning once more to the equations (89) and (90), we have to find what terms are produced by the preceding value of R_2. If, in these equations, we omit the derivatives with respect to t' in the second members, the effect of which we have just computed, substitute for the derivatives of R those with respect to R_2 just found, and for the factors $\left(\frac{d\,v}{d\,k_e}\right)$, etc., their numerical values found in (96) the equations (89) become

$$\begin{aligned}
\frac{d\,v}{d\,t}&=\mu_4\,a^3\,n\,\{1.0112\,i-.0146\,i'+.0037\,i''\}\,\{\mathrm{K}\cos(\mathrm{N}\pm\mathrm{U})-\mathrm{K}'\sin(\mathrm{N}\pm\mathrm{U})\}\\
e\frac{d\,e}{d\,t}&=\mu_4\,a^3\,n\,\{-.0005\,i-.5257\,i'+.0000\,i''\}\,\{\mathrm{K}\cos(\mathrm{N}\pm\mathrm{U})-\mathrm{K}'\sin(\mathrm{N}\pm\mathrm{U})\}\quad(109)\\
\gamma\frac{d\,\gamma}{d\,t}&=\mu_4\,a^3\,n\,\{-.0005\,i-.0005\,i'-.1250\,i''\}\,\{\mathrm{K}\cos(\mathrm{N}\pm\mathrm{U})-\mathrm{K}'\sin(\mathrm{N}\pm\mathrm{U})\}
\end{aligned}$$

where we put, for brevity,

$$\mu_4=\frac{m_4}{m_2+m_3}$$

or the ratio of the mass of the planet to the sum of the masses of the Earth and Moon. Integrating these equations, and putting, as before, ν, the quotient of the Moon's mean motion by the coefficient of the time in the angle $\mathrm{N}\pm\mathrm{U}$, we have

$$\begin{aligned}
\delta\,v&=\mu_4\,a^3\,\nu\,\{1.0112\,i-.0146\,i'+.0037\,i''\}\,\{\mathrm{K}\sin(\mathrm{N}\pm\mathrm{U})+\mathrm{K}'\cos(\mathrm{N}\pm\mathrm{U})\}\\
e\,\delta\,e&=\mu_4\,a^3\,\nu\,\{-.0005\,i-.5257\,i'+.0000\,i''\}\,\{\mathrm{K}\sin(\mathrm{N}\pm\mathrm{U})+\mathrm{K}'\cos(\mathrm{N}\pm\mathrm{U})\}\quad(110)\\
\gamma\,\delta\,\gamma&=\mu_4\,a^3\,\nu\,\{-.0005\,i-.0005\,i'-.1250\,i''\}\,\{\mathrm{K}\sin(\mathrm{N}\pm\mathrm{U})+\mathrm{K}'\cos(\mathrm{N}\pm\mathrm{U})\}
\end{aligned}$$

There can be no terms corresponding to the case of $\mathrm{N}\pm\mathrm{U}=0$, because U contains neither ε, π, nor θ, and therefore in this case N can contain neither ε, π, nor θ, and therefore i, i', and i'' will all be zero, so that the coefficient of the angle will vanish.

These values of $\delta\,\varpi$, $\delta\,e$, and $\delta\,\gamma$ are now to be substituted in the derivatives of n, π_1, and θ_1, and the latter are then to be integrated. If we put

$$\delta\,n = \frac{d\,n}{d\,\varpi}\,\delta\,\varpi$$

$$\delta\,\pi_1 = \frac{d\,\pi_1}{d\,\varpi}\,\delta\,\varpi + \frac{d\,\pi_1}{d\,e}\,\delta\,e + \frac{d\,\pi_1}{d\,\gamma}\,\delta\,\gamma$$

$$\delta\,\theta_1 = \frac{d\,\theta_1}{d\,\varpi}\,\delta\,\varpi + \frac{d\,\theta_1}{d\,e}\,\delta\,e + \frac{d\,\theta_1}{d\,\gamma}\,\delta\,\gamma$$

the values of $\delta\,\varepsilon$, $\delta\,\pi$, and $\delta\,\theta$ will be given by the equations

$$\delta\,\varepsilon = \int \delta\,n\;d\,t - \int \frac{d\,\mathrm{R}}{d\,k_\varepsilon}\,d\,t$$

$$\delta\,\pi = \int \delta\,\pi_1\,d\,t - \int \frac{d\,\mathrm{R}}{d\,k_\pi}\,d\,t$$

$$\delta\,\theta = \int \delta\,\theta_1\,d\,t - \int \frac{d\,\mathrm{R}}{d\,k_\theta}\,d\,t$$

Substituting the numerical values of $\frac{d\,n}{d\,\varpi}$, $\frac{d\,\pi_1}{d\,\varpi}$, etc., we have

$$\delta\,n = -\frac{3}{2}\,n\,\delta\,\varpi$$

$$\delta\,\pi_1 = n\,\{\;\;.02223\,\delta\,\varpi - .0195\,e\,\delta\,e - .0994\,\gamma\,\delta\,\gamma\}$$

$$\delta\,\theta_1 = n\,\{-.00561\,\delta\,\varpi - .0236\,e\,\delta\,e + .0149\,\gamma\,\delta\,\gamma\}$$

Substituing the preceding values of $\delta\,\varpi$, $e\,\delta\,e$, and $\gamma\,\delta\,\gamma$, and integrating, we have

$$\delta\,\varepsilon = \mu_4\,a^3\,\nu^2\,\{-1.517\,i + .022\,i' - .005\,i''\}\,\{-\mathrm{K}\cos(\mathrm{N}\pm\mathrm{U}) + \mathrm{K}'\sin(\mathrm{N}\pm\mathrm{U})\}$$
$$-\int \frac{d\,\mathrm{R}}{d\,k_\varepsilon}\,d\,t$$

$$\delta\,\pi = \mu_4\,a^3\,\nu^2\,\{\;\;.0225\,i + .0099\,i' + .0124\,i''\}\,\{-\mathrm{K}\cos(\mathrm{N}\pm\mathrm{U}) \qquad (111)$$
$$+\mathrm{K}'\sin(\mathrm{N}\pm\mathrm{U})\} - \int \frac{d\,\mathrm{R}}{d\,k_\pi}\,d\,t$$

$$\delta\,\theta = \mu_4\,a^3\,\nu^2\,\{-\;.0057\,i + .0124\,i' - .0019\,i''\}\,\{-\mathrm{K}\cos(\mathrm{N}\pm\mathrm{U}) + \mathrm{K}'\sin(\mathrm{N}\pm\mathrm{U})\}$$
$$-\int \frac{d\,\mathrm{R}}{d\,k_\theta}\,d\,t$$

We are now to find the values of $\frac{dR}{dk_{\epsilon}}$, etc., from the equations

$$\frac{dR}{dk_{\epsilon}}=\left(\frac{d\alpha}{dk_{\epsilon}}\right)\frac{dR}{d\varpi}+\left(\frac{de}{dk_{\epsilon}}\right)\frac{dR}{de}+\left(\frac{d\gamma}{dk_{\epsilon}}\right)\frac{dR}{d\gamma}$$

etc. etc. etc. etc.

Substituting the numerical values of $\frac{d\varpi}{dk_{\epsilon}}$ and the values (108) of $\frac{dR_2}{d\varpi}$, etc., we have, for any one term of R_2,

$$\frac{dR_2}{dk_{\epsilon}}=\mu_4 a^3 n\left\{\quad 2.023\,K+1.011\frac{dK}{d\varpi}-.0005\frac{dK}{e\,de}-.0005\frac{dK}{\gamma\,d\gamma}\right\}\sin(N\pm U)$$

$$+\mu_4 a^3 n\left\{\quad 2.023\,K'+1.011\frac{dK'}{d\varpi}-.0005\frac{dK'}{e\,de}-.0005\frac{dK'}{\gamma\,d\gamma}\right\}\cos(N\pm U)$$

$$\frac{dR_2}{dk_{\pi}}=\mu_4 a^3 n\left\{-.0293\,K-.0147\frac{dK}{d\varpi}-0.526\frac{dK}{e\,de}-.0005\frac{dK}{\gamma\,d\gamma}\right\}\sin(N\pm U)$$

$$+\mu_4 a^3 n\left\{-.0293\,K'-.0147\frac{dK'}{d\varpi}-0.526\frac{dK'}{e\,de}-.0005\frac{dK'}{\gamma\,d\gamma}\right\}\cos(N\pm U)$$

$$\frac{dR_2}{dk_{\theta}}=\mu_4 a^3 n\left\{+.0075\,K+.0037\frac{dK}{d\varpi}+.00004\frac{dK}{e\,de}-0.1250\frac{dK}{\gamma\,d\gamma}\right\}\sin(N\pm U)$$

$$+\mu_4 a^3 n\left\{+.0075\,K'+.0037\frac{dK'}{d\varpi}+.00004\frac{dK'}{e\,de}-0.1250\frac{dK'}{\gamma\,d\gamma}\right\}\cos(N\pm U)$$

Integrating, and substituting in the preceding expressions for $\delta\epsilon$, $\delta\pi$, and $\delta\theta$, we find that the important terms of these expressions may be expressed in the form

$$\begin{aligned}\delta\epsilon&=\epsilon_c\cos(N\pm U)+\epsilon_s\sin(N\pm U)\\ \delta\pi&=\pi_c\cos(N\pm U)+\pi_s\sin(N\pm U)\\ \delta\theta&=\theta_c\cos(N\pm U)+\theta_s\sin(N\pm U)\end{aligned}\qquad(112)$$

where we have

$$\epsilon_c=\mu_4 a^3\left\{\left(\quad 1.517\,i-.022\,i'+.005\,i''\right)\nu^2 K+\left(\quad 2.023\,K+1.011\frac{dK}{d\varpi}\right)\nu\right\}$$

$$\epsilon_s=\mu_4 a^3\left\{\left(-1.517\,i+.022\,i'-.005\,i''\right)\nu^2 K'+\left(-2.023\,K'-1.011\frac{dK'}{d\varpi}\right)\nu\right\}$$

$$\pi_c=\mu_4 a^3\left\{\left(-.0226\,i-.0100\,i'-.0124\,i''\right)\nu^2 K+\left(-.0293\,K-.0147\frac{dK}{d\varpi}\right.\right.$$
$$\left.\left.-.526\frac{dK}{e\,de}\right)\nu\right\}\qquad(113)$$

$$\pi_s = \mu_4 a^3 \Big\{ (\quad .0226\, i + .0100\, i' + .0124\, i'')\, \nu^2 K' + \Big(+.0295\, K' + .0147 \frac{d K'}{d \varpi} + .526 \frac{d K'}{c\, d e} \Big) \nu \Big\}$$

$$\theta_c = \mu_4 a^3 \Big\{ (\quad 0057\, i - .0124\, i' + .0019\, i'')\, \nu^2 K + (\quad .0074\, K + .0037 \frac{d K}{d \varpi} - .1250 \frac{d K}{\gamma d \gamma})\, \nu \Big\} \qquad (113)$$

$$\theta_s = \mu_4 a^3 \Big\{ (-.0057\, i + .0124\, i' - .0019\, i'')\, \nu^2 K' + (-.0074\, K' - .0037 \frac{d K'}{d \varpi} + .1250 \frac{d K'}{\gamma d \gamma})\, \nu \Big\}$$

These expressions do not apply to the case of $N \pm U = 0$, since ν will then become infinite. In this case i, i', i'' all vanish, as already shown, and the values of $\frac{d \varepsilon}{d t}$, etc., reduce to constants. Integration then gives

$$\delta \varepsilon = \mu_4 a^3 \Big\{ -2.023\, K'_0 - 1.011 \frac{d K'_0}{d \varpi} + .0005 \frac{d K'_0}{e\, d e} \Big\} n t$$

$$\delta \pi = \mu_4 a^3 \Big\{ \quad .0293\, K'_0 + .0147 \frac{d K'_0}{d \alpha} + 0.526 \frac{d K'_0}{e\, d e} \Big\} n t \qquad (114)$$

$$\delta \theta = \mu_4 a^3 \Big\{ -.0074\, K'_0 - .0037 \frac{d K'_0}{d \varpi} + 0.125 \frac{d K'_0}{\gamma\, d \gamma} \Big\} n t$$

We have next to find accurate or approximate values of the various quantities which enter in these expressions. Let us begin with the quantities $\varkappa_1$, $\varkappa_2$, etc., which enter into the values of K. The quantities are by (104) the coefficients of cos N or sin N in the development of $\frac{x^2}{a^2}$, $\frac{y^2}{a^2}$, etc., which powers and products are to be derived from the values of x, y, and z in (81) by the substitution of the analytical values of k_1 and k_2 given in the same section. If we take the plane of the ecliptic at the mean epoch for that of xy, the value of γ' will be of the order of the disturbing forces, and may be neglected entirely when attention is confined to quantities of the first order relatively to the disturbing forces. Putting $\gamma' = 0$, the values of x, y, and z become

$$\frac{x}{a} = k_1 \cos \Big\{ (1 + i + j)\, l - i \pi - (j + j')\, \theta + (i' + j')\, l' - i'\, \pi' \Big\}$$

$$+ k_2 \cos \Big\{ (1 - i - j)\, l + i \pi + (j + j')\, \theta - (i' + j')\, l' + i'\, \pi' \Big\}$$

$$\frac{y}{a} = k_1 \sin \Big\{ (1 + i + j)\, l - i \pi - (j + j')\, \theta + (i' + j')\, l' - i'\, \pi' \Big\}$$

$$+ k_2 \sin \Big\{ (1 - i - j)\, l + i \pi + (j + j')\, \theta - (i' + j')\, l' + i'\, \pi' \Big\}$$

$$\frac{z}{a} = c \sin \Big\{ (i_1 + j_1)\, l - i_1\, \pi - (j_1 + j'_1)\, \theta + (i'_1 + j'_1)\, l' - i'_1\, \pi' \Big\}$$

The position of the axis of x being entirely arbitrary we may suppose all the longitudes to be counted from the solar perigee. This will be effected by putting $\pi'=0$, which will give $l'=g'$. Thus, x, y, z, etc., will be expressed in terms of the four varying angles

$$l,\ \pi,\ \theta,\ g',$$

the three quantities l, π, and θ representing the mean longitude of the Moon, and the longitudes of its perigee and node, all counted from the solar perigee, and g' the mean anomaly of the Sun or of the Earth. We shall first give the development of x, y, and z, retaining the solar perigee, π', so that the axis of X remains arbitrary. The first column gives the indices i, i', j, and j', or the coefficients of g, g', λ, and λ'. The next gives the coefficients of l, π, θ, l', and π', derived from those indices. We then have the coefficients k_1 and k_2, which, being multiplied by the *cosines* of the angles, give the terms of $\frac{x}{a}$, and by the *sines* the terms of $\frac{y}{a}$

i,	i'	j,	j'	l,	π,	θ,	l'	π'	k_1 and k_2
0	0	0	0	1	0	0	0	0	$1-\frac{1}{2}e^2-\gamma^2-\frac{1}{6}m^2+\frac{1}{2}e^2\gamma^2-\frac{931}{384}e^2m^2$
1	0	0	0	2	-1	0	0	0	$+\frac{1}{2}e-\frac{3}{8}e^3-\frac{1}{2}e\gamma^2+\frac{7}{24}em^2$
				0	1	0	0	0	$-\frac{3}{2}e+\frac{3}{2}e\gamma^2+\frac{5}{8}em^2$
2	0	0	0	3	-2	0	0	0	$+\frac{3}{8}e^2-\frac{3}{8}e^4-\frac{3}{8}e^2\gamma^2+\frac{7}{32}e^2m^2$
				-1	2	0	0	0	$+\frac{1}{8}e^2+\frac{1}{24}e^4+\frac{9}{8}e^2\gamma^2-\frac{5}{96}e^2m^2$
3	0	0	0	4	-3	0	0	0	$+\frac{1}{3}e^3$
				-2	3	0	0	0	$+\frac{1}{24}e^3$
-1	1	0	0	0	1	0	1	-1	$-\frac{27}{16}ee'm$
				2	-1	0	-1	1	$+\frac{33}{16}ee'm$
0	1	0	0	1	0	0	1	-1	$-\frac{3}{2}e'm+\frac{3}{4}e'm^2$
				1	0	0	-1	1	$+\frac{3}{2}e'm+\frac{3}{4}e'm^2$

i	i'	j	j'	l	π	θ	l'	π'	k_1 and k_2
1	1	0	0	2	−1	0	1	−1	$-\frac{33}{16}ee'm$
				0	1	0	−1	1	$+\frac{27}{16}ee'm$
−1	0	2	0	2	1	−2	0	0	$-\frac{5}{4}e\gamma^2$
				0	−1	2	0	0	$+\frac{9}{4}e\gamma^2$
0	0	2	0	3	0	−2	0	0	$\left(-\frac{15}{8}e^2\gamma^2\right)$
				−1	0	2	0	0	$+\gamma^2$
1	0	2	0	4	−1	−2	0	0	0
				−2	1	2	0	0	$+\frac{1}{2}e\gamma^2$
−2	0	2	−2	1	2	0	−2	0	$\frac{15}{32}e^2m$
				1	−2	0	2	0	$-\frac{75}{32}e^2m$
−1	0	2	−2	2	1	0	−2	0	$\frac{15}{16}em+\frac{337}{64}em^2$
				0	−1	0	2	0	$-\frac{45}{16}em-\frac{735}{64}em^2$
0	0	2	−2	3	0	0	−2	0	$+\frac{3}{16}m^2+\frac{7}{8}m^3+\frac{45}{32}e^2m$
				−1	0	0	2	0	$-\frac{19}{16}m^2-\frac{97}{24}m^3+\frac{15}{32}e^2m+\frac{3}{4}\gamma^2m$
1	0	2	−2	4	−1	0	−2	0	$+\frac{3}{8}em^2$
				−2	1	0	2	0	$-\frac{1}{16}em^2$
−1	−1	2	−2	2	1	0	−3	1	$+\frac{35}{16}ee'm$
				0	−1	0	3	−1	$-\frac{105}{16}ee'm$
0	−1	2	−2	3	0	0	−3	1	$+\frac{21}{32}e'm^2$
				−1	0	0	3	−1	$-\frac{133}{32}e'm^2$

i,	i',	j,	j'	l,	π,	θ,	l',	π'	k_1 and k_2
−1	1	2	−2	2	1	0	−1	−1	$-\frac{15}{16}ee'm$
				0	−1	0	1	1	$+\frac{45}{16}ee'm$
0	1	2	−2	3	0	0	−1	−1	$-\frac{3}{32}e'm^2$
				−1	0	0	1	1	$+\frac{19}{32}e'm^2$
0	0	0	2	1	0	−2	2	0	$-\frac{3}{2}\gamma^2 m$
				1	0	2	−2	0	$+\frac{3}{4}\gamma^2 m$
0	0	1	−1	2	0	0	−1	0	$-\frac{15}{32}\alpha m$
				0	0	0	1	0	$+\frac{45}{32}\alpha m$
0	1	1	−0	2	0	0	0	−1	$+\frac{5}{8}e'\alpha$
				0	0	0	0	1	$-\frac{15}{8}e'\alpha$
0	2	0	0	1	0	0	2	−2	$-\frac{9}{8}e'^2 m$
				1	0	0	−2	2	$+\frac{9}{8}e'^2 m$

$$\frac{z}{a}=\Sigma\, c \sin \mathrm{N}$$

i,	i',	j,	j'	l,	π,	θ,	l',	π'	c
−2	0	1	0	−1	2	−1	0	0	$-e^2\gamma$
−1	0	1	0	0	1	−1	0	0	$-3e\gamma$
0	0	1	0	1	0	−1	0	0	$+2\gamma-e^2\gamma-\gamma^3-\frac{1}{3}\gamma m^2$
1	0	1	0	2	−1	−1	0	0	$+e\gamma$
2	0	1	0	3	−2	−1	0	0	$+\frac{3}{4}e^2\gamma$
0	−1	1	0	1	0	−1	−1	1	$+\frac{3}{4}e'\gamma m$

$\frac{z}{a} = \Sigma\, c \sin \mathrm{N}$—(Continued.)

i	i'	j	j'	l	π	θ	l'	π'	c
0	1	1	0	1	0	−1	1	−1	$-\frac{3}{4} e' \gamma m$
−1	0	3	−2	2	1	−1	−2	0	$+\frac{15}{8} e \gamma m$
0	0	3	−2	3	0	−1	−2	0	$+\frac{3}{8} \gamma m^2$
0	−1	1	−2	1	0	1	−3	1	$+\frac{7}{4} e' \gamma m$
−1	0	1	−2	0	1	1	−2	0	$+\frac{9}{2} e \gamma m$
0	0	1	−2	1	0	1	−2	0	$+\frac{3}{4} \gamma m + \frac{41}{16} \gamma m^2$
1	0	1	−2	2	−1	1	−2	0	$+\frac{3}{8} e \gamma m$
0	1	1	−2	1	0	1	−1	−1	$-\frac{3}{4} e' \gamma m$

We next form the products x^2, y^2, z^2, $x\,y$, $y\,z$, $x\,z$, so as to obtain the expressions for $\varkappa_1$, $\varkappa_2$, $\varkappa_3$, $\varkappa_4$, $\varkappa_5$, $\varkappa_6$ defined in (104) and used in (107). The complete values to quantities of the third order, inclusive, are given in the following tables. We now replace l, π, θ, l' by the quantities

$$(l) = l - \pi'$$
$$(\pi) = \pi - \pi'$$
$$(\theta) = \theta - \pi'$$
$$g' = l' - \pi'$$

omitting the multiplier of π' entirely, which will be equivalent to supposing all the longitudes counted from the Sun's perihelion.

The computation stops at terms of the third order, because this is about the limit of possibly sensible terms of long period, and our immediate object is rather to detect the few such terms that may possibly exist than to make an accurate computation of their values. If necessary a more accurate determination of them can then be made after they are found.

(l) (π) (θ) g'	Values of $\varkappa_1$	Values of $\varkappa_2$
0 0 0 0	$\frac{1}{2}+\frac{3}{4}e^2-\gamma^2-\frac{1}{6}m^2-\frac{3}{2}e^2\gamma^2+\gamma^4$ $+\frac{969}{768}e^2m^2+\frac{37}{192}\gamma^2m^2-\frac{1}{4}e'^2m^2$ $+\frac{407}{288}m^4+\frac{4995}{512}e^2m^3$	$\frac{1}{2}+\frac{3}{4}e^2-\gamma^2-\frac{1}{6}m^2-\frac{3}{2}e^2\gamma^2+\gamma^4$ $+\frac{969}{768}e^2m^2+\frac{37}{192}\gamma^2m^2-\frac{1}{4}e'^2m^2$ $+\frac{407}{288}m^2+\frac{4995}{512}e^2m^3$
0 0 0 1	$+\frac{67}{32}e'm^2$	$+\frac{29}{32}e'm^2$
0 0 0 2	$-\frac{19}{16}m^2-\frac{97}{24}m^3+\frac{75}{16}e^2m+\frac{3}{4}\gamma^2m$	$+\frac{19}{16}m^2+\frac{97}{24}m^3-\frac{75}{16}e^2m-\frac{3}{4}\gamma^2m$
0 0 0 3	$-\frac{133}{32}e'm^2$	$+\frac{133}{32}e'm^2$
0 0 2 −2	$-\frac{3}{4}\gamma^2m$	$-\frac{3}{4}\gamma^2m$
0 0 2 0	$+\gamma^2$	$-\gamma^2$
0 2 0 −2	$+\frac{45}{16}e^2m$	$+\frac{45}{16}e^2m$
0 2 0 0	$+\frac{5}{4}e^2-\frac{5}{4}e^2\gamma^2-\frac{97}{96}e^2m^2$	$-\frac{5}{4}e^2+\frac{5}{4}e^2\gamma^2+\frac{97}{96}e^2m^2$
1 0 0 −1	$+\frac{15}{16}\alpha m$	$+\frac{15}{16}\alpha m$
1 0 0 0	$-\frac{5}{4}e'\alpha$	$+\frac{5}{8}e'\alpha$
1 0 0 1	$+\frac{45}{32}\alpha m$	$-\frac{45}{32}\alpha m$
1 −3 0 0	$-\frac{7}{48}e^3$	$+\frac{7}{48}e^3$
1 −1 0 −2	$+\frac{55}{32}em^2$	$-\frac{55}{32}em^2$
1 −1 0 −1	$-\frac{21}{8}ee'm$	$-\frac{21}{8}ee'm$
1 −1 0 0	$-e+\frac{1}{8}e^3+2e\gamma^2+\frac{13}{12}em^2$	$-e+\frac{1}{8}e^3+2e\gamma^2+\frac{13}{12}em^2$

$(l)\,(\pi)\,(\theta)\,g'$	Values of $\varkappa_1$	Values of $\varkappa_2$
1 −1 0 1	$+\frac{87}{16}ee'm$	$-\frac{3}{16}ee'm$
1 −1 0 2	$-\frac{45}{16}em-\frac{773}{64}em^2$	$+\frac{45}{16}em+\frac{773}{64}em^2$
1 −1 0 3	$-\frac{105}{16}ee'm$	$+\frac{105}{16}ee'm$
1 −1 −2 0	$-e\gamma^2$	$+e\gamma^2$
1 −1 2 0	$+\frac{11}{4}e\gamma^2$	$-\frac{11}{4}e\gamma^2$
1 1 0 −3	$-\frac{35}{8}ee'm$	$-\frac{35}{8}ee'm$
1 1 0 −2	$-\frac{15}{8}em-\frac{139}{32}em^2$	$-\frac{15}{8}em-\frac{139}{32}em^2$
1 1 0 −1	. . . .	$+\frac{39}{16}ee'm$
1 1 0 0	$-\frac{3}{2}e+\frac{13}{16}e^3+3e\gamma^2+\frac{7}{8}em^2$	$+\frac{3}{2}e-\frac{13}{16}e^3-3e\gamma^2-\frac{7}{8}em^2$
1 1 0 1	$+\frac{9}{16}ee'm$	$-\frac{9}{16}ee'm$
1 1 −2 0	$-\frac{1}{2}e\gamma^2$	$-\frac{1}{2}e\gamma^2$
2 0 0 −3	$-\frac{7}{2}e'm^2$	$-\frac{7}{2}e'm^2$
2 0 0 −2	$-m^2-\frac{19}{6}m^3+\frac{15}{32}e^2m+\frac{3}{4}\gamma^2m$	$-m^2-\frac{19}{6}m^3+\frac{15}{32}e^2m+\frac{3}{4}\gamma^2m$
	$+\frac{9}{8}e'^2m$	$-\frac{9}{8}e'^2m$
2 0 0 −1	$\frac{3}{2}e'm+\frac{5}{4}e'm^2$	$-\frac{3}{2}e'm-\frac{1}{4}e'm^2$
2 0 0 0	$\frac{1}{2}-2e^2-\gamma^2-\frac{1}{6}m^2$	$-\frac{1}{2}+\frac{5}{4}e^2+\gamma^2+\frac{1}{6}m^2$
2 0 0 1	$-\frac{3}{2}e'm+\frac{3}{4}e'm^2$	$+\frac{3}{2}e'm-\frac{3}{4}e'm^2$

(l)	(π)	(θ)	g'	Values of $\varkappa_1$	Values of $\varkappa_2$
2	0	0	2	$-\frac{45}{32}e^2m$	$\frac{9}{8}e'^2m-\frac{45}{32}e^2m$
2	0	−2	0	$+\gamma^2$	$+\gamma^2$
2	0	−2	2	$-\frac{3}{2}\gamma^2m$	$+\frac{3}{2}\gamma^2m$
2	0	2	−2	$+\frac{3}{4}\gamma^2m$	$-\frac{3}{4}\gamma^2m$
2	−2	0	0	$-\frac{1}{4}e^2$	$-\frac{1}{4}e^2$
2	−2	0	2	$-\frac{15}{4}e^2m$	$+\frac{15}{4}e^2m$
2	2	0	−2	$-\frac{15}{16}e^2m$	$+\frac{15}{16}e^2m$
3	0	0	−1	$-\frac{15}{32}\alpha m$	$+\frac{15}{32}\alpha m$
3	0	0	0	$-\frac{5}{4}e'\alpha$	$-\frac{5}{8}e'\alpha$
3	−1	0	−2	$-\frac{9}{16}e^2m$	$-\frac{9}{16}em^2$
3	−1	0	−1	$+\frac{33}{16}ee'm$	$-\frac{45}{16}ee'm$
3	−1	0	0	$-\frac{1}{2}e-\frac{19}{16}e^3-e\gamma^2+\frac{5}{24}em^2$	$-\frac{1}{2}e+\frac{19}{16}e^3+e\gamma^2-\frac{5}{24}em^2$
3	−1	0	1	$-\frac{21}{16}ee'm$	$+\frac{45}{16}ee'm$
3	−1	−2	−0	$+e\gamma^2$	$+e\gamma^2$
3	1	0	−3	$+\frac{35}{16}ee'm$	$-\frac{35}{16}ee'm$
3	1	0	−2	$+\frac{15}{16}em+\frac{319}{64}em^2$	$-\frac{15}{16}em-\frac{319}{64}em^2$
3	1	0	−1	$-\frac{3}{16}ee'm$	$+\frac{15}{16}ee'm$

(l)	(π)	(θ)	g'	Values of κ_1	Values of κ_2
3	1	−2	0	$-\frac{5}{4}e\gamma^2$	$+\frac{5}{4}e\gamma^2$
3	−3	0	0	$-\frac{1}{8}e^3$	$-\frac{1}{8}e^3$
4	0	0	−3	$+\frac{21}{32}e'm^2$	$-\frac{21}{32}e'm^2$
4	0	0	−2	$+\frac{3}{16}m^2+\frac{21}{24}m^3+\frac{15}{8}e^2m$	$-\frac{3}{16}m^2-\frac{21}{24}m^3-\frac{15}{8}e^2m$
4	0	0	−1	$-\frac{3}{32}ee'm$	$+\frac{3}{32}e'm^2$
4	−2	0	0	$+\frac{3}{8}e^2$	$-\frac{1}{2}e^2$
				Values of κ_3	Values of κ_4
0	0	0	0	$2\gamma^2$	0
0	0	0	1		$+\frac{19}{32}e'm^2$
0	0	0	2		$-\frac{19}{16}m^2-\frac{97}{24}m^3+\frac{75}{16}e^2m+\frac{3}{4}\gamma^2m$
0	0	0	3		$-\frac{133}{32}e'm^2$
0	0	2	−2	$+\frac{3}{2}\gamma^2m$	
0	0	2	0		$+\gamma^2$
0	2	0	−2		
0	2	0	0		$+\frac{5}{4}e^2-\frac{5}{4}e^2\gamma^2-\frac{97}{96}e^2m^2$
1	0	0	−1		
1	0	0	0		$-\frac{15}{8}e'\alpha$
1	0	0	1		$+\frac{45}{32}\alpha m$

$(l)(\pi)(\theta)\,g'$	Values of $\varkappa_3$	Values of $\varkappa_4$
1 −1 0 −2		$-\frac{55}{32}\,e\,m^2$
1 −1 0 −1		
1 −1 0 0	$-4\,e\,\gamma^2$	
1 −1 0 1		$+\frac{45}{16}\,e\,e'\,m$
1 −1 0 2		$-\frac{45}{16}\,e\,m-\frac{773}{64}\,e\,m^2$
1 −1 0 3		$-\frac{105}{16}\,e\,e'\,m$
1 −1 −2 0		$-e\,\gamma^2$
1 −1 2 0		$+\frac{11}{4}\,e\,\gamma^2$
1 1 0 −3		
1 1 0 −2		
1 1 0 −1		$-\frac{9}{16}\,e\,e'\,m$
1 1 0 0		$-\frac{3}{2}\,e+\frac{13}{16}\,e^3+3\,e\,\gamma^2+\frac{7}{8}\,e\,m^2$
1 1 0 1		$+\frac{9}{16}\,e\,e'\,m$
1 1 −2 0	$+6\,e\,\gamma^2$	
1 −3 0 0		$+\frac{7}{48}\,e^3$
2 0 0 −3		
2 0 0 −2	$-\frac{3}{2}\,\gamma^2\,m$	$+\frac{9}{8}\,e'^2\,m$
2 0 0 −1		$+\frac{3}{2}\,e'\,m+\frac{3}{4}\,e'\,m^2$

(l)	(π)	(θ)	g'	Values of ${}_3$	Values of x_4
2	0	0	0		$+\frac{1}{2}-\frac{5}{4}e^2-\gamma^2-\frac{1}{6}m^2$
2	0	0	1		$-\frac{3}{2}e'm+\frac{3}{4}e'm^2$
2	0	0	2		$-\frac{9}{8}e'^2 m$
2	0	−2	0	$-2\gamma^2$	
2	0	−2	2		$-\frac{3}{2}\gamma^2 m$
2	0	2	−2		$+\frac{3}{4}\gamma^2 m$
2	−2	0	0		
2	−2	0	2		$-\frac{15}{4}e^2 m$
2	2	0	−2		$-\frac{15}{16}e^2 m$
3	0	0	−1		$-\frac{15}{32}\alpha m$
3	0	0	0		$+\frac{5}{8}e'\alpha$
3	−1	0	−2		
3	−1	0	−1		$+\frac{45}{16}ee'm$
3	−1	0	0		$+\frac{1}{2}e-\frac{19}{16}e^3-e\gamma^2+\frac{5}{24}em^2$
3	−1	0	1		$-\frac{45}{16}ee'm$
3	−1	−2	0	$-2e\gamma^2$	
3	1	0	−3		$+\frac{35}{16}ee'm$

$(l)\ (\pi)\ (\theta)\ g'$	Values of $\varkappa_3$	Values of $\varkappa_4$
3 1 0 −2		$+\frac{15}{16}e\,m+\frac{319}{64}e\,m^2$
3 1 0 −1		$-\frac{15}{16}e\,e'\,m$
3 1 −2 0		$-\frac{5}{4}e\,\gamma^2$
3 −3 0 0		
4 0 0 −3		
4 0 0 −2		
4 0 0 −1		
4 −2 0 0		$+\frac{3}{8}e^2$
	Values of $2\,\varkappa_5$	Values of $2\,\varkappa_6$
0 0 1 −3	$+\frac{7}{4}e'\,\gamma\,m$	$+\frac{7}{4}e'\,\gamma\,m$
0 0 1 −2	$+\frac{3}{4}\gamma\,m+\frac{79}{16}\gamma\,m^2$	$\frac{3}{4}\gamma\,m$
0 0 1 −1	$+\frac{3}{2}e'\,\gamma\,m$	$-3\,e'\,\gamma\,m$
0 0 1 0	$-2\,\gamma-3\,e^2\gamma+5\,\gamma^3+\frac{2}{3}\gamma\,m^2$	$-2\,\gamma-3\,e^2\gamma+5\,\gamma^3+\frac{2}{3}\gamma\,m^2$
0 0 1 1	$-\frac{9}{4}e'\,\gamma\,m$	$+\frac{9}{4}e'\,\gamma\,m$
0 2 −1 0	$-\frac{15}{4}e^2\,\gamma$	$+\frac{15}{4}e^2\,\gamma$
1 −1 −1 0	$-2\,e\,\gamma$	$-2\,e\,\gamma$
1 −1 −1 2	$-\frac{21}{2}e\,\gamma\,m$	$-\frac{21}{2}e\,\gamma\,m$
1 −1 1 −2	$-\frac{3}{4}e\,\gamma\,m$	$-\frac{3}{4}e\,\gamma\,m$

$(l)(\pi)(\theta)\ g'$	Values of $2\,\varkappa_5$	Values of $2\,\varkappa_6$
1 −1 1 0	$+2e\gamma$	$+2e\gamma$
1 1 −1 −2	$-\frac{15}{4}e\gamma m$	$-\frac{15}{4}e\gamma m$
1 1 −1 0	$+6e\gamma$	$-6e\gamma$
1 1 1 −2	$-\frac{3}{2}e\gamma m$	$+\frac{3}{2}e\gamma m$
2 −2 −1 0	$-\frac{1}{2}e^2\gamma$	$-\frac{1}{2}e^2\gamma$
2 −2 1 0	$+\frac{7}{4}e^2\gamma$	$+\frac{7}{4}e^2\gamma$
2 0 −3 0	$+2\gamma^3$	$+2\gamma^3$
2 0 −1 −2	$-2\gamma m^2$	$-2\gamma m^2$
2 0 −1 −1	$-\frac{15}{4}e'\gamma m$	$+\frac{15}{4}e'\gamma m$
2 0 −1 0	$-2\gamma+5e^2\gamma+3\gamma^3+\frac{2}{3}\gamma m^2$	$+2\gamma-5e^2\gamma-3\gamma^3-\frac{2}{3}\gamma m^2$
2 0 −1 1	$-\frac{15}{4}e'\gamma m$	$-\frac{15}{4}e'\gamma m$
2 0 1 −3	$-\frac{7}{4}e'\gamma m$	$+\frac{7}{4}e'\gamma m$
2 0 1 −2	$-\frac{3}{4}\gamma m-\frac{35}{16}\gamma m^2$	$+\frac{3}{4}\gamma m+\frac{35}{16}\gamma m^2$
2 0 1 −1	$+\frac{3}{4}e'\gamma m$	$-\frac{3}{4}e'\gamma m$
3 −1 −1 0	$-2e\gamma$	$+2e\gamma$
3 −1 1 −2	$-\frac{3}{4}e\gamma m$	$+\frac{3}{4}e\gamma m$
3 1 −1 −2	$-\frac{15}{4}e\gamma m$	$+\frac{15}{4}e\gamma m$
4 −2 −1 0	$-2e^2\gamma$	$+2e^2\gamma$
4 0 −1 −2	$-\frac{3}{4}\gamma m^2$	$+\frac{3}{4}\gamma m^2$

The values (107) of K and K′ and of such of their derivatives as enter into the equations (112) to (114) are next to be investigated. Owing to the great number of the separate factors $\varkappa_i$, K_i, and K'_i, and the small number of combinations which give rise to sensible inequalities, we shall adopt the course of deferring the reduction of the $\varkappa_i$ to numbers until we find what values of those quantities are required. It is, however, practically inconvenient to compute individual values of K and K′; we shall therefore show how all sensible values of these quantities may be found by developing the expressions (103) in numerical series.

Although the numerical method adopted in the next chapter seems to me that best adapted to the detection of inequalities of long period, it is quite possible that, after the argument of such an inequality is established, its coefficient can be best found by the method of analytic development.

CHAPTER III.

NUMERICAL DEVELOPMENTS FOR DETERMINING THE ACTION OF THE SEPARATE PLANETS ON THE MOON.

§ 17.

DEVELOPMENT OF THOSE FACTORS IN THE PERTURBATIVE FUNCTION WHICH DEPEND UPON THE COORDINATES OF THE SUN AND OF THE PLANET.

Having found the factors κ_i in the expression (107) we have next to find K_1, K_2, etc., as defined in (103), and used in (107). The first and most difficult operation under this head is the development of $\frac{1}{\rho^3}$ and $\frac{1}{\rho^5}$, ρ being the distance of the planet from the center of gravity of the Earth and Moon. It is necessary to carry this development to much higher multiples of the mean anomalies, and to higher powers of the eccentricities, than are necessary in computing the perturbations of the planets. The labor of computing the numerical values of the terms of the development from their analytical expressions would be enormous. I have therefore adopted the numerical method, in which the development is first effected in terms of the eccentric anomalies.

The original inventor of this method was, I believe, CAUCHY, who explained it in a series of short papers published in the *Comptes Rendus*, between 1843 and 1845. A systematic exposition of it, according to the plan of CAUCHY, is given by PUISEUX in *Annales de l'Observatoire Impériale de Paris*, *Tome VII*, where, however, it is applied only to the computation of particular terms which may give rise to inequalities of long period and not to the general development. The same volume contains a paper by BOURGET, in which it is applied to the general development of the perturbative function. The adaptation of the method to the numerical computation of the general development of the perturbative function and its derivatives is mainly due to HANSEN, who has developed it very copiously in his *Auseinanderse tzung einer zweckmässigen Methode zur Berechnung der absoluten Störungen der kleinen Planeten : Erste Abhandlung*, found in the *Abhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften, Band III*, *Leipzig, 1857*. HANSEN'S exposition of the method is so well adapted to numerical computation that I have generally followed it, and the papers of both HANSEN and PUISEUX are so copious that it is not necessary to present a detailed development of the

formulæ employed. But, as I have found it convenient to adopt forms of computation not expressly given by HANSEN, which may possibly be useful in the general computation of planetary perturbations, I shall present a brief derivation of the formulæ as I have used them. As it is convenient to use a notation in some respects peculiar to the present chapter, we present that actually adopted. Let us put

f, f', the true anomalies of any two planets.

$u\ u'$, their eccentric anomalies.

$g\ g'$, their mean anomalies.

w, w', the distances of their perihelia from their common node.

ρ, their linear distance.

γ, the mutual inclination of their orbits.

φ, φ', their angles of eccentricity, so that $e = \sin \varphi$

We then have for the square of their distance

$$\rho^2 = r^2 + r'^2 - 2\, r\, r' \{ \cos (f+w) \cos (f'+w') + \cos \gamma \sin (f+w) \sin (f'+w') \}$$

If we substitute for r', $r' \cos f'$, and $r' \sin f'$ their values in terms of the eccentric anomaly

$$r' = a' (1 - e' \cos u')$$

$$r' \cos f' = a' (\cos u' - e')$$

$$r' \sin f' = a' \cos \varphi' \sin u'$$

and retain r, f, and w, we find that the value of ρ^2 may be put in the form

$$\rho^2 = \mathrm{H} - \mathrm{K} \cos \psi \cos u' - \mathrm{K} \sin \psi \sin u' + s \cos^2 u' \tag{116}$$

where H, K $\cos \psi$, K $\sin \psi$, and s have the following values:

$$\mathrm{H} = r^2 + a'^2 + 2a'\, e'\, r \{ \cos w' \cos (f+w) + \cos \gamma \sin w' \sin (f+w) \}$$

$$\mathrm{K} \cos \psi = 2\, a'\, r \{ \cos w' \cos (f+w) + \cos \gamma \sin w' \sin (f+w) \} + 2\, e'\, a'^2$$

$$\mathrm{K} \sin \psi = 2\, a'\, r \cos \varphi' \{ - \sin w' \cos (f+w) + \cos \gamma \cos w' \sin (f+w) \}$$

$$s = a'^2\, e'^2$$

The computation of these quantities may be simplified by introducing the auxiliary quantities k_1, k_2, and the angles K_1 and K_2 determined by the equations

$$\begin{aligned} k_1 \sin K_1 &= \cos \gamma \sin w' \\ k_1 \cos K_1 &= \cos w' \\ k_2 \sin K_2 &= \sin w' \\ k_2 \cos K_2 &= \cos \gamma \cos w' \end{aligned} \tag{117}$$

We then have

$$\begin{aligned} H &= r^2 + a'^2 + 2\, a'\, e'\, k_1\, r \cos (f + w - K_1) \\ K \cos \psi &= 2\, a'\, k_1\, r \cos (f + w - K_1) + 2\, e'\, a'^2 \\ K \sin \psi &= 2\, a'\, r\, k_2 \cos \varphi' \sin (f + w - K_2) \\ s &= e'^2\, a'^2 \end{aligned} \tag{118}$$

These four quantities are functions of the elements of both planets, and of the coordinates of the unaccented planet, but do not contain the coordinates of the accented planet. Thus, by the equation (116) we have the theorem:

If the position of a planet be given, the square of its distance from any other planet may be expressed as a rational and entire function of the sine and cosine of the eccentric anomaly of the latter.

On this principle is based the CAUCHY-HANSEN method of developing the perturbative function.

Both CAUCHY and HANSEN express the coefficients H, K cos ψ, etc., as functions of the eccentric anomaly u, but for our present purposes I have deemed it more convenient to retain the true anomaly f. Having computed them for any assumed value of f, we have next to transform the expression (116) into the form

$$\rho^2 = \{ C - q \cos (Q - u') \} \{ 1 - q_1 \cos (Q + u') \} \tag{119}$$

Developing this product and comparing with (116) we find that C, Q, q, and q_1 are to be determined by the equations

$$\begin{aligned} q_1 &= \frac{s}{q} \\ C &= H + s \sin^2 Q \\ q \cos Q &= K \cos \psi - C\, q_1 \cos Q \\ q \sin Q &= K \sin \psi + C\, q_1 \sin Q \end{aligned} \tag{120}$$

Let us substitute in the last members of these equations the first approximate values of C, q, and Q, obtained by putting $q_1 = 0$, namely, H, K, and φ. We shall then have, for a second approximation, which includes the first powers of s only

$$q \cos Q = K \cos \psi \left(1 - \frac{Hs}{K^2}\right)$$

$$q \sin Q = K \sin \psi \left(1 + \frac{Hs}{K^2}\right)$$

$$q = K \left(1 - \frac{Hs}{K^2} \cos 2\psi\right) \tag{121}$$

$$\sin Q = \sin \psi \left(1 + \frac{2Hs}{K^2} \cos^2 \psi\right)$$

The substitution of the last two expressions in the first two of (120) will give the following values of q and C, true to terms of the second order in s

$$q_1 = \frac{s}{K}\left(1 + \frac{Hs}{K^2} \cos 2\psi\right)$$

$$C = H + s\left(1 + 4\frac{Hs}{K^2} \cos^2 \psi\right) \sin^2 \psi \tag{122}$$

It is most convenient to compute C from this formulæ before proceeding to compute q and Q. If we retain C in the last members of the last two equations (120) and substitute for q_1 its value just given, we shall have $q \cos Q$ and $q \sin Q$ to terms of the second order in s. First, we obtain from (121)

$$C q_1 \cos Q = \frac{C q_1}{q} K \cos \psi \left(1 - \frac{Hs}{K^2}\right)$$

$$C q_1 \sin Q = \frac{C q_1}{q} K \sin \psi \left(1 + \frac{Hs}{K^2}\right)$$

But from (121) and (122) we have to terms of the second order in s

$$\frac{q_1}{q} = \frac{s}{K^2}\left(1 + \frac{2Hs}{K^2} \cos 2\psi\right)$$

which, being substituted in the last equations, gives

$$C q_1 \cos Q = \frac{Cs}{K} \cos \psi \left(1 + \frac{Hs}{K^2} (2 \cos 2\psi) - 1\right)$$

$$= \frac{Cs}{K} \cos \psi \left(1 + \frac{Hs}{K^2} (1 - 4 \sin^2 \psi)\right)$$

$$C q_1 \sin Q = \frac{Cs}{K} \sin \psi \left(1 + \frac{Hs}{K^2} (1 + 2 \cos 2\psi)\right)$$

$$= \frac{Cs}{K} \sin \psi \left(1 - \frac{Hs}{K^2} (1 - 4 \cos^2 \psi)\right)$$

The substitution of these expressions in the last two equations (120) gives for the definitive values of q cos Q and q sin Q to terms of the second order in s

$$\left.\begin{aligned} q\cos Q &= K\cos\psi\left\{1 - s\frac{C}{K^2}\left(1+\frac{Hs}{K^2}-\frac{4Hs}{K^2}\sin^2\psi\right)\right\} \\ q\sin Q &= K\sin\psi\left\{1 + s\frac{C}{K^2}\left(1-\frac{Hs}{K^2}+\frac{4Hs}{K^2}\cos^2\psi\right)\right\} \end{aligned}\right\} \qquad (123)$$

In these expressions the only quantities neglected are of the third order with respect to s, and therefore of the sixth order with respect to the eccentricity, and no case is likely to occur in which they can become sensible. The logarithmic computation of C, q, Q, and q_1 may be conveniently effected as follows:

Compute the five quantities s_1, F_1, F_2, F_3, and ζ by means of the equations

$$\begin{aligned} s_1 &= \frac{H}{K^2}s \\ F_1 &= 1+4s_1\cos^2\psi \\ F_2 &= 1-4s_1\sin^2\psi \\ F_3 &= 1+s_1 \\ \zeta &= sF_1\sin^2\psi \end{aligned} \qquad (124)$$

We then have

$$\begin{aligned} C &= H+\zeta \\ q\sin Q &= \left(1+s\frac{C}{K^2}\frac{F_1}{F_3}\right)K\sin\psi \\ q\cos Q &= \left(1-s\frac{C}{K^2}F_2F_3\right)K\cos\psi \\ q_1 &= \frac{s}{q} \end{aligned} \qquad (125)$$

I put the factors into the form $1 \pm h$, because they are then very easily computed by ZECH's Table of addition and subtraction logarithms. By entering the addition table with the argument $\frac{1}{h}$ we take out the logarithm of $1+h$, and by entering the subtraction table we take out $\log \frac{1}{1-h}$

Having thus, for any assumed position of the one planet, expressed the square of its distance from the other in the form (119), we have next to develop the negative and uneven powers of ρ. Here, again, I make use of the method of HANSEN, which seems to be based upon the investigations of GAUSS.* Let us put

* The developments and rules on this and the next two pages are, to a great extent, superseded by those of Vol. III, pages 42–53 and 60–75. I have, however, retained them in the present form, as pertaining to the work, and as equally convenient with the more general ones of Vol. III, which apply to a wider range of cases. (See *post* p. 237.)

$$\rho_0^2 = C - q\cos(Q - u') \tag{126}$$

so that we have

$$\rho^2 = \rho_0^2 \{1 - q_1 \cos(Q + u')\}$$

the last factor differing very little from unity.

We have then

$$\rho^{-n} = \rho_0^{-n} \{1 - q_1 \cos(Q + u')\}^{-\frac{n}{2}}$$

The first factor is to be developed into the form

$$\rho_0^{-n} = \frac{1}{2} \sum_{i=-\infty}^{i=\infty} b_n^{(i)} \cos i(Q - u')$$

If the cofficients $b_n^{(i)}$ be developed in powers of $\frac{q}{C}$ their general expressions will be

$$\frac{1}{2} b_n^{(i)} = \frac{n(n+2)(n+4) \ldots (n+2i-2)}{2.4.6 \ldots 2i} \frac{q^i}{C^i} F\left(\frac{n}{4} + \frac{i+1}{2}, \frac{n}{4} + \frac{i}{2}, i+1, \frac{q^2}{C^2}\right)$$

Here we use GAUSS' notation for the hypergeometric series

$$F(\alpha, \beta, \gamma, x) = 1 + \frac{\alpha\beta}{\gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)} x^2 + \text{etc.}$$

We have also, by putting $i+1$ for i in the above value of b_n^i, and noticing that

$$F(\alpha, \beta, \gamma, x) = F(\beta, \alpha, \gamma, x)$$

$$\frac{1}{2} b_n^{(i+1)} = \frac{n(n+2)(n+4) \ldots (n+2i)}{2.4.6 \ldots (2i+2)} \frac{q^{i+1}}{C^{i+1}} F\left(\frac{n}{4} + \frac{i+1}{2}, \frac{n}{4} + \frac{i}{2} + 1, i+2, \frac{q^2}{C^2}\right)$$

The ratio $\frac{b_n^{i+1}}{b_n^{(i)}}$ of two consecutive coefficients may now be developed in the form of a continued fraction by the method of GAUSS, as given in his *Disquisitiones generales circa seriem infinitam, etc., Sectio Secunda.* Calling this ratio p_{i+1}, we have

$$p_i = \cfrac{\frac{n+2i-2}{4i} \frac{q}{C}}{1 - \cfrac{a}{1 - \cfrac{b}{1 - \cfrac{c}{\text{etc.}}}}}$$

where

$$a=\frac{(n+2i)(2i+2-n)}{16\,i(i+1)}\frac{q^2}{C^2}$$

$$b=\frac{(n+2i+2)(2i+4-n)}{16(i+1)(i+2)}\frac{q^2}{C^2}$$

$$c=\frac{(n+2i+4)(2i+6-n)}{16(i+2)(i+3)}\frac{q^2}{C^2}$$

etc.,

each term being formed from the preceding by increasing i by unity. We thus have the form given by Hansen

$$p_i=\cfrac{F_i}{1-\cfrac{\lambda_{i+1}}{1-\cfrac{\lambda_{i+2}}{1-\cfrac{\lambda_{i+3}}{1-\text{etc.}}}}} \tag{127}$$

where

$$\left.\begin{aligned}\lambda_i&=\frac{(n+2i-2)(2i-n)}{16\,i(i-1)}\frac{q^2}{C^2}\\ F_i&=\frac{2i+n-2}{2i}\frac{q}{C}\end{aligned}\right\} \tag{128}$$

Owing to the slow convergence of the continued fraction (127) it is necessary to compute its value for the highest value of i independently. To derive the necessary equations we must develop $b_n^{(i)}$ in powers of α, a quantity to be determined by the equation

$$\frac{2\alpha}{1+\alpha^2}=\frac{q}{C}$$

the expression (126) being put in the form

$$\rho_0^2=k(1-2\alpha\cos(Q-u')+\alpha^2)$$

which is La Place's form for development.

I deem it unnecessary to go over the details of the derivation, since I have obtained the result given by Hansen, whose formulæ may be most conveniently employed as follows:

Compute χ from the equation

$$\sin\chi=\frac{q}{c}$$

when we have

$$\alpha = \tan\frac{1}{2}\chi \tag{129}$$

For the highest value of i to which it is deemed necessary to carry the development, compute γ_i by means of the continued fraction

$$\gamma_i = \cfrac{\sec^2\frac{1}{2}\chi}{1 - \cfrac{a_1}{1 - \cfrac{b_1}{1 - \cfrac{a_2}{1 - \cfrac{b_2}{1 - \text{etc.}}}}}}$$

where

$$a_1 = \frac{n(2-n)}{4i(i+1)}\alpha^2 \qquad b_1 = \frac{(n+2i)(2i+2-n)}{4(i+1)(i+2)}\alpha^2$$

$$a_2 = \frac{(n+1)(4-n)}{4(i+2)(i+3)}\alpha^2 \qquad b_2 = \frac{(n+2i+2)(2i+4-n)}{4(i+3)(i+4)}\alpha^2$$

$$a_3 = \frac{(n+2)(6-n)}{4(i+4)(i+5)}\alpha^2 \qquad b_3 = \frac{(n+2i+4)(2i+6-n)}{4(i+5)(i+6)}\alpha^2$$

etc. etc.

Put also

$$\begin{aligned} \theta_n^{(i)} &= \frac{n+2i-2}{4i} \\ \mu_n^{(i)} &= \frac{(2i-n)(n+2i-2)}{16i(i-1)} = \frac{2i-n}{4(i-1)}\theta_n^{(i)} \end{aligned} \tag{130}$$

A general table of the values of $\theta_n^{(i)}$ and μ_n^i for all values of i and n commonly used is easy to form, and is given for $n=3$ and $n=5$ in the following section.

Then compute, for all values of i

$$\begin{aligned} F_i &= \theta_n^{(i)} \sin\chi \\ \lambda_i &= \mu_n^{(i)} \sin^2\chi \end{aligned} \tag{131}$$

Commencing with the highest value of γ_i compute in succession all the other values by the formulæ

$$\gamma_i = \frac{1}{1 - \lambda_{i+1}\gamma_{i+1}} \tag{132}$$

when we have

$$\begin{aligned} p_i &= F_i \gamma_i \\ b_n^{(i)} &= p_i b_n^{(i-1)}. \end{aligned} \tag{133}$$

The logarithms of γ_i in (132) can be computed with great facility by the table of subtraction-logarithms. Then having all the values of $\log p_i$ we require the value of $\log b_n^{(0)}$. This can be found most conveniently from RUNKLE'S *Tables for determining the values of the Coefficients in the Perturbative Function of planetary motion which depend on the ratio of the mean distances.* (Smithsonian Contributions to Knowledge, Washington, 1855.) These tables give, for the argument α, the coefficients $b_s^{(i)}$ of $\cos i\,\varphi$ in the development of

$$(1-2\alpha\cos\varphi+\alpha^2)^{-s}=\tfrac{1}{2}\Sigma b_s^{(i)}\cos i\,\varphi.$$

The development we actually seek is that of

$$(\mathrm{C}-q\cos\varphi)^{-\frac{n}{2}}=\tfrac{1}{2}\Sigma b_n^{(i)}\cos i\,\varphi.$$

To assimilate it to RUNKLE'S development we must determine α and k so as to satisfy the equation

$$\mathrm{C}-q\cos\varphi=k\,(1+\alpha^2-2\,\alpha\cos\varphi)$$

Introducing the auxiliary χ,

$$\sin\chi=\frac{q}{\mathrm{C}},$$

we have

$$\left.\begin{aligned}\alpha&=\tan\tfrac{1}{2}\chi\\ k&=\mathrm{C}\cos^2\tfrac{1}{2}\,\chi=\frac{q}{2\tan\frac{1}{2}\chi}\end{aligned}\right\}\qquad(134)$$

Then, comparing like terms in the two developments,

$$b_n^{(0)}=\sec^n\tfrac{1}{2}\,\chi\,\mathrm{C}^{-\frac{n}{2}}\,b_s^{(0)}$$

provided that $b_s^{(0)}$ be taken from RUNKLE'S tables with the argument α. In the cases of $s=\frac{3}{2}$ and $s=\frac{5}{2}$ the quantities actually tabulated are,

$$\text{for } s=\frac{3}{2};\;(1-\alpha^2)^2\,b_s^{(0)}=\frac{\cos^2\chi}{\cos^4\frac{1}{2}\,\chi}\,b_s^{(0)}$$

$$\text{for } s=\frac{5}{2};\;(1-\alpha^2)^4\,b_s^{(0)}=\frac{\cos^4\chi}{\cos^8\frac{1}{2}\,\chi}\,b_s^{(0)}.$$

If, then, we represent by $\beta_{\frac{3}{2}}^{(0)}$ and $\beta_{\frac{5}{2}}^{(0)}$ the quantities actually taken from RUNKLE'S tables when $s=\frac{3}{2}$ and $s=\frac{5}{2}$, we have

$$\left.\begin{aligned}b_3^{(0)}&=\mathrm{C}^{-\frac{3}{2}}\sec^2\chi\cos\tfrac{1}{2}\,\chi\,\beta_{\frac{3}{2}}^{(0)}\\ b_5^{(0)}&=\mathrm{C}^{-\frac{5}{2}}\sec^4\chi\cos^3\tfrac{1}{2}\,\chi\,\beta_{\frac{5}{2}}^{(0)}\end{aligned}\right\}\qquad(135)$$

These quantities may also be computed with equal ease, though, perhaps, not always with equal accuracy, by the use of the arithmetico-geometrical mean. Representing this mean in the usual way by M we have, first,

$$b_1^{(0)} = \frac{2}{\sqrt{C}\, M(\cos \tfrac{1}{2}\chi, \sqrt{\cos\chi})} = \frac{2}{\sqrt{C\cos\chi}\, M\left(1, \frac{\cos\frac{1}{2}\chi}{\sqrt{\cos\chi}}\right)} \tag{136}$$

Then $b_3^{(0)}$ and $b_5^{(0)}$ may be obtained by the formulæ

$$\begin{aligned} b_3^{(0)} &= \frac{b_1^{(0)}}{C - q\, p_3^{(1)}} = \frac{b_1^{(0)}}{C\,(1 - p_3^{(1)} \sin\chi)} \\ b_5^{(0)} &= \frac{b_3^{(0)}}{C - q\, p_5^{(1)}} = \frac{b_3^{(0)}}{C\,(1 - p_5^{(1)} \sin\chi)} \end{aligned} \tag{137}$$

When $p_3^{(1)} \sin \chi$ and $p_5^{(1)} \sin\chi$ approach to unity, these formulæ will give $b_3^{(0)}$ and $b_5^{(0)}$ with only very limited accuracy and others must be used.

The result of the preceding operations is that, for any assumed position of the unaccented planet in its orbit, we may develop the value of ρ_0^{-n} in the form

$$\rho_0^{-n} = \tfrac{1}{2} \sum_{-\infty}^{\infty} b_n^{(i)} \cos i\,(Q - u') \tag{138}$$

We now compute this development for a number of positions corresponding to equidistant values of the mean anomaly, g, of the planet, and thus obtain a separate value of $b_n^{(i)}$ and of Q for each value of the mean anomaly. Then, any functions of these quantities may be developed in a periodic series, proceeding according to the multiples of the mean anomaly by the well-known process of mechanical development. The most convenient functions to develop thus are $b_n^{(i)} \cos i\,(Q-g)$ and $b_n^{(i)} \sin i\,(Q-g)$. If we put

$$\begin{aligned} Q' &= Q - g \\ A &= u' - g \end{aligned} \tag{139}$$

we have

$$\begin{aligned} \rho_0^{-n} &= \tfrac{1}{2} \Sigma\, b_n^{(i)} \cos (i\,Q' - i\,A) \\ &= \tfrac{1}{2} \Sigma\, b_n^{(i)} \cos i\,Q' \cos i\,A + \tfrac{1}{2} \Sigma\, b_n^{(i)} \sin i\,Q' \sin i\,A. \end{aligned}$$

Let us, then, develop $b_n^{(i)} \cos i\ Q'$ and $b_n^{(i)} \sin i\ Q'$ in a periodic series of the form

$$\begin{aligned} b_n^{(i)} \cos i\,Q' &= c_0^{(i)} + c_1^{(i)} \cos g + c_2^{(i)} \cos 2g +, \text{etc.} \\ &\quad + s_1^{(i)} \sin g + s_2^{(i)} \sin 2g +, \text{etc.} \\ b_n^{(i)} \sin i\,Q' &= c'^{(i)}_0 + c'^{(i)}_1 \cos g + c'^{(i)}_2 \cos 2g +, \text{etc.} \\ &\quad + s'^{(i)}_1 \sin g + s'^{(i)}_2 \sin 2g +, \text{etc.} \end{aligned} \tag{140}$$

Then, let us compute values of $(a_{i,j})$ and $(b_{i,j})$ for all positive values of i and j from the formulæ

$$(a_{0,0})=\frac{1}{2}c_0^{(0)}$$

$$(a_{0,j})\quad\frac{1}{2}c_j^{(0)}\qquad\qquad(b_{0,j})=\frac{1}{2}s_i^{(0)}$$

$$(a_{i,0})=\frac{1}{2}c_j^{(i)}\qquad\qquad(b_{i,0})=c'^{(i)}_0\tag{141}$$

$$(a_{i,j})=\frac{1}{2}(c_j^{(i)}-s'^{(i)}_j)\qquad\qquad(b_{i,j})=\frac{1}{2}(c'^{(i)}_j+s_j^{(i)})$$

$$(a_{i,-j})=\frac{1}{2}(c_j^{(i)}+s'^{(i)}_j)\qquad\qquad(b_{i,-j})=\frac{1}{2}(c'^{(i)}_j-s_j^{(i)})$$

and we shall have the numerical expression for ρ_0^{-n} for all positions of the two planets by the formula

$$\rho_0^{-n}=\sum_0^{\infty}{}_i\sum_{-\infty}^{+\infty}{}_j\left\{(a_{i,j})\cos(iA+jg)+(b_{ij})\sin(iA+jg)\right\}\tag{142}$$

To obtain ρ^{-n}, this development is still to be multiplied by the factor

$$F=\{1-q_1\cos(Q+u')\}^{-\frac{2}{n}}=\{1-q_1\cos Q'\cos(A+2g)+q_1\sin Q'\sin(A+2g)\}^{-\frac{2}{n}}$$

Owing to the minuteness of the factor q_1 we may nearly always neglect its powers, and so put

$$F=1+\frac{n}{2}q_1\cos Q'\cos(A+2g)-\frac{n}{2}q_1\sin Q'\sin(A+2g)$$

Let us develop as before $\frac{n}{2}q_1\cos Q'$ and $\frac{n}{2}q_1\sin Q'$ into a periodic series of the form

$$\begin{aligned}\frac{n}{2}q_1\cos Q'&=p_0+p_1\cos g+p_2\cos 2g+\text{etc.}\\&\quad+q_1\sin g+q_2\sin 2g+\text{etc.}\\\frac{n}{2}q_1\sin Q'&=p'_0+p'_1\cos g+p'_2\cos 2g+\text{etc.}\\&\quad+q'_1\sin g+q'_2\sin 2g+\text{etc.}\end{aligned}\tag{143}$$

and we shall have

$$\begin{aligned}F&=1+p_0\cos(A+2g)-p'_0\sin(A+2g)\\&+\Sigma\frac{1}{2}(p_i+q'_i)\cos\big(A+(2+i)g\big)+\Sigma\frac{1}{2}(p_i-q'_i)\cos\big(A+(2-i)g\big)\\&-\Sigma\frac{1}{2}(p'_i-q_i)\sin\big(A+(2+i)g\big)-\Sigma\frac{1}{2}(p'_i+q_i)\sin\big(A+(2-i)g\big)\end{aligned}\tag{144}$$

Multiplying the series (142) and (144) term by term, we shall have the value of ρ^{-n} expressed in the form (142). The development will be given in terms of the mean anomaly of the one planet and the eccentric anomaly of the other.

The next process is to change it into a development in terms of the mean anomalies of both planets. This is done by means of the BESSELIAN transcendents, tables and formulæ for which are given by BESSEL in the *Berlin Memoirs* for 1824, while the mode of applying them is fully explained by HANSEN in his "*Auseinandersetzung.*"

Having thus effected the development of $\frac{1}{\rho^5}$ in terms of the mean anomalies of the Earth and planet, we have only to express x', y', z', the rectangular coordinates of the Sun relatively to the center of gravity of the Earth and Moon, and x_4, y_4, z_4, the rectangular coordinates of the planet relatively to the Sun, in terms of the respective mean anomalies. The factors depending on the coordinates of the planet and Sun will then be formed by multiplication. As the complete formation of these terms would be very laborious, we shall only actually develop those of which we find ourselves in need, and these can not be selected until we proceed with the more particular consideration of the action of the separate planets.

§ 18.

NUMERICAL DETAILS OF QUANTITIES DEPENDING ON THE POSITION OF THE PLANETS MERCURY, VENUS, AND MARS.

The computations of which the results are now given were made in the years 1871–'73, before I had prepared the more general discussion of the method found in Vol. III, Part I, Chapters II and IV. The latter does not, however, involve any material change in the method so far as the present application is concerned, where no deviation from HANSEN's procedure is necessary. The object of the present section is to present such details as will enable the results to be verified, corrected, or reproduced with the least amount of labor.

The computations were made with more or less fullness for Mercury, Venus, and Mars. In the case of Mercury, however, the development was in error from the use of the longitude of the solar perigee, when that of the Earth's perihelion should have been used. As this error would but slightly change the larger terms in the development, and therefore would not invalidate the general conclusion that the action of Mercury produce no inequalities of importance in the Moon's motion, it was not corrected by a recomputation.

In the case of Mars the developments were not completed beyond the point where the eccentric has to be replaced by the mean anomaly, because the order of magnitude of the coefficients in the development according to the eccentric anomaly would suffice to draw general conclusions as to the possibility of hidden terms of long period.

In making the computations the first question is of which eccentric anomaly the development shall be a function. Theoretically this is indifferent; in practice, however, it will be convenient to choose the eccentric anomaly of the orbit having the

smaller eccentricity, because the BESSELIAN transformation to mean anomaly will then be simpler, as will the change of ρ_0^{-n} into ρ^{-n}.

It will then be better to assume special equidistant values of the mean anomaly of the other planet rather than of its eccentric anomaly, because the necessity of a transformation to mean anomaly of this planet will then be avoided. For each position of the planet thus arising the quantities K, ψ, C, etc., are to be computed. No example of the way of effecting this computation is necessary. It is different, however, with the coefficients $b_n^{(i)}$, the computation of which may be greatly facilitated by a suitable arrangement of the work, as will be shown in a specimen of the computation. On page 247 is given a transcript from the computing book of all the figures made in computing a set of values of $b_n^{(i)}$ up to $i=20$, except the computation of $b_n^{(0)}$, $p_n^{(1)}$, and the last value of γ_i. The process is as follows:

The values of log λ_i are formed by adding to each $\mu^{(i)}$ the logarithm of sin $^2\chi$

$$\log \sin^2\chi = 9.9583004$$

and are written in the second column opposite the corresponding values of i.

The last value of log γ_i (in the computation log γ_{32}, but, in printing, log γ_{20}) is then written under the corresponding value of λ_i. The sum of the two logarithms is written opposite i in the third column and immediately under this sum its arithmetical complement, to serve as the argument in ZECH's table of subtraction logarithms. With this argument the value of log γ_{i-1} is taken from the table and written under log λ_{i-1}. This process is repeated upward until the value of γ_1 is reached.

To render easy the detection of an error the remainder of the work is done by the continual addition of differences. Since

$$\log p_i = \log \sin \chi + \log \theta_i + \log \gamma_i$$

we have

$$\triangle \log p_i = \triangle \log \theta_i + \triangle \log \gamma_i$$

The values of $\triangle$ log γ_i are formed by subtraction of the consecutive values of log γ_i and those of log θ_i by taking the differences of the Table of values of log θ_i. The remainder of the operation of forming log $b^{(i)}$ needs no explanation.

The above computation has to be made for each of a certain number of equidistant values of g, extending through the circle. In the case of Venus and Mars I have taken

$$g = 0^\circ;\ 15^\circ;\ 30^\circ;\ \ldots\ldots\ 345^\circ$$

thus dividing the circle into 24 parts.

The next step is, for each value of g and of i, to compute

$$b^{(i)} \cos i\,(Q-g) \text{ and } b^{(i)} \sin i\,(Q-g) \tag{145}$$

where, in each case, the special numerical value of g, used in computing $b^{(i)}$ and Q, is to be used.

With the 24 special values of each of the quantities (145) its general expression in

a trigonometric series, proceeding according to sines and cosines of g, is to be formed. I deem ENCKE's method of forming this series to be the most convenient. For the case of $n=24$ the formulæ are as follows:

We put

(m)=numerical value of the function of $g=15^\circ\times m$ in (145)

We compute

$$\left.\begin{array}{ll} C^0=(0); & C^k=(k)+(24-k) \\ C^{12}=(12); & S^k=(k)-(24-k) \end{array}\right\} (k=1;\ 2;\ 3;\ \ldots\ 11)$$

$$C^6_+=C^6; \quad C^k_+=C^k+C^{12-k}; \quad C_-=C^k-C^{12-k} \quad (k=0;\ 1;\ 2;\ 3;\ 4;\ 5)$$

$$S^6_+=S^6; \quad S^k_+=S_k+S^{12-k}; \quad S^k_-=S^k-S^{12-k} \quad (k=1;\ 2;\ 3;\ 4;\ 5)$$

$$C^k_{++}=C^k_++C^{6-k}_+; \quad C^k_{+-}=C^k_+-C^{6-k}_+ \quad (k=0;\ 1;\ 2)$$

$$C^k_{-+}=C^k_-+C^{6-k}_-; \quad C^k_{--}=C^k_--C^{6-k}_- \quad (k=1)$$

$$S^k_{++}=S^k_++S^{6-k}_+; \quad S^k_{+-}=S^k_+-S^{6-k}_+ \quad (k=1)$$

$$S^k_{-+}=S^k_-+S^{6-k}_-; \quad S^k_{--}=S^k_--S^{6-k}_- \quad (k=1;\ 2)$$

$$a_0=C^1_{--}-C^3_- \qquad b_0=S^1_{+-}+S^3_+$$

$$a_1=C^0_-+\frac{1}{2}C^4_- \qquad b_1=\frac{1}{2}\sqrt{\frac{3}{2}}S^1_{++}$$

$$a_2=\frac{1}{2}\sqrt{3}\,C^2_- \qquad b_2=\sqrt{\frac{1}{2}}\left(\frac{1}{2}S^1_{+-}-S^3_+\right)$$

$$a_3=\frac{1}{2}\sqrt{\frac{3}{2}}C^1_{-+} \qquad b_3=\frac{1}{2}S^2_++S^6$$

$$a_4=\sqrt{\frac{1}{2}}\left(\frac{1}{2}C^1_{--}+C^3_-\right) \qquad b_4=\frac{1}{2}\sqrt{3}\,S^4_+$$

$$12\,(c_0+c_{12})=C^0_{++}+C^2_{++}$$

$$12\,(c_0-c_{12})=C^1_{++}+C^3_+$$

$$6\,(c_1+c_{11})=a_1+a_2 \qquad 6\,(s_1+s_{11})=b_1-b_2$$

$$6\,(c_1-c_{11})=a_3+a_4 \qquad 6\,(s_1-s_{11})=b_3+b_4$$

$$6\,(c_2+c_{10})=C^0_{+-}+\frac{1}{2}C^2_{+-} \qquad 6\,(s_2+s_{10})=\frac{1}{2}S^1_{-+}+S^3_-$$

$$6\,(c_2-c_{10})=\frac{1}{2}\sqrt{3}\,C^1_{+-} \qquad 6\,(s_2-s_{10})=\frac{1}{2}\sqrt{3}\,S^2_{-+}$$

$$6\,(c_3+c_9)=C^0_--C^4_- \qquad 6\,(s_3+s_9)=\sqrt{\frac{1}{2}}\,b_0$$

$$6\,(c_3 - c_9) = \sqrt{\frac{1}{2}}\,a_0 \qquad\qquad 6\,(s_3 - s_9) = S^2_{+} - S^6$$

$$6\,(c_4 + c_8) = C^0_{++} - \frac{1}{2}\,C^2_{++} \qquad\qquad 6\,(s_4 + s_8) = \frac{1}{2}\sqrt{3}\,S^1_{--}$$

$$6\,(c_4 - c_8) = \frac{1}{2}\,C^1_{++} - C^3_{+} \qquad\qquad 6\,(s_4 - s_8) = \frac{1}{2}\sqrt{3}\,S^2_{--}$$

$$6\,(c_5 + c_7) = a_1 - a_2 \qquad\qquad 6\,(s_5 + s_7) = b_1 + b_2$$

$$6\,(c_5 - c_7) = a_3 - a_4 \qquad\qquad 6\,(s_5 - s_7) = b_3 - b_4$$

$$12\,c_6 = C^0_{+-} - C^2_{+-} \qquad\qquad 12\,s_6 = S^1_{-+} - S^3_{-}$$

It should be noted that the indices of C and S are not powers. The preceding formulæ may be controlled at pleasure by the general conditions

$$c_i = (0) + (1)\cos i\,.\,15^\circ + (2)\cos i\,.\,30^\circ + (3)\cos i\,.\,45^\circ + \quad . \quad . \quad .$$

$$s_i = \qquad (1)\sin i\,.\,15^\circ + (2)\sin i\,.\,30^\circ + (3)\sin i\,.\,45^\circ + \quad . \quad . \quad .$$

or by

$$c_0 + c_1\cos n\,.\,15^\circ + c_2\cos 2\,n\,.\,15^\circ + c_3\cos 3\,n\,.\,15^\circ + \quad . \quad . \quad .$$

$$+ s_1\sin n\,.\,15^\circ + s_2\sin 2\,n\,.\,15^\circ + s_3\sin 3\,n\,.\,15^\circ \ + \quad . \quad . \qquad = (n)$$

The values of c_i and s_i obtained in this way from the special values of $b_n^{(i)}\cos i\,Q'$, and the accented coefficients c' and s' obtained from $b_n^{(i)}\sin i\,Q'$ for each value of i, being substituted in (141), give the required development of ρ_0^{-n} in the form (142).

The following form for the principal part of the computations was used:

	0	1	2	3	4	5	6
	(0)	(1)	(2)	(3)	(4)	(5)	
		(23)	(22)	(21)	(20)	(19)	
		(11)	(10)	(9)	(8)	(7)	(6)
	(12)	(13)	(14)	(15)	(16)	(17)	(18)
C	C^0	C^1	C^2	C^3	C^4	C^5	C^6
	C^{12}	C^{11}	C^{10}	C^9	C^8	C^7	
S		S^1	S^2	S^3	S^4	S^5	S^6
		S^{11}	S^{10}	S^9	S^8	S^7	
$\mathbf{C}_+$	C^0_+	C^1_+	C^2_+	C^3_+			
	C^6_+	C^5_+	C^4_+				
			$\log C^2_-$				
$\mathbf{C}_-$	C^0_-	C^1_-	C^2_-	C^3_-			
	$\frac{1}{2}C^4_-$	C^5_-	C^4_-	$\frac{1}{2}C^1_{--}$			
			C^0_-				
$\mathbf{C}_{++}$	C^0_{++}	C^1_{++}	C^2_{++}	C^3_+			
	$\frac{1}{2}C^2_{++}$	C^3_+	C^0_{++}	$\frac{1}{2}C^1_{++}$			
$\mathbf{C}_{+-}$	C^0_{+-}	C^1_{+-}	C^2_{+-}				
	$\frac{1}{2}C^2_{+-}$	$\log C^1_{+-}$	C^0_{+-}				
$\mathbf{C}_{-+}$		C^1_{-+}					
		$\log C^1_{-+}$					
$\mathbf{C}_{--}$		C^1_{--}					
		C^3_-	S^6				
$\mathbf{S}_+$		S^1_+	S^2_+	S^3_+			
	S^6	S^5_+	S^4_+				
	$\frac{1}{2}S^2_+$		$\log S^4_+$				
$\mathbf{S}_-$		S^1_-	S^2_-	S^3_-			
		S^5_-	S^4_-	$\frac{1}{2}S^1_{-+}$			
$\mathbf{S}_{++}$		S^1_{++}	$\log S^1_{++}$				
$\mathbf{S}_{+-}$		S^1_{+-}					
		S^3_+					
$\mathbf{S}_{-+}$		S^1_{-+}	S^2_{-+}				
		S^3_-					
$\mathbf{S}_{--}$		S^1_{--}	S^2_{--}				

In presenting the following numerical data and results the writer has been guided by the consideration that future investigators may desire to repeat his work with a higher degree of numerical precision than that which he has himself reached. In this connection it may be remarked that complete numerical consistency in the use of the adopted elements seems to be necessary. For example, serious error might result in the terms of very high order if one value of the eccentricity were used in one part of the computation and another value in another, and this, although neither of the two values should be seriously in error. It is also to be remembered that small changes in the elements, especially the positions of the perihelia and nodes, may make considerable changes in the coefficients of the terms in question. The epoch 1800 was therefore chosen as that to which the developments should refer, because this was fairly near to the mean of the time through which observations of the Moon's position were available. If the elements for 1900 were used, it is possible that slightly different results might be found for the inequalities of very long period, as, for instance, HANSEN'S inequality due to the action of Venus.

In the actual computations the true anomalies and radii vectores of the Earth and Mars for different values of their mean anomaly enter as fundamental numerical data in the computation. These anomalies and radii vectores were taken for 1800 out of LE VERRIER'S tables. On the other hand, where the eccentricities enter into the formulæ, they were reduced to 1800 with LE VERRIER'S data. It is not impossible that in this way numerical inconsistencies, amounting to a large fraction of a second of arc, have crept into the computation, especially as account was not taken of the minute error in the secular variation of the Earth's radius vector which exists in LE VERRIER'S tables of the Sun. It must not, therefore, be understood that the whole computation is absolutely consistent with the following numerical elements which were intended to form their basis. I shall therefore present such derived quantities as will enable the general consistency of the work to be most easily tested.

In the notation of § 17 it will be seen that the position of the planet whose elements are accented remains indeterminate, while the computations are made for special values of the mean anomaly which is unaccented. Now, in the respective cases of Venus-Earth and Earth-Mars, it is most convenient to let the outer planet play the part of the unaccented one, since its eccentricity is in each case the largest.

The values of the fundamental elements actually used in the computations were as follows:

Adopted quantities for 1800.

	Venus and Earth.			Earth and Mars.		
	°	′	″	°	′	″
π	99	29	56	332	22	42
π'	128	46	1	99	29	56
θ	254	52	28	48	0	33
γ	3	23	32.5	1	51	3.5
$\pi - \theta = \omega$	204	37	28	284	22	9
$\pi' - \theta = \omega'$	233	53	33	51	29	23
$\omega - K_1$	330	41	3	232	53	38.3
$\omega - K_2$	330	46	48	232	51	53.7
e'	0.0068703			0.016792		
$\log k_1$	9.9997360			9.999861		
$\log k_2$	9.9995030			9.999912		
$\log 2\, e'\, a'^2$	7.856682			8.52614		
$\log s$	5.3926			6.4502		

The data and principal results of the computations made by the preceding method are given in the pages next following with some fullness, not only on account of their bearing upon the problem immediately in hand, but of their possible usefulness in the theories of Venus, the Earth, and Mars. In the Hansenian method of developing the perturbative function, the inverse first, third, and fifth powers of the distances of the two planets must first be developed as the basis of the whole computation. From this development the function itself and its derivatives is derived by the process called by Hansen "mechanical multiplication;" that is to say, the factors by which the inverse powers are multiplied are expressed as functions of the mean anomaly of one or the other of the planets, developable in sines or cosines of multiples of that quantity. The coefficients in this development are pure numbers, so that only a numerical multiplication is necessary.

We first have to compute the values of the coefficients which enter into the expressions (100), (116), (119), and (126). This computation is made by the formulæ (124) and (125).

For the separate computations the circle has been divided into 24 parts; a number greater is rigorously necessary, but advisable in order that the general consistency of the results may afford a check upon the accuracy of the computations.

VENUS AND THE EARTH.

Values of the principal quantities for each 15° of the Earth's mean anomaly.

g	log K sin ψ	log K cos ψ	C	Q	log q	log $b_3^{(0)}$	log $b_5^{(0)}$	log sin χ
°				° ′ ″				
0	9.8426013n	0.0959176	1.4984309	330 50 5.3	0.1547869	1.0393590	2.0106083	9.9791502
15	9.5307811n	0.1424754	1.5005595	346 15 34.0	0.1550778	1.0321281	1.9961461	9.9788246
30	8.6167445	0.1555235	1.5042569	1 39 24.0	0.1556971	1.0219812	1.9759630	9.9783751
45	9.6225171	0.1372497	1.5092620	16 59 52.6	0.1566407	1.0105490	1.9533528	9.9778760
60	9.8851684	0.0850808	1.5152267	32 15 22.8	0.1578726	0.9992563	1.9311747	9.9773950
75	0.0263142	9.9897552	1.5217372	47 24 35.2	0.1593188	0.9890811	1.9113730	9.9769792
90	0.1085707	9.8261023	1.5283468	62 26 37.7	0.1608716	0.9805013	1.8948515	9.9766495
105	0.1517266	9.5027179	1.5346068	77 21 13.2	0.1624003	0.9735970	1.8817050	9.9764032
120	0.1634551	8.7370093n	1.5400942	92 8 42.6	0.1637673	0.9681470	1.8714128	9.9762200
135	0.1458188	9.6266576n	1.5444443	106 50 2.9	0.1648478	0.9638634	1.8633261	9.9760756
150	0.0965588	9.8829568n	1.5473683	121 26 41.9	0.1655454	0.9605464	1.8569903	9.9759517
165	0.0075048	0.0228020n	1.5486731	136 0 28.2	0.1658028	0.9581767	1.8523352	9.9758430
180	+9.8571886	0.1055524n	1.5482730	150 33 21.5	0.1656082	0.9569734	1.8497980	9.9757607
195	9.5745021	0.1501937n	1.5461945	165 7 21.1	0.1649943	0.9573378	1.8502075	9.9757301
210	7.8244550	0.1640344n	1.5425743	179 44 16.2	0.1640312	0.9597802	1.8546328	9.9757851
225	9.5592682n	0.1489068n	1.5376514	194 25 38.0	0.1628151	0.9647424	1.8640192	9.9759572
240	9.8498682n	0.1023996n	1.5317533	209 12 33.3	0.1614556	0.9724792	1.8789404	9.9762668
255	0.0025697n	0.0163044n	1.5252760	224 5 42.6	0.1600604	0.9828938	1.8992569	9.9767119
270	0.0921850n	9.8694460n	1.5186557	239 5 20.2	0.1587229	0.9954332	1.9238996	9.9772636
285	0.1407563n	9.5928653n	1.5123460	254 11 16.3	0.1575166	1.0090239	1.9507358	9.9778654
300	0.1564598n	8.1880642n	1.5067814	269 23 1.8	0.1564928	1.0221600	1.9767483	9.9784426
315	0.1412940n	9.5590788	1.5023484	284 39 51.5	0.1556843	1.0330654	1.9983638	9.9789136
330	0.0925806n	9.8542650	1.4993584	300 0 47.2	0.1551152	1.0401244	2.0123302	9.9792098
345	0.0011484n	0.0073922	1.4980207	315 24 38.9	0.1548076	1.0422776	2.0165188	9.9792898

MARS AND THE EARTH.

Values of the principal quantities for each 15° of the mean anomaly of Mars.

g	log K sin ψ	log K cos ψ	C	log q	Q	log $b_3^{(0)}$	log q	log sin χ
°					° ′ ″			
0	0.342858n	0.212979n	2.881190	0.438017	233 26 53.9	0.599924	6.0122	9.978445
15	0.418731n	9.939190n	2.910317	0.441407	251 39 51.6	0.575359	6.0088	9.977466
30	0.448288n	8.384265n	2.971771	0.448350	269 30 20.5	0.525201	6.0018	9.975335
45	0.439091n	9.917232	3.060116	0.457926	286 43 58.9	0.457934	5.9923	9.972189
60	0.391572n	0.207560	3.167944	0.469034	303 12 14.5	0.382842	5.9812	9.968257
75	0.298636n	0.357601	3.286834	0.480623	318 52 19.0	0.307729	5.9696	9.963845
90	0.137266n	0.444640	3.408158	0.491815	333 45 49.5	0.238109	5.9584	9.959295
105	9.821323n	0.492325	3.523778	0.501949	347 57 24.5	0.177420	5.9483	9.954940
120	8.944271	0.510437	3.626471	0.510555	1 33 20.6	0.127466	5.9396	9.951070
135	9.921101	0.502957	3.710223	0.517327	14 40 44.9	0.089206	5.9329	9.947927
150	0.185703	0.470233	3.770379	0.522066	27 27 0.6	0.062981	5.9281	9.945680
165	0.332616	0.408997	3.803729	0.524655	39 59 34.1	0.048896	5.9255	9.944445
180	0.424058	0.310220	3.808538	0.525041	52 25 48.2	0.047016	6.9252	9.944282
195	0.480036	0.151066	3.784559	0.523210	64 53 5.7	0.057224	5.9270	9.945195
210	0.508707	9.855220	3.733036	0.519199	77 28 52.3	0.079588	5.9310	9.947137
225	0.513048	8.291540n	3.656695	0.513098	90 20 38.3	0.113981	5.9371	9.950010
240	0.492674	9.876442n	3.559711	0.505068	103 36 0.2	0.160083	5.9451	9.953654
255	0.443756	0.158005n	3.447645	0.495379	117 22 30.6	0.217243	5.9548	9.957857
270	0.356911	0.308213n	3.327367	0.484444	131 47 19.0	0.283875	5.9658	9.962343
285	0.209610	0.396210n	3.206725	0.472863	146 56 28.9	0.357230	5.9773	9.966801
300	9.929867	0.441856n	3.094241	0.461452	162 53 52.3	0.432631	5.9887	9.970898
315	8.223455n	0.451265n	2.998513	0.451227	179 39 39.0	0.503153	5.9990	9.974321
330	9.912794n	0.423601n	2.927385	0.443299	197 8 46.0	0.560062	6.0069	9.976819
345	0.199031n	0.351198n	2.887057	0.438679	215 10 3.2	0.594434	6.0115	9.978223

Table of the valves of log $\theta_n^{(i)}$ and log $\mu_n^{(i)}$ for the computation of $b_n^{(i)}$.

i	$n=3$			$n=5$		
	log $\theta^{(i)}$	Δ log $\theta^{(i)}$	log $\mu^{(i)}$	log $\theta^{(i)}$	Δ log $\theta^{(i)}$	log $\mu^{(i)}$
1	9.8750613	791813	. . . . . .	0.0969100	1549020	. . . .
2	9.7958800	299632	9.1938200	9.9420080	669467	9.3399480*n*
3	9.7659168	. . . . .	9.3399481	9.8750613	377886	8.9719713
4	9.7501225	157943	9.3699113	9.8372727	243593	9.2352128
5	9.7403627	97598	9.3813407	9.8129134	170333	9.3077634
6	9.7337322	66305	9.3869447	9.7958801	125892	9.3399481
7	9.7289333	47989	9.3901148	9.7832909	96873	9.3573222
8	9.7252989	36344	9.3920843	9.7736036	76868	9.3678383
9	9.7224511	28478	9.3933924	9.7659168	62490	9.3747102
10	9.7201593	22918	9.3943057	9.7596678	51805	9.3794566
11	9.7182751	18842	9.3949687	9.7544873	43647	9.3828762
12	9.7166988	15763	9.3954654	9.7501226	37279	9.3854235
13	9.7153605	13383	9.3958471	9.7463947	32210	9.3873728
14	9.7142100	11505	9.3961467	9.7431737	28111	9.3888982
15	9.7132104	9996	9.3963862	9.7403626	24746	9.3901146
16	9.7123339	8765	9.3965806	9.7378880	21952	9.3911005
17	9.7115591	7748	9.3967408	9.7356928	19607	9.3919108
18	9.7108692	6899	9.3968742	9.7337321	17618	9.3925849
19	9.7102510	6182	9.3969865	9.7319703	15918	9.3931517
20	9.7096939	5571	9.3970820	9.7303785	14453	9.3936329
21	9.7091892	5047	9.3971638	9.7289332	. . . . . .	9.3940449
22	9.7087298	4594	9.3972344	9.7276152	13180	9.3944005
23	9.7083101	4197	9.3972959	9.7264083	12069	9.3947095
24	9.7079249	3852	9.3973496	9.7252990	11093	9.3949797
25	9.7075702	3547	9.3973969	9.7242759	10231	9.3952172
26	9.7072426	3276	9.3974387	9.7233294	9465	9.3954273
27	9.7069389	3037	9.3974758	9.7224511	8783	9.3956139
28	9.7066569	2820	9.3975090	9.7216340	8171	9.3957804
29	9.7063940	2629	9.3975387	9.7208718	7622	9.3959297
30	9.7061486	2454	9.3975655	9.7201593	7125	9.3960640
31	9.7059188	2298	9.3975896	9.7194917	6676	9.3961854
32	9.7057034	2154	9.3976115	9.7188648	6269	9.3962948

The following is a complete copy of the computation of $b_5^{(i)}$ for Venus when $g=0^\circ$, showing every figure that was made, as explained on pages 233–234:

VENUS: *Computation of $b_5^{(i)}$ for g (Earth)$=0°$.*

	$\log \lambda_i$ $\log \gamma_i$	$\log \lambda_i \gamma_i$	$\Delta \log \gamma_i$ $\Delta \log p_i$	$\log b_5^{(i)}$ $\log p_i$
	. . . .	. . . . . .	. . . . . .	2.0106083
1	. . . .	. . . . . .	. . . . . .	9.9887235
	9.9126633	. . . . . .	−1369007	1.9993318
2	−9.2982484	9.3478124	−180013	9.9707222
	0.0495640	0.6521876	529996	1.9700540
3	+8.9302717	9.0328353	−139471	9.9567751
	0.1025636	0.9671647	268591	1.9268291
4	9.1935132	9.3229359	−109295	9.9458456
	0.1294227	0.6770641	156343	1.8726747
5	9.2660638	9.4111208	−87250	9.9371206
	0.1450570	0.5888792	99373	1.8097953
6	9.2982485	9.4532428	−70960	9.9300246
	0.1549943	0.5467572	67197	1.7398199
7	9.3156226	9.4773366	−58695	9.9241551
	0.1617140	0.5226634	47596	1.6639750
8	9.3261387	9.4926123	−49277	9.9192274
	0.1664736	0.5073877	34956	1.5832024
9	9.3330106	9.5029798	−41912	9.9150362
	0.1699692	0.4970202	26434	1.4982386
10	9.3377570	9.5103696	−36056	9.9114306
	0.1726126	0.4896304	20475	1.4096692
11	9.3411766	9.5158366	−31330	9.9082976
	0.1746600	0.4841634	16182	1.3179668
12	9.3437239	9.5200022	−27465	9.9055511
	0.1762783	0.4799978	13010	1.2235179
13	9.3456732	9.5232524	−24269	9.9031242
	0.1775792	0.4767476	10617	1.1266421
14	9.3471986	9.5258395	−21593	9.9009649
	0.1786409	0.4741605	8777	1.0276070
15	9.3484150	9.5279346	−19334	9.8990315
	0.1795186	0.4720654	7339	0.9266385
16	9.3494009	9.5296534	−17407	9.8972908
	0.1802525	0.4703466	6198	0.8239293
17	9.3502112	9.5310835	−15754	9.8957154
	0.1808723	0.4689165	5282	0.7196447
18	9.3508853	9.5322858	−14325	9.8942829
	0.1814005	0.4677142	4537	0.6139276
19	9.3514521	9.5333063	−13081	9.8929748
	0.1818542	0.4666937	3925	0.5069024
20	9.3519333	9.5341801	−11993	9.8917755
	0.1822468	0.4658199	3424	0.3986779

VENUS AND THE EARTH.

Numerical development of 24 times the inverse third power of the distance of Venus from the Earth in terms of the mean anomalies of the two planets in their elliptic orbits for 1900.

$$\frac{1}{\rho^3}=\Sigma\left\{a\cos\left(i'g'+i_4 g_4\right)+b\sin\left(i'g'+i_4 g_4\right)\right\}$$

i' i_4	$24a$	$24b$	i' i_4	$24a$	$24b$
0 0	+ 118.3612	. . . . . .	− 4 − 1	− 0.0025	+ 0.0028
+ 1	+ 5.7884	− 5.3696	0	+ .0036	+ 0.0298
2	− 0.5389	− 1.9642	1	+ .2163	+ 0.1177
3	− 0.1207	− .0838	2	+ 2.1331	− 0.6524
4	− 0.0190	+ .0090	3	+ 3.4226	− 13.2752
5	− 0.0011	+ .0018	4	− 50.2777	− 98.3635
− 1 − 4	+ .0028	− .0014	5	− 1.1882	− .7570
− 3	− .0232	+ .0018	6	− 0.8552	+ .8260
− 2	− .0787	+ .1582	7	− 0.0124	+ .0545
− 1	+ .5543	+ 2.1122	8	+ 0.0098	+ .0062
0	+ 10.5927	+ 2.3432	9	+ 0.0005	+ .0005
+ 1	+ 183.0618	− 102.6121	− 5 + 1	+ 0.0195	+ 0.0248
2	+ 1.7962	− 5.1687	2	+ .2559	− .0058
3	− 1.3062	− 1.2980	3	+ 1.4302	− 1.5819
4	− .1188	− .0134	4	− 3.3224	− 12.3912
5	− .0102	+ .0154	5	− 71.4518	− 47.5899
6	+ .0002	− .0003	6	− 0.7918	− .1739
− 2 − 3	− .0025	− .0006	7	− 0.2824	+ .9536
− 2	− .0209	+ .0145	8	+ .0106	+ .0404
− 1	+ .0084	+ .2047	9	+ .0102	+ .0005
0	+ 1.6027	+ 1.5935	10	+ .0005	− .0004
+ 1	+ 12.5252	− 3.6702	− 6 + 1	+ 0.0009	+ 0.0034
2	+ 90.9943	− 148.7331	2	+ .0302	+ 0.0125
3	− .4977	− 3.6032	3	+ .2204	− .1339
4	− 1.5665	− 0.4238	4	+ .3934	− 1.9542
5	− .0875	+ 0.0352	5	− 8.0905	− 8.2552
6	− .0010	+ 0.0160	6	− 65.8513	− 5.0742
7	+ .0008	− 0.0035	7	− .4463	+ .0008
− 3 − 2	− .0027	+ 0.0005	8	+ .1845	+ .8016
− 1	− .0112	+ 0.0255	9	+ .0206	+ .0234
0	+ .1208	+ 0.1948	10	+ .0075	− .0042
+ 1	+ 2.2085	+ 0.5489	11	+ .0006	+ .0002
2	+ 9.6563	− 9.9992	− 7 + 2	+ .0034	+ 0.0030
3	− 5.3594	− 140.0914	3	+ .0325	− 0.0052
4	− 1.2859	− 1.9297	4	+ .1215	− .2214
5	− 1.3566	+ 0.3500	5	− .6007	− 1.7245
0	− 0.0474	+ 0.0562	6	− 9.7205	− 2.8233
7	+ 0.0064	+ 0.0122	7	− 45.7518	+ 21.1876
8	+ 0.0011	+ .0006	8	− .2665	− .0015

VENUS AND THE EARTH—Continued.

Numerical development of 24 times the inverse third power, etc.—Continued.

i' i_4	$24a$	$24b$	i' i_4	$24a$	$24b$
− 7 +9	+ .4540	+ .4971	−11+13	+ .0963	− .2625
10	+ .0211	+ .0085	14	+ .0016	− .0059
11	+ .0033	− .0053	15	− .0034	− .0006
12	+ .0004	− .0004	16	− .0003	0000
− 8+ 3	+ .0042	+ 0.0005	−12+ 7	− 0.0014	+ 0.0008
4	+ .0255	− .0206	8	− .0250	− 0.0085
5	− .0055	− .2418	9	− .1452	+ .0888
6	− 1.2516	− 1.0642	10	− .1604	+ .9208
7	− 8.4424	+ 1.9735	11	+ 2.6107	+ 3.0627
8	− 22.4300	+ 30.9883	12	+ 12.0025	+ 1.8825
9	− .2175	− .0212	13	− .0779	+ 0.1720
10	+ .5188	+ .1705	14	− .0350	− .2180
11	+ .0155	− .0011	15	− .0004	− .0042
12	− .0002	− .0062	16	− .0024	+ .0009
13	− .0005	− .0007	17	− .0002	0000
− 9+ 4	− 0.0006	− 0.0015	−13+ 8	− 0.0031	− 0.0024
5	+ .0112	− .0296	9	− .0242	+ .0045
6	− .1184	− .1940	10	− .0722	+ .1313
7	− 1.4322	− .2547	11	+ .2694	+ .7423
8	− 5.3796	+ 4.9292	12	+ 3.0353	+ 1.1206
9	− 3.3436	+ 28.6854	13	+ 8.4873	− 3.1441
10	− .2202	+ .0054	14	+ .0056	+ .1654
11	+ .4307	− .0860	15	− .1084	− .1357
12	+ .0105	− .0048	16	− .0011	− .0026
13	− .0024	− .0045	17	− .0012	+ .0016
14	0000	+ .0001	18	0000	+ .0002
−10+ 5	+ 0.0025	− 0.0032	−14+ 9	− 0.0026	− 0.0002
6	− .0053	− .0302	10	− .0170	+ .0143
7	− .1835	− .1005	11	+ .0020	+ .1315
8	− 1.1930	+ .4365	12	+ .5045	+ .4256
9	− 1.8885	+ 5.7498	13	+ 2.5500	− .4050
10	+ 8.3438	+ 20.0448	14	+ 4.3630	− 5.1199
11	− .2082	+ .0699	15	+ .0685	+ .1226
12	+ .2669	− .2289	16	− .1263	− .0514
13	+ .0053	− .0062	17	− .0012	− .0016
14	− .0034	− .0024	18	− .0002	+ .0017
15	− .0003	+ .0012	19	− .0003	− .0004
−11+ 6	+ 0.0007	− 0.0021	−15+10	− 0.0028	+ 0.0012
7	− 0.0183	− .0217	11	− .0065	+ .0179
8	− .1899	+ .0042	12	+ .0590	+ .0971
9	− .7018	+ .8408	13	+ .5388	+ .0992
10	+ .9582	+ 4.8665	14	+ 1.6070	− 1.2697
11	+ 12.7915	+ 10.0493	15	+ .9693	− 4.8945
12	− .1571	+ .1368	16	+ .0989	+ .0652

VENUS AND THE EARTH—Continued.

Numerical development of 24 times the inverse third power, etc.—Continued.

i' i_4	24 a	24 b	i' i_4	24 a	24 b
−15+17	− .1052	+ .0133	−19+21	+ .0220	+ .0312
18	− .0012	− .0008	22	− .0010	+ .0010
19	+ .0003	+ .0015	23	+ .0006	− .0002
20	− .0001	− .0002	24	− .0008	− .0002
−16+11	− 0.0019	− 0.0014	−20+15	+ 0.0017	+ 0.0013
12	+ .0038	+ .0173	16	+ .0076	− .0069
13	+ .0850	+ .0475	17	0000	− .0498
14	+ .4264	− .1495	18	− .1454	− .1314
15	+ .6158	− 1.4937	19	− .5933	+ .0490
16	− 1.1430	− 3.5132	20	− .7741	+ .7724
17	+ .0966	+ .0094	21	− .0135	− .0381
18	− .06638	+ .04913	22	+ .0257	+ .0135
19	− .0012	− .0002	23	− .0006	+ .0006
20	+ .0009	+ .0005	24	+ .0005	− .0010
21	+ .0002	+ .0004	25	+ .0002	0000
−17+12	− 0.0006	+ 0.0027	−21+16	+ 0.0015	− 0.0006
13	+ .0103	+ .0116	17	+ .0030	− .0082
14	+ .0835	− .0010	18	− .0203	− .0355
15	+ .2425	− .2790	19	− .1534	− .0342
16	− .1522	− 1.2574	20	− .3810	+ .2566
17	− 2.0054	− 1.8541	21	− .2198	+ .7735
18	+ .0732	− 0300	22	− .0235	− .0220
19	− .0264	+ .0584	23	+ .0225	− .0005
20	− .0010	0000	24	− .0002	+ .0014
21	+ .0010	− .0003	25	− .0001	− .0004
22	− .0003	+ .0009	26	0000	− .0004
−18+13	+ 0.0013	+ 0.0025	−22+17	+ 0.0007	− 0.0011
14	+ .0130	+ .0045	18	− .0014	− .0072
15	+ .0608	− .0358	19	− .0285	− .0174
16	+ .0592	− .2955	20	− .1200	+ .0367
17	− .5833	− .7981	21	− .1603	+ .3160
18	− 1.9597	− .4708	22	+ .1359	+ .5745
19	+ .0408	− .0494	23	− .0235	− .0062
20	+ .0042	+ .0488	24	+ .0140	− .0096
21	− .0005	+ .0004	25	+ .0002	+ .0003
22	+ .0006	+ .0001	26	− .0002	− .0002
23	− .0001	0000	27	− .0003	− .0002
−19+14	+ 0.0016	+ 0.0013	−23+18	+ 0.0004	− 0.0014
15	+ .0118	− 0.0020	19	− .0041	− .0050
16	+ .0298	− .0509	20	− .0276	− .0005
17	− .0758	− .2314	21	− .0690	+ .0725
18	− .6988	− .3196	22	+ .0110	+ .2710
19	− 1.4302	+ .4021	23	+ .2931	+ .3202
20	+ .0095	− .0500	24	− .0180	+ .0040

VENUS AND THE EARTH—Continued.

Numerical development of 24 times the inverse third power, etc.—Continued.

i' i_4	$24a$	$24b$
−23+25	+ .0060	− .0116
26	+ .0002	.0000
27	− .0002	− .0002
−24+19	− 0.0001	− 0.0010
20	− .0048	− .0018
21	− .0198	+ .0106
22	− .0201	+ .0762
23	+ .1092	+ .1768
24	+ .3016	+ .1000
25	− .0109	+ .0096
26	.0000	− .0102
27	+ .0002	.0000
−25+20	− 0.0009	− 0.0008
21	− .0044	+ .0013
22	− .0101	+ .0150
23	+ .0168	+ .0610
24	+ .1382	+ .0765
25	+ .2283	− .0440
26	− .0036	+ .0108
27	− .0037	− .0065
−26+21	− 0.0008	− 0.0004
22	− .0034	+ .0017
23	− .0007	+ .0148
24	+ .0352	+ .0348
25	+ .1205	− .0004

i' i_4	$24a$	$24b$
−26+26	+ .1301	− .1101
27	+ .0018	+ .0086
−27+22	− 0.0009	− 0.0010
23	− .0012	+ .0028
24	+ .0056	+ .0116
25	+ .0369	+ .0105
26	+ .0796	− .0454
27	+ .0438	− .1170
−28+23	0.0000	+ 0.0003
24	− .0002	+ .0021
25	+ .0080	+ .0046
26	+ .0294	− .0077
27	+ .0362	− .0600
−29+24	+ 0.0004	0.0000
25	+ .0013	+ .0015
26	+ .0078	+ .0008
27	+ .0172	− .0160
−30+25	. . . . . .	. . . . . .
26	. . . . . .	. . . . . .
27	. . . . . .	. . . . . .
−31+26	. . . . . .	. . . . . .
27	. . . . . .	. . . . . .
−32+27	. . . . . .	. . . . . .

VENUS AND THE EARTH.

Numerical development of 24 times the inverse fifth power of the distance of Venus from the Earth in terms of the mean anomalies of the two planets in their elliptic orbits for 1800.

$$\frac{1}{\rho^5}=\Sigma\left\{a\cos(i'\,g'+i_4\,g_4)+b\sin(i'\,g'+i_4\,g_4)\right\}$$

i' i_4	$24a$	$24b$	i' i_4	$24a$	$24b$
0 0	+1006.4170	....	−3 −2	− .0981	+ .0387
+1	+ 119.0931	− 106.3205	−1	− .3112	+ .7738
2	− 8.8087	− 38.0411	0	+ 3.1062	+ 4.8577
3	− 3.4431	− 2.3838	+1	+ 43.1818	+ 9.4122
4	− 0.5932	+ 0.2470	2	+ 151.6440	− 157.4512
5	+ .0854	+ .0702	3	+ 62.5029	−1631.3784
6	+ .0045	− .0039	4	− 51.3790	− 66.0423
7	+ .0006	− .0090	5	− 28.7837	+ 6.4600
−1 −6	+ .0024	− .0010	6	− 1.5760	+ 2.0155
−5	+ .0129	− .0071	7	+ .1773	+ 0.4129
−4	− .0673	− .0532	8	+ .0441	+ .0199
−3	− .7002	+ .0931	9	+ .0034	− .0010
−2	− 2.0962	+ 4.2720	+10	− .0005	− .0008
−1	+ 12.1686	+ 39.6173	−4 −3	+ .0036	+ .0001
0	+ 181.1343	+ 40.4850	−2	− .0104	− .0124
+1	+1706.5824	− 956.5143	−1	− .0682	− .1264
2	+ 43.7904	− 124.9798	0	+ .1310	+ .8778
3	− 24.6545	− 26.4264	+1	+ 5.4265	+ 2.8633
4	− 3.5975	− 0.3571	2	+ 42.1948	− 13.8210
5	− 0.3468	+ 0.4573	3	+ 55.8009	− 215.1555
6	+ .0081	+ .0650	4	− 649.5585	−1270.9491
7	+ .0054	+ .0028	5	− 58.7327	− 25.9984
8	− .0008	− .0030	6	− 19.2802	+ 17.5802
−2 −5	+ .0067	+ .0009	7	− .3446	+ 2.0762
−4	− .0050	− .0063	8	+ .3152	+ 0.2342
−3	− .0886	− .0163	9	+ .0411	− 0.0018
−2	− .6092	+ .4736	10	+ .0023	− .0022
−1	+ .2904	+ 5.2895	+11	− .0007	+ .0009
0	+ 31.7583	+ 29.2754	−5 −2	+ .0024	− .0018
+1	+ 197.4674	− 59.2818	−1	− .0078	+ .0076
2	+ 948.9104	−1550.8436	0	− .0205	+ .1192
3	− 17.4126	− 105.1862	+1	+ .5847	+ .7284
4	− 31.4258	− 9.6859	2	+ 6.3901	− .2083
5	− 2.8180	+ 1.2074	3	+ 28.9857	− 32.8389
6	− .0606	+ .5064	4	− 55.8416	− 210.7800
7	+ .0336	+ .0390	5	−1016.7720	− 677.2520
8	+ .0051	− .0004	6	− 48.1612	+ 3.4306
9	− .0021	− .0002	7	− 7.0478	+ 21.6610
−3 −4	+ .0039	− 0.0004	8	+ 0.5576	+ 1.6135
−3	− .0083	− .0112	9	+ .3400	+ 0.0402

VENUS AND THE EARTH—Continued.

Numerical development of 24 times the inverse fifth power, etc.—Continued.

i' i_4	$24a$	$24b$	i' i_4	$24a$	$24b$
−5+10	+ .0278	− .0199	−8+12	+ 0.0256	− 0.2130
11	− .0001	− .0030	13	− 0.0092	− 0.0160
12	− .0013	+ .0006	14	− 0.0024	+ 0.0003
−6−1	.0000	. . . .	15	+ 0.0001	+ 0.0006
0	− .0002	+ 0.0003	−9+2	− 0.0044	− 0.0018
+1	+ .0460	− 0.0898	3	+ 0.0143	+ 0.0132
2	+ .9020	+ .3424	4	+ 0.1349	− 0.0494
3	+ 5.5517	− 3.4215	5	+ 0.3370	− 0.9244
4	+ 8.1177	− 41.6650	6	− 3.2152	− 5.2920
5	− 144.7228	− 148.7746	7	− 34.5082	− 6.2933
6	−1024.8068	− 78.9782	8	− 115.3828	+ 104.7718
7	− 30.3002	+ 18.7156	9	− 65.5964	+ 563.2880
8	+ 3.9405	+ 19.2787	10	+ 4.6548	+ 11.8749
9	+ 1.0080	+ 0.9265	11	+ 11.7690	− 2.0918
10	+ 0.2701	− .1169	12	+ 0.4940	− 0.4355
11	+ .0138	− .0262	13	− 0.0715	− 0.1671
12	− .0014	− .0028	14	− 0.0133	− 0.0080
13	− .0001	+ .0003	15	− 0.0016	+ 0.0012
−7 0	0.0000	0.0000	16	+ 0.0009	+ 0.0001
+1	+ 0.0024	+ 0.0120	−10+3	− 0.0034	− 0.0011
2	+ 0.1040	+ 0.0844	4	+ 0.0185	− 0.0022
3	+ 0.9728	− 0.1614	5	− 0.0078	− 0.1103
4	+ 3.1142	− 5.7330	6	− 0.1714	− 0.9483
5	− 13.3523	− 38.1902	7	− 5.1623	− 2.8388
6	− 184.6949	− 54.4921	8	− 30.1037	+ 10.8131
7	− 773.6470	+ 358.3035	9	− 43.2380	+ 129.7126
8	− 13.3186	+ 22.0683	10	+ 175.3675	+ 421.1381
9	+ 11.0498	+ 12.6665	11	+ 6.2918	+ 6.0300
10	+ 1.0504	+ 0.2704	12	+ 7.6830	− 6.3400
11	+ 0.1498	− 0.2015	13	+ 0.1820	− 0.4718
12	− 0.0001	− 0.0235	14	− 0.1218	− 0.0920
13	− 0.0022	− 0.0012	15	− 0.0126	− 0.0005
14	− 0.0001	+ 0.0007	16	− 0.0005	+ 0.0016
−8+1	+ 0.0010	+ 0.0066	17	+ 0.0013	+ 0.0007
2	+ 0.0080	− 0.0006	−11+4	− 0.0013	+ 0.0024
3	+ 0.1435	+ 0.0258	5	− 0.0006	− 0.0108
4	+ 0.7643	− 0.6336	6	+ 0.0236	− 0.1406
5	− 0.1532	− 6.4062	7	− 0.6030	− 0.7098
6	− 28.8606	− 24.6526	8	− 5.5253	+ 0.1065
7	− 170.4302	+ 39.0604	9	− 18.6115	+ 21.9767
8	− 409.7310	+ 566.1468	10	+ 22.3335	+ 116.1812
9	− 1.4276	+ 18.2510	11	+ 286.4366	+ 224.9622
10	+ 13.4256	+ 4.7282	12	+ 5.4702	+ 2.0475
11	+ 0.8253	− 0.2005	13	+ 2.9600	− 7.6945

VENUS AND THE EARTH—Continued.

Numerical development of 24 times the inverse fifth power, etc.—Continued.

i' i_4	$24a$	$24b$	i' i_4	$24a$	$24b$
−11+14	− 0.0420	− 0.3824	−14+18	− 0.0102	+ 0.0723
15	− 0.1272	− 0.0159	19	+ 0.0015	+ 0.0047
16	− 0.0090	+ 0.0040	20	+ 0.0006	− 0.0006
17	+ 0.0001	+ 0.0013	21	− 0.0002	+ 0.0013
18	− 0.0017	+ 0.0002	−15+ 8	+ 0.0014	+ 0.0001
−12+ 5	+ 0.0098	+ 0.0024	9	− 0.0166	− 0.0064
6	+ 0.0070	− 0.0064	10	− 0.1158	+ 0.0439
7	− 0.0489	− 0.1380	11	− 0.2402	+ 0.6828
8	− 0.8374	− 0.2978	12	+ 1.9330	+ 3.2702
9	− 4.3874	+ 2.6358	13	+ 16.7141	+ 3.2108
10	− 4.5948	+ 25.2182	14	+ 46.8696	− 36.7282
11	+ 65.2679	+ 77.2590	15	+ 27.0458	− 136.8284
12	+ 285.3858	+ 44.6933	16	+ 0.9186	− 0.8614
13	+ 3.8562	− 0.0322	17	− 3.6249	+ 0.4142
14	− 0.9870	− 6.7076	18	− 0.1100	+ 0.0450
15	− 0.1595	− 0.2420	19	+ 0.0232	+ 0.0555
16	− 0.0983	+ 0.0408	20	+ 0.0032	+ 0.0031
17	− 0.0047	+ 0.0070	21	+ 0.0019	− 0.0003
18	+ 0.0006	+ 0.0008	22	− 0.0004	0.0000
19	+ 0.0001	− 0.0007	−16+ 9	+ 0.0010	+ 0.0007
−13+ 6	+ 0.0006	− 0.0016	10	− 0.0169	+ 0.0005
7	− 0.0012	− 0.0180	11	− 0.0735	+ 0.0906
8	− 0.1048	− 0.0859	12	+ 0.1229	+ 0.6504
9	− 0.8258	+ 0.1542	13	+ 2.9312	+ 1.6659
10	− 2.2782	+ 4.0909	14	+ 13.8108	− 4.7136
11	+ 7.5528	+ 21.2754	15	+ 18.8912	− 45.3175
12	+ 80.1228	+ 29.9663	16	− 33.6045	− 103.0926
13	+ 213.5595	− 79.1750	17	+ 0.5979	− 0.8240
14	+ 2.4126	− 0.7986	18	− 2.3821	+ 1.7223
15	− 3.3950	− 4.3820	19	− 0.0594	+ 0.0637
16	− 0.1882	− 0.1086	20	+ 0.0383	+ 0.0306
17	− 0.0542	+ 0.0745	21	+ 0.0020	− 0.0008
18	+ 0.0022	+ 0.0057	22	+ 0.0002	− 0.0006
19	+ 0.0012	+ 0.0003	23	− 0.0005	+ 0.0003
20	− 0.0008	+ 0.0002	−17+10	+ 0.0008	− 0.0004
−14+ 7	+ 0.0010	− 0.0057	11	− 0.0140	+ 0.0091
8	− 0.0106	− 0.0147	12	− 0.0168	+ 0.1066
9	− 0.1292	− 0.0195	13	+ 0.3910	+ 0.4555
10	− 0.5932	+ 0.5106	14	+ 2.9703	− 0.0100
11	+ 0.0550	+ 4.2398	15	+ 8.2208	− 9.3011
12	+ 14.9414	+ 12.8237	16	− 4.6588	− 39.9350
13	+ 70.8306	− 10.9530	17	− 61.7008	− 56.9620
14	+ 115.8324	− 136.0136	18	+ 0.3273	− 0.8206
15	+ 1.4579	− 0.9192	19	− 0.9874	+ 2.1408
16	− 4.1575	− 1.7500	20	− 0.0205	+ 0.0601
17	− 0.1608	− 0.0103	21	+ 0.0386	+ 0.0072

VENUS AND THE EARTH—Continued.

Numerical development of 24 times the inverse fifth power, etc.—Continued.

i' i_4	$24a$	$24b$	i' i_4	$24a$	$24b$
−17+22	+ 0.0018	− 0.0002	−20+24	+ 0.0036	− 0.0215
23	0.0000	− 0.0003	25	− 0.0006	− 0.0008
24	− 0.0004	+ 0.0003	26	− 0.0004	+ 0.0001
−18+11	− 0.0013	− 0.0008	27	− 0.0001	0.0000
12	− 0.0071	+ 0.0148	−21+14	+ 0.0003	+ 0.0010
13	+ 0.0355	+ 0.0927	15	+ 0.0114	+ 0.0058
14	+ 0.5083	+ 0.1795	16	+ 0.0692	− 0.0258
15	+ 2.2469	− 1.2944	17	+ 0.1268	− 0.3562
16	+ 2.1494	− 10.2641	18	− 0.8055	− 1.4460
17	− 19.2032	− 26.4948	19	− 5.9333	− 1.3774
18	− 63.0207	− 15.1096	20	− 14.2028	+ 9.4897
19	+ 0.0514	− 0.7883	21	− 7.9673	+ 28.1476
20	+ 0.1501	+ 1.8777	22	− 0.4890	− 0.2235
21	+ 0.0042	+ 0.0471	23	+ 0.9396	− 0.0418
22	+ 0.0309	− 0.0115	24	+ 0.0173	− 0.0021
23	+ 0.0013	− 0.0004	25	− 0.0055	+ 0.0810
24	− 0.0001	0.0000	26	+ 0.0001	+ 0.0166
25	− 0.0006	+ 0.0002	27	− 0.0002	+ 0.0015
−19+12	− 0.0010	+ 0.0002	28	+ 0.0004	0.0000
13	+ 0.0010	+ 0.0150	−22+15	+ 0.0002	+ 0.0008
14	+ 0.0703	+ 0.0576	16	+ 0.0139	− 0.0006
15	+ 0.4734	− 0.0872	17	+ 0.0076	− 0.0520
16	+ 1.1342	− 1.9401	18	− 0.0590	− 0.3217
17	− 2.6885	− 8.3803	19	− 1.2038	− 0.7290
18	− 24.0400	− 11.1036	20	− 4.8200	+ 1.4254
19	− 47.9844	+ 13.5290	21	− 6.1820	+ 12.1431
20	− 0.2098	− 0.6725	22	+ 5.1793	+ 21.7115
21	+ 0.8373	+ 1.2426	23	− 0.4595	+ 0.0088
22	+ 0.0158	+ 0.0302	24	+ 0.6297	− 0.3905
23	+ 0.0171	− 0.0205	25	+ 0.0144	+ 0.0002
24	+ 0.0003	− 0.0015	26	− 0.0108	− 0.0090
25	− 0.0004	− 0.0003	27	− 0.0008	− 0.0006
26	0.0000	+ 0.0009	28	− 0.0001	+ 0.0003
−20+13	− 0.0010	+ 0.0005	29	+ 0.0004	0.0000
14	+ 0.0077	+ 0.0114	−23+16	+ 0.0010	+ 0.0005
15	+ 0.0815	+ 0.0129	17	+ 0.0087	− 0.0071
16	+ 0.3246	− 0.2755	18	+ 0.0127	− 0.0385
17	+ 0.0198	− 1.9348	19	− 0.1815	− 0.2166
18	− 5.4015	− 4.9725	20	− 1.1884	− 0.0322
19	− 21.2524	+ 1.6657	21	− 2.8686	+ 2.9670
20	− 27.0530	+ 27.0268	22	+ 0.4315	+ 10.8178
21	− 0.4065	− 0.4705	23	+ 11.5092	+ 12.5025
22	+ 1.0620	+ 0.5356	24	− 0.3418	+ 0.1165
23	+ 0.0182	+ 0.0161	25	+ 0.2809	+ 0.2164

VENUS AND THE EARTH—Continued.

Numerical development of 24 times the inverse fifth power, etc.—Continued.

i' i_4	$24a$	$24b$
−23+26	+ 0.0107	+ 0.0588
27	− 0.0112	+ 0.0008
28	− 0.0001	− 0.0002
29	− 0.0002	− 0.0001
30	0.0000	0.0000
−24+17	+ 0.0002	− 0.0004
18	+ 0.0044	− 0.0086
19	− 0.0182	− 0.0485
20	− 0.2287	− 0.0856
21	− 0.8847	+ 0.4722
22	− 0.8460	+ 3.2665
23	+ 4.4825	+ 7.3042
24	+ 12.2651	+ 4.0198
25	− 0.1860	+ 0.2620
26	− 0.0037	− 0.4572
27	+ 0.0066	− 0.0055
28	− 0.0083	+ 0.0028
29	− 0.0009	+ 0.0001
30	+ 0.0001	+ 0.0002
31	0.0000	0.0000
−25+18	− 0.0008	− 0.0001
19	− 0.0004	− 0.0090
20	− 0.0361	− 0.0290
21	− 0.2060	+ 0.0349
22	− 0.4475	+ 0.7081
23	+ 0.7090	+ 2.6536
24	+ 5.9176	+ 3.2976
25	+ 9.2026	− 1.8788
26	+ 4.4205	+ 0.2617
27	+ 0.2220	− 0.3098
28	+ 0.0222	− 0.0068
29	− 0.0050	+ 0.0052
30	− 0.0004	+ 0.0005
31	+ 0.0003	+ 0.0001
32	0.0000	0.0000
−26+19	+ 0.0004	− 0.0010
20	− 0.0041	− 0.0070
21	− 0.0395	− 0.0086
22	− 0.1390	+ 0.1130
23	− 0.0342	+ 0.6967
24	+ 1.5590	+ 1.5903
25	+ 5.3153	− 0.0059
26	+ 5.6498	− 4.7876
27	+ 0.0365	+ 0.2030
28	+ 0.1322	− 0.1417

i' i_4	$24a$	$24b$
−26+29	+ 0.0367	− 0.0058
30	+ 0.0012	+ 0.0054
31	− 0.0003	+ 0.0002
32	− 0.0001	− 0.0001
−27+20	+ 0.0006	− 0.0009
21	− 0.0077	− 0.0030
22	− 0.0329	+ 0.0129
23	− 0.0550	+ 0.1397
24	+ 0.2448	+ 0.5785
25	+ 2.1273	− 0.2278
26	− 0.8251	− 2.1119
27	+ 1.5586	− 4.9265
28	+ 0.1114	− 3.1374
29	− 0.2166	− 0.3201
30	− 0.0015	− 0.0207
31	+ 0.0012	+ 0.0045
32	− 0.0002	− 0.0002
−28+21	+ 0.0004	+ 0.0010
22	− 0.0060	+ 0.0002
23	− 0.0188	+ 0.0226
24	+ 0.0193	+ 0.1278
25	+ 0.3984	+ 0.2640
26	+ 1.4068	− 0.3300
27	+ 1.6985	− 2.7895
28	− 0.6368	− 3.9551
29	+ 0.1357	− 2.1250
30	− 0.1490	− 0.1375
31	− 0.0025	− 0.0146
32	+ 0.0023	+ 0.0019
−29+22	− 0.0006	− 0.0003
23	− 0.0046	+ 0.0024
24	− 0.0044	+ 0.0348
25	+ 0.0670	− 0.0128
26	+ 0.3912	+ 0.0340
27	+ 0.8526	− 1.1025
28	+ 0.0972	+ 0.7070
29	− 1.9505	− 2.0906
30	+ 0.1078	− 0.9708
31	− 0.0712	+ 0.0102
32	− 0.0025	− 0.0068
−30+23	− 0.0004	− 0.0007
24	− 0.0025	+ 0.0042
25	+ 0.0078	+ 0.0215
26	+ 0.0825	+ 0.0348

VENUS AND THE EARTH—Concluded.

Numerical development of 24 times the inverse fifth power, etc.—Continued.

i' i_4	$24a$	$24b$	i' i_4	$24a$	$24b$
$-30+27$	+ 0.3264	− 0.1291	$-31+30$	− 1.2740	+ 0.1109
28	− 0.0875	− 1.1000	31	− 1.7805	+ 0.2964
29	− 0.9342	+ 0.4018	32	+ 0.0259	− 0.0514
30	− 2.2000	− 0.7114	$-32+25$	+ 0.0002	+ 0.0007
31	+ 0.0654	− 0.0480	26	+ 0.0018	+ 0.0029
32	− 0.0066	+ 0.0995	27	+ 0.0155	+ 0.0038
$-31+24$	− 0.0003	+ 0.0005	28	+ 0.0505	− 0.0388
25	+ 0.0001	+ 0.0052	29	+ 0.0205	− 0.2132
26	+ 0.0142	+ 0.0034	30	− 0.3852	− 0.4444
27	+ 0.0758	− 0.0108	31	− 1.1826	− 0.0906
28	+ 0.1510	− 0.2112	32	− 1.0808	+ 0.7648
29	− 0.1400	− 0.8247			

MARS AND THE EARTH.

In the following developments of the inverse third power of the distance of Mars from the Earth, I have given the development for ρ_0 as well as for ρ itself, in order that a comparison of the two sets of numbers might afford a general control upon their accuracy, and facilitate a judgment as to the magnitude of the difference between them. It will be seen that, notwithstanding the considerable eccentricity of Mars, the two developments are appreciably different only for the smaller indices, and in nearly all cases so small as to be almost without influence in the planetary theories.

As already remarked, the change from eccentric to mean anomaly was completed only for a portion of the work. So far as completed the corresponding development is given in the margin of the table.

MARS AND THE EARTH.

Development of 24 times the inverse third power of the distance of Mars from the Earth in terms of the eccentric anomaly of the Earth and the mean anomaly of Mars, and also in terms of the mean anomalies of both planets in their elliptical orbits for 1800.

$$\left(\frac{1}{\rho_0}\right)^3 = \sum_{i=0}^{i=\infty} \sum_{j=-\infty}^{j=+\infty} \left\{ a''_{i,j} \cos\left(i\,u' + (j-i)\,g_4\right) + b''_{i,j} \sin\left(i\,u' + (j-i)\,g_4\right) \right\}$$

$$\left(\frac{1}{\rho}\right)^3 = \sum_{i=0}^{i=\infty} \sum_{=-\infty}^{j=+\infty} \left\{ a'_{i,j} \cos\left(i\,u' + (j-i)\,g_4\right) + b'_{i,j} \sin\left(i\,u' + (j-i)\,g_4\right) \right\}$$

$$\left(\frac{1}{\rho}\right)^3 = \sum_{i=0}^{i=\infty} \sum_{i'-\infty}^{i'=\infty} \left\{ a_{i,i'} \cos\left(i\,g_4 + i'\,g'\right) + b_{i,i'} \cos\left(i\,g_4 + i'\,g'\right) \right\}$$

$i,\ j$	$24a''$	$24b''$	$24a'$	$24b'$
0 0	+25.9965	. . . . . .	+25.9967	. . . . .
1	+16.0792	− 1.3987	+16.0799	− 1.3985
2	+ 4.3143	− 0.7339	4.3173	− 0.7325
3	+ 1.0612	− 0.2657	1.0626	− 0.2653
4	+ 0.2457	− 0.0806	0.2461	− 0.0805
5	+ 0.0544	− 0.0215	+ 0.0544	− 0.0215
1 −5	− 0.0278	− 0.0960	− 0.0279	− 0.0961
−4	− 0.1420	− 0.3932	− 0.1422	− 0.3935
−3	− 0.6703	− 1.5065	− 0.6710	− 1.5070
−2	− 2.8373	− 5.2622	2.8387	− 5.2626
−1	−10.2120	−15.8520	−10.2124	−15.8519
0	−26.5592	−34.8581	−26.5595	−34.8582
1	− 7.5230	− 8.0352	− 7.5237	− 8.0357
2	− 2.0451	− 1.8484	− 2.0473	− 1.8511
3	− 0.5149	− c.3985	− 0.5160	− 0.3996
4	− 0.1220	− 0.0820	− 0.1223	− 0.0822
5	− 0.0272	− .0164	− 0.0272	− 0.0164
2 −5	− .0918	+ 0.1264	− 0.0919	+ 0.1265
−4	− 0.3252	+ 0.5150	− 0.3254	+ 0.5154
−3	− 1.0379	+ 1.9367	− 1.0384	+ 1.9376
−2	− 2.8729	+ 6.4649	− 2.8734	+ 6.4661
−1	− 6.3452	+17.7462	− 6.3453	+17.7462
0	− 8.9240	+32.5040	− 8.9242	+32.5043
1	− 0.9780	+ 6.7278	− 0.9785	+ 6.7289
2	− 0.1043	+ 1.6219	− 0.1051	+ 1.6248
3	+ 0.0037	+ 0.3675	+ 0.0037	+ 0.3684
4	+ .0062	+ 0.0794	+ 0.0062	+ 0.0796
5	+ .0022	+ 0.0160	+ 0.0022	+ 0.0160
3 −5	+ 0.2253	− 0.0056	+ 0.2255	− 0.0057
−4	+ 0.8029	− 0.0758	+ 0.8033	− 0.0761
−3	+ 2.6005	− 0.4338	+ 2.6015	− 0.4344

$i,\ i'$	$24a$	$24b$
0 0	+26.0599	. . . . .
1	− 7.5211	− 8.0626
2	− 0.1722	+ 1.5602
3	+ 0.1525	− 0.0987
4	− 0.0161	− 0.0061
−1 −2	− 0.0135	− 0.3535
−1	− 2.0472	+ 1.8572
0	+16.3201	+ 1.6763
1	−26.5411	−34.9688
2	− 1.2202	+ 6.4469
3	+ 0.7365	− 0.4131
4	− 0.0832	− 0.0358
−2 −1	− 0.5160	+ 0.3996
0	+ 4.4030	+ 0.8656
1	−10.0615	−16.3968
2	− 9.0903	+32.4121
3	+ 3.1512	− 1.4397
4	− 0.4008	− 0.1888
5	+ 0.0075	+ 0.0120
−3 0	+ 1.0850	+ 0.3095
1	− 2.7298	− 5.5612
2	− 6.9476	+17.9297
3	+22.7839	− 8.9235
4	− 1.2838	− 0.8245
5	+ 0.0463	+ 0.2290
−4 1	− 0.6211	− 1.6149
2	− 3.2924	+ 6.5805
3	+16.7239	− 4.8102
4	−14.5642	− 8.9784
5	− 0.1007	+ 0.4246

MARS AND THE EARTH—Continued.

Development of 24 times the inverse third power of the distance, etc.—Continued.

$i,\ j$	$24a''$	$24b''$	$24a'$	$24b'$	$i,\ i'$	$24a$	$24b$
3−2	+ 7.3054	− 1.7673	+ 7.3065	− 1.7680			
−1	+16.2795	− 5.2186	+16.2797	− 5.2187	−5 2	− 1.2267	+ 1.9740
0	+22.8424	− 9.2464	+22.8428	− 9.2466	3	+ 7.6854	− 1.4500
1	+ 3.2869	− 1.9896	+ 3.2882	− 1.9900	4	−11.7801	− 8.8138
2	+ 0.7506	− 0.5225	+ 0.7528	− 0.5235	5	+ 1.0515	+11.5392
3	+ 0.1541	− 0.1248	+ 0.1545	− 0.1253	6	− 0.0018	+ 0.2386
4	+ 0.0285	− 0.0276	+ 0.0285	− 0.0277			
5	+ 0.0039	− 0.0054	+ 0.0039	− 0.0054	−6 2	− 0.3928	+ 0.5241
					3	+ 2.7871	− 0.2638
4−5	− 0.1847	− 0.2282	− 0.1848	− 0.2282	4	− 5.6791	− 5.1302
−4	− 0.6650	− 0.7136	− 0.6654	− 0.7136	5	+ 0.0558	+11.8322
−3	− 2.1220	− 1.9780	− 2.1226	− 1.9780	6	+ 5.6758	− 5.0841
−2	− 5.7326	− 4.6278	− 5.7333	− 4.6279	. . . .	. . . . .	. . . . .
−1	−11.9165	− 8.2846	−11.9166	− 8.2848	. . . .	. . . . .	. . . . .
0	−14.9733	− 8.8283	−14.9737	− 8.8285	. . . .	. . . . .	. . . . .
1	− 1.8541	− 0.5862	− 1.8551	− 0.5867	. . . .	. . . . .	. . . . .
2	− 0.4792	− 0.1452	− 0.4807	− 0.1461	. . . .	. . . . .	. . . . .
3	− 0.1016	− 0.0240	− 0.1019	− 0.0241	. . . .	. . . . .	. . . . .
4	− 0.0200	− 0.0034	− 0.0200	− 0.0034	. . . .	. . . . .	. . . . .
5	− 0.0043	− 0.0010	− 0.0043	− 0.0010	. . . .	. . . . .	. . . . .
5−5	− 0.0831	+ 0.3494	− 0.0831	+ 0.3493	. . . .	. . . . .	. . . . .
−4	− 0.1811	+ 1.0855	− 0.1811	+ 1.0852	. . . .	. . . . .	. . . . .
−3	− 0.2861	+ 2.9698	− 0.2861	+ 2.9694	. . . .	. . . . .	. . . . .
−2	− 0.1758	+ 6.7908	− 0.1757	+ 6.7905	. . . .	. . . . .	. . . . .
−1	+ 0.5333	+11.7270	+ 0.5334	+11.7272	. . . .	. . . . .	. . . . .
0	+ 1.4477	+11.8261	+ 1.4476	+11.8265	. . . .	. . . . .	. . . . .
1	+ 0.3981	+ 0 7178	+ 0.3981	+ 0.7186	. . . .	. . . . .	. . . . .
2	+ 0.0966	+ 0.2528	+ 0.0967	+ 0.2540	. . . .	. . . . .	. . . . .
3	+ 0.0219	+ 0.0478	+ 0.0220	+ 0.0479	. . . .	. . . . .	. . . . .
4	+ 0.0043	+ 0.0091	+ 0.0043	+ 0.0091	. . . .	. . . . .	. . . . .
5	+ 0.0005	+ 0.0018	+ 0.0005	+ 0.0018	. . . .	. . . . .	. . . . .
6−5	+ 0.3778	− 0.1693	+ 0.3778	− 0.1693	. . . .	. . . . .	. . . .
−4	+ 1.0323	− 0.5497	+ 1.0323	− 0.5497	. . . .	. . . . .	. . . . .
−3	+ 2.4634	− 1.5352	+ 2.4634	− 1.5352	. . . .	. . . . .	. . . . .
−2	+ 4.8336	− 3.5029	+ 4.8336	− 3.5029	. . . .	. . . . .	. . . . .
−1	+ 7.0007	− 5.8793	+ 7.0009	− 5.8794	. . . .	. . . . .	. . . . .
0	+ 5.6749	− 5.5859	+ 5.6753	− 5.5862	. . . .	. . . . .	. . . . .
1	− 0.0531	− 0.2505	− 0.0525	− 0.2510	. . . .	. . . . .	. . . . .
2	+ 0.0812	− 0.1263	+ 0.0818	− 0.1269	. . . .	. . . . .	. . . . .
3	+ 0.0112	− 0.0220	+ 0.0112	− 0.0220	. . . .	. . . . .	. . . . .
4	+ 0.0029	− 0.0043	+ 0.0029	− 0.0043	. . . .	. . . . .	. . . . .
5	− 0.0002	− 0.0005	− 0.0002	− 0.0005	. . . .	. . . . .	. . . . .

MARS AND THE EARTH—Continued.

Development of 24 times the inverse third power of the distance, etc.— Continued.

i, *j*	24*a''*	24*b''*	24*a'*	24*b'*	*i*, *i'*	24*a*	24*b*
7−5	− 0.3956	− 0.2145	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
−4	− 1.0763	− 0.4919	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
−3	− 2.5265	− 0.9514	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
−2	− 4.8311	− 1.4461	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
−1	− 6.7382	− 1.4989	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
0	− 5.1504	− 0.7046	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
1	+ 0.1456	+ 0.1729	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
2	− 0.0895	+ 0.0009	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
3	− 0.0113	+ 0.0020	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
4	− 0.0021	+ 0.0003	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
5	− 0.0002	+ 0.0001	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
8−5	+ 0.073032	+ 0.462748	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
−4	+ 0.254889	+ 1.118259	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
−3	+ 0.696699	+ 2.313555	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
−2	+ 1.460949	+ 3.856621	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
−1	+ 2.139512	+ 4.604768	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
0	+ 1.655980	+ 2.871084	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
1	− 0.050326	− 0.300910	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
2	+ 0.032403	+ 0.051113	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
3	+ 0.003523	+ 0.002867	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
4	+ 0.000481	+ 0.000689	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
5	− 0.000018	+ 0.000154	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
9−6	+ 0.134032	− 0.123011	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
−5	+ 0.323465	− 0.339440	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
−4	+ 0.685544	− 0.824099	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
−3	+ 1.225756	− 1.696901	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
−2	+ 1.730192	− 2.786337	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
−1	+ 1.690033	− 3.234444	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
0	+ 0.776387	1.910446	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
1	− 0.184523	+ 0.252098	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
2	+ 0.020274	− 0.042087	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
3	− 0.000384	− 0.001129	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
4	+ 0.000240	− 0.000323	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
5	+ 0.000109	+ 0.000022	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
10−6	− 0.184043	− 0.032609	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
−5	− 0.450811	− 0.049228	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
−4	− 0.969895	− 0.040121	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
−3	− 1.762504	+ 0.047783	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
−2	− 2.533631	+ 0.247898	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
−1	− 2.529947	+ 0.443574	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
0	− 1.200805	+ 0.344079	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
1	+ 0.276643	− 0.021898	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
2	− 0.038781	+ 0.005340	. . . . .	. . . . .	. . . .	. . . . .	. . . . .

MARS AND THE EARTH—Continued.

Development of 24 times the inverse third power of the distance, etc.—Continued.

$i,\ j$	$24a''$	$24b''$	$24a'$	$24b'$	$i,\ i'$	$24a$	$24b$
10+3	+ 0.000972	+ 0.000201	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
4	− 0.000165	− 0.000065	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
5	− 0.000025	− 0.000056	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
11−7	+ 0.02926	+ 0.06708	. . . . .	. . . .	. . . .	. . . . .	. . . . .
−6	+ 0.08532	+ 0.16503	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
−5	+ 0.22006	+ 0.36400	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
−4	+ 0.49059	+ 0.70020	. . . . .	. . . .	. . . .	. . . . .	. . . . .
−3	+ 0.91084	+ 1.12669	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
−2	+ 1.31959	+ 1.41357	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
−1	+ 1.30754	+ 1.19726	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
0	+ 0.59625	+ 0.42722	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
1	− 0.15734	− 0.16330	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
2	+ 0.02311	+ 0.02453	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
3	− 0.00116	− 0.00135	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
4	− 0.00001	+ 0.00016	. . . . .	. . . . .	. . . .	. . . . .	. . . . .
5	− 0.00006	+ 0.00004	. . . . .	. . . . .	. . . .	. . . . .	. . . . .

THE EARTH AND VENUS.

Coefficients of Cosines.

$g'\ g_4$	$(x'+x_4)^2$	$(y'+y_4)^2$	z_4^2	$2(x'+x^4)(y'+y^4)$	$2(y'+y_4)z_4$	$2(x'+x_4)z_4$
0 0	+0.761716	+ .760708	+ .000916	− .000823	− .028096	− .012862
1	+ .025365	− .000771	− 18	+ .008596	+ .000310	− .000930
2	+ .136256	− .136439	+ 279	+ .445509	+ .003702	− .030734
3	+ .000936	− .000938	+ 1	+ .003060	+ 26	− 212
4	+ .000007	− .000006	......	+ 20	0	0
1−3	− .000011	− .000010	......	0	0	0
−2	− .002166	− .002165	......	+ .000003	+ .000087	+ .000119
−1	− .630490	− .630184	......	+ .001214	+ .025218	+ .034576
0	− .028982	− .008394	......	+ .007269	0	− .000712
1	− .630490	+ .630184	......	− .706530	− .025218	+ .034576
2	− .002166	+ .002165	......	− .002427	− .000087	+ .000119
3	− .000011	+ .000010	......	− .000012	0	0
2−2	− .000018	− .000018	......	0	+ 1	+ .000001
−1	− .005293	− .005291	......	+ .000011	+ .000212	+ .000290
0	+ .499687	− .499718	......	+ .000061	0	− .000006
1	− .005293	+ .005291	......	− .005933	− 212	+ .000290
2	− .000018	+ .000018	......	− .000020	0	+ .000001
3−1	− .000067	− .000067	......	0	+ .000003	+ .000004
−0	+ .008391	− .008391	......	+ .000001	0	0
1	− .000067	+ .000067	......	− .000074	− .000003	+ .000004
4 0	+ .000141	− .000141	......	0	0	0
			Coefficients of Sines.			
0 1	− .013230	− .004582	− .000017	+ .026132	+ .000634	− .001195
2	− .223220	+ .222286	+ 872	+ .272693	− .030690	− .003668
3	− .001534	+ .001528	6	+ .001874	− .000208	− .000026
4	− .000008	+ .000008	0	+ .000012	0	0
1−3	− .000006	− .000006	......	− .000001	+ .000119	0
−2	− .001216	− .001212	......	− .000001	+ .034573	− .000087
−1	− .353896	− .352634	......	− .000217	+ .000712	− .025220
0	0	+ .007268	......	− .037374	+ .034573	0
1	+ .353896	− .352634	......	−1.260673	+ .000119	+ .025220
2	+ .001216	− .001212	......	− .004331	+ .000119	+ .000087
3	+ .000006	− .000006	......	− .000021	0	0
2−2	− .000010	− .000010	......	0	0	0
−1	− .002971	− .002961	......	− .000002	+ .000290	− .000212
0	0	+ .000061	......	+ .999405	− .000006	0
1	+ .002971	− .002961	......	− .010584	+ .000290	+ .000212
2	+ .000010	− .000010	......	− .000036	0	0
3−1	− .000037	− .000037	......	0	+ .000004	− .000003
0	0	0	......	+ .016783	0	0
1	+ .000037	− .000037	......	− .000134	+ .000004	+ .000003
4 0	0	0	......	+ .000282	0	0

The accurate determination with all necessary numerical precision of the coefficients K_1, K_2, etc., of the preceding theory, forms one of the most difficult parts of the present problem. The analytic developments which the author has generally used in the planetary theories seem to be unsatisfactory here, owing to the great number of terms to be included and the slow convergence of the series to be used. He therefore determined to make use of the CAUCHY-HANSEN method, as developed in the preceding section. But in practically applying this method, an unforeseen difficulty occurred. In forming the necessary products of the terms of ρ^5, by $(x+x')^2$, etc., it was found that the final coefficients were much smaller in absolute value than the separate value of the terms whose sum made them up. For example, in the case of HANSEN'S great inequality, the addition of the several numbers which gave the principal coefficients for the argument $18n_4-16n'$ were as follows:

	$\frac{(x_4+x')^2}{\rho^5}$		$\frac{(y_4+y')^2}{\rho^5}$	
	cos	sin	cos	sin
Positive terms .	+63.0215	+46.6846	+62.9561	+46.6238
Negative terms	−63.1753	−46.6273	−63.0257	−46.5312
Difference . . .	− 0.1538	+ 0.0573	− 0.0696	+ 0.0926

We shall hereafter see that the degree of precision thus obtained is not entirely satisfactory. Where each quantity depends on the result of so great a number of operations and additions as is involved in the numerical processes which have to be gone through with, entire accuracy can be secured only by carrying the individual computations to a higher degree of precision than is expected in the final result. It must therefore be regarded as an open question whether the developments in powers of the inclination, as used by DELAUNAY, or the preceding numerical method of HANSEN, is the best. Notwithstanding the difficulties the author inclines to prefer the latter, on the ground that the numerical work may be done by ordinary computers, and believes that, had he been aware of the difficulty in question before the computations were commenced, he could have had them so conducted as to lead to undoubted results.

As the degree of precision obtained is practically sufficient for the present needs of the lunar theory, he has not deemed it necessary to go through so complete a revision of the numerical work as would be necessary to make the expressions reliable beyond the fourth decimal figure. He believes that the coefficients given in the preceding section for inverse powers of the distances of Venus and Mars from the Earth are correct within a very few units in the fourth place of decimals.

It is to be remarked in this connection that the developments of $b_n^{(i)} \sin i\,Q$, and $b_n^{(i)} \cos i\,Q$, when the numerical values of these quantities are computed for as many as 24 equidistant intervals admit of a very certain numerical control, in that the sum of the values of any of these functions corresponding to even multiples of 15 degrees, should be nearly equal to the corresponding sum of the odd multiples, the equality probably holding true up to at least the fourth place of decimals. This amounts to the same thing as saying that in the developments the coefficients of the sines and cosines of multiples

of g', higher than the 11th, should vanish. From the manner in which these coefficients diminish it would seem that this should be true to at least the fourth place of decimals. As a matter of fact, the actual delopments give coefficients for the cosine of $12g'$, generally amounting to two or three units of the fourth place. This limit, therefore, seems to be that of the precision obtained in the final results.

Of one substantial result there can be no doubt. The developments in question are carried far enough to detect any inequality of long period which can possibly arise, and to assign a limit to the probable order of magnitude of its amplitude. In the present state of the lunar theory, when observations of the Moon seem to show inequalities still unaccounted for, this result may be regarded as a point gained.

§ 19.

INEQUALITIES INDEPENDENT OF R.

In the preceding theory the inequalities of the lunar elements determined by the three equations (94) reduced to numbers in (96)

$$\begin{aligned} \delta\, n &= -.0096 \quad \delta n' + .003\,04 \quad \delta . e'^2 \\ \delta\, e^2 &= -.000\,58\ \delta n' - .000\,09 \quad \delta . e'^2 \\ \delta . \gamma^2 &= \qquad\qquad\qquad -.000\,011\ \delta . e'^2 \end{aligned}$$

are not confined to the secular variation. We must, in fact, substitute for $\delta n'$ and $\delta . e'^2$ the entire inequalities of these elements of the Earth's orbit produced by the action of the planets. As this operation is quite independent of determining the other inequalities, I have brought the results together in a single section. The computation of $\delta n'$ and $\delta . e'^2$ is so simple that I do not deem it necessary to enter into details of the computation, and shall therefore only give the results.

Putting λ_4, λ', the mean longitudes of Venus and the Earth, counted from the ascending node of the orbit of Venus, I have found the following results:

From the action of Venus.

$$\begin{aligned} \delta n' = &+1.''52 \quad \cos(\lambda' - \lambda_4) \\ &-0.''83 \quad \cos(2\lambda' - 2\lambda_4) \\ &-0.''51 \quad \cos(3\lambda' - 3\lambda_4) \\ &+0.''06 \quad \cos(2\lambda' - \lambda_4) - 0.''01 \quad \sin(2\lambda' - \lambda_4) \\ &-0.''29 \quad \cos(3\lambda' - 2\lambda_4) - 0.''07 \quad \sin(3\lambda' - 2\lambda_4) \\ &+0.''09 \quad \cos(5\lambda' - 3\lambda_4) - 0.''01 \quad \sin(5\lambda' - 3\lambda_4) \\ &-0.''0045 \cos(13\lambda' - 8\lambda_4) - 0.''0029 \sin(13\lambda' - 8\lambda_4) \end{aligned} \tag{146}$$

$$
\begin{aligned}
\delta e' = &+0.''88 \cos \lambda_4 + 0''.41 \sin \lambda_4 \\
&+0.''80 \cos \lambda' + 0''.37 \sin \lambda' \\
&+1.''52 \cos (\ 2\lambda' - \lambda_4) + 0''.71 \sin (\ 2\lambda' - \lambda_4) \\
&-3.''66 \cos (\ 3\lambda' - 2\lambda_4) - 1''.71 \sin (\ 3\lambda' - 2\lambda_4) \\
&+0.''48 \cos (\ 5\lambda' - 3\lambda_4) + 0''.37 \sin (\ 5\lambda' - 3\lambda_4) \\
&-0.''010 \cos (13\lambda' - 8\lambda_4) - 0''.019 \sin (13\lambda' - 8\lambda_4)
\end{aligned}
$$

From the action of Mars.

$$
\begin{aligned}
\delta v' = &-0.''04 \cos (\ \lambda_4 - \lambda') \\
&-0.''08 \cos (2\lambda_4 - 2\lambda') \\
&+0.''032 \cos (2\lambda_4 - \lambda' - \varpi') \\
&-0.''069 \cos (2\lambda_4 - \lambda' - \varpi_4) \\
\delta e' = &-0.''95 \cos (2\lambda_4 - \lambda' - \varpi')
\end{aligned}
$$

From the action of Jupiter.

$$
\begin{aligned}
\delta v' = &+0.''24 \cos (\ \lambda_4 - \lambda') \\
&-2.''33 \cos (2\lambda_4 - 2\lambda') \\
&+0.''42 \cos (3\lambda_4 - 2\lambda' - \varpi_4) \\
\delta e' = &-0.73 \cos (\ \lambda' - \varpi') \\
&-3.09 \cos (\ \lambda_4 - \varpi_4) \\
&+3.88 \cos (2\lambda_4 - \lambda' - \varpi')
\end{aligned}
$$

The preceding formulæ then give

Action of Venus on the Moon.

$$
\begin{aligned}
\delta \varepsilon = &+0.''46 \sin (\ \lambda' - \lambda_4) \\
&-0.''13 \sin (\ 2\lambda' - 2\lambda_4) \\
&-0.''05 \sin (\ 3\lambda' - 3\lambda_4) \\
&+0.''22 \sin (\ 3\lambda' - 2\lambda_4) \quad -0.''05 \cos (3\lambda' - 2\lambda_4) \\
&+0.''14 \sin (\ 5\lambda' - 3\lambda_4) \quad +0.''02 \cos (5\lambda' - 3\lambda_4) \\
&+0.''03 \sin (\ 2\lambda' - \lambda_4)
\end{aligned}
\qquad (147)
$$

$$+0.''20 \sin(13\lambda' - 8\lambda_4) \qquad -0.''12 \cos(13\lambda' - 8\lambda_4)$$

$$\delta\pi = -1.''12 \sin(\lambda' - \lambda_4)$$
$$+0.''32 \sin(2\lambda' - 2\lambda_4)$$
$$-0.''07 \sin(2\lambda' - \lambda_4)$$
$$+0.''12 \sin(3\lambda' - 3\lambda_4)$$
$$-0.''54 \sin(3\lambda' - 2\lambda_4) \qquad +0.''12 \cos(3\lambda' - 2\lambda_4)$$
$$-0.''35 \sin(5\lambda' - 3\lambda_4) \qquad -0.''05 \cos(5\lambda' - 3\lambda_4)$$
$$-0.''50 \sin(13\lambda' - 8\lambda_4) \qquad +0.''30 \cos(13\lambda' - 8\lambda_4)$$
$$\delta\theta = +0.''37 \sin(\lambda' - \lambda_4)$$

Action of Mars on the Moon.

$$\begin{aligned}
\delta\varepsilon &= +0.''02 \sin(\lambda_4 - \lambda') \\
&\quad +0.''13 \sin(2\lambda_4 - \lambda' - \varpi') \\
&\quad -0.''21 \sin(2\lambda_4 - \lambda' - \varpi_4) \\
\delta\pi &= -0.''04 \sin(\lambda_4 - \lambda') \\
&\quad -0.''31 \sin(2\lambda_4 - \lambda' - \varpi') \\
&\quad +0.''51 \sin(2\lambda_4 - \lambda' - \varpi_4)
\end{aligned} \qquad (148)$$

Action of Jupiter on the Moon.

$$\begin{aligned}
\delta\varepsilon &= -0.''05 \sin(\lambda_4 - \lambda') \\
&\quad +0.''25 \sin(2\lambda_4 - 2\lambda') \\
&\quad +0.''08 \sin(\lambda_4 - \varpi') \\
&\quad -0.''05 \sin(3\lambda_4 - \lambda' - \varpi_4) \\
\delta\pi &= -0.''60 \sin(2\lambda_4 - 2\lambda')
\end{aligned} \qquad (149)$$

If, in these expressions we consider that part of $\delta\varepsilon$ which arises from δv, we shall see that it is necessarily less than the corresponding term in the mean longitude of the Sun. For instance, the term

$$a_e \cos \mathrm{N}$$

in $\delta v'$ leads in $\delta\varepsilon'$ or in $\int n'dt$ to the term

$$-\frac{3}{2}\frac{n'}{b}\,a_e \sin \mathrm{N},$$

while the term produced in $\delta\varepsilon$ is

$$.0144\,\frac{n}{b}\,a_e \sin \mathrm{N},$$

which is about one-eighth that produced in $\delta\varepsilon'$. Hence, if, in our theory, we have omitted sensible terms from this source, terms eight times as great must be omitted in the theory of the Sun. In this proposition it is assumed that the perturbations of the Earth's eccentricity are so minute that in the expression

$$.000\,155\,\nu\,e_e$$

no omitted term can possibly become sensible.

§ 20.

PRELIMINARY DATA FOR INEQUALITIES ARISING THROUGH R_2.

The following values of the masses of the planets were adopted in the computations:

Mercury,	$\frac{1}{5\,000\,000}$
Venus,	$\frac{1}{420\,000}$
Earth,	$\frac{1}{326\,800}$
Earth + Moon,	$\frac{1}{322\,800}$
Mars,	$\frac{1}{3\,000\,000}$
Jupiter,	$\frac{1}{1047.8}$

The expressions for the action of the planets on the Moon all contain the factor

$$\mu^4\,\alpha^3 = \frac{m_4}{m_2+m_3}\,\frac{a^3}{a'^3}$$

where a is the mean distance of the Moon from the Earth, and a' the mean distance of the Earth from the Sun. Of this expression the factor

$$\frac{1}{m_2+m_3}\,\frac{a^3}{a'^3}$$

admits of being found with greater precision than we can find its component parts, because the quantity $\frac{m_2 a'^3}{a^3}$ is derivable from the length of the seconds pendulum, combined with the dimensions of the Earth. These data give the equations

$$\log m_2 a'^3 = 7\ 58838$$
$$\log a = 1.78044$$

taking for $m_3 : m_2$ the value $1 : 81.4$ we then have

$$\log \frac{m_1}{m_2 + m_3} \frac{a_3}{a'^3} = 7.74763 - 10$$

a purely numerical quantity, independent of the adopted units of mass and length.

Since we represent by m_4 the ratio of the mass of the planet to that of the Sun we have for the value of the common factor sought

$$\mu_4 \frac{a^3}{a'^3} = [7.74763 - 10]\, m_4$$

Referring to the equations (73) and (74) it will be seen that the magnitude of the coefficient of any inequality depends very largely on the value of ν, the ratio of the period of an argument to that of revolution of the Moon. The magnitude of the coefficients $k_1, k_2 \quad . \quad . \quad . \quad k_6$ which enter into K is also to be considered. To facilitate the search for inequalities which may become sensible in consequence of a large value of ν the following table of motions of the lunar arguments N, and of the principal term in K_1 and K_5, has been prepared:

Daily motions of the lunar arguments for all terms to those of the third order inclusive, with order of magnitude of the corresponding coefficients.

Argument.	Daily motion.	Principal term of $\varkappa_1$	log $\varkappa_1$
$l,\ \pi,\ \theta$	$''$		
0, 0, 2	−381	$+\gamma^2$	−2.7
0, 2, 0	802	$+\frac{5}{4}e^2$	−2.4
1, 0, 0	47435	$+\frac{45}{32}\alpha m$	−3.6
1, −3, 0	46232	$-\frac{7}{48}e^3$	−4.6
1, −1, 0	47034	$-e$	−1.3
1, −1, −2	47416	$-e\gamma^2$	−4.0
1, −1, 2	46652	$+\frac{11}{4}e\gamma^2$	−3.6
1, 1, 0	47836	$-\frac{3}{2}e$	−1.1
1, 1, −2	48217	$-\frac{1}{2}e\gamma^2$	−4.3
2, 0, 2	94488	$\frac{3}{4}\gamma^2 m$	−3.9
2, 0, 0	94870	$+\frac{1}{2}$	−0.3
2, 0, −2	95251	$+\gamma^2$	−2.7
2, −2, 0	94068	$-\frac{1}{4}e^2$	−3.1
2, 2, 0	95672	$-\frac{15}{16}e^2 m$	−3.7
3, 0, 0	142305	$-\frac{5}{4}\alpha e'$	−4.3
3, −1, 0	141904	$-\frac{1}{2}e$	−1.6
3, −1, −2	142285	$+e\gamma^2$	−4.0
3, 1, 0	142706	$+\frac{15}{16}em$	−2.4

Daily motions of the lunar arguments, etc.—Continued.

Argument.	Daily motion.	Principal term of $\varkappa_1$	log $\varkappa_1$
$l,\ \pi,\ \theta$	$''$		
3, 1, −2	143087	$-\frac{5}{4} e\gamma^2$	−3.9
3, −3, 0	141102	$-\frac{1}{8} e^3$	−4.7
4, 0, 0	189740	$+\frac{3}{16} m^2$	−3.0
4, −2, 0	188938	$+\frac{3}{8} e^2$	−4.2

Argument.	Daily motion	Principal term of 2 $\varkappa_5$	log 2 $\varkappa_5$
0, 0, 1	−191	-2γ	−1.1
0, 2, −1	993	$-\frac{15}{4} e\gamma^2$	−3.3
1, −1, −1	47225	$-2e\gamma$	−2.3
1, −1, 1	46843	$+2e\gamma$	−2.3
1, 1, −1	48027	$+6e\gamma$	−1.8
1, 1, 1	47645	$-\frac{3}{2} e\gamma m$	−3.6
2, −2, −1	94259	$-\frac{1}{2} e^2\gamma$	−4.2
2, −2, 1	93877	$+\frac{7}{4} e^2\gamma$	−3.6
2, 0, −3	95442	$+2\gamma^3$	−3.7
2, 0, 1	94680	$-\frac{3}{4}\gamma m$	−2.6
2, 0, −1	95061	-2γ	−1.1

Daily motions of the lunar arguments, etc.—Continued.

Argment.	Daily motion.	Principal term of $2\,\varkappa_5$	log $2\,\varkappa_5$
$l,\ \ \pi,\ \ \theta,$	$''$		
$3, -1, -1$	142095	$-2\,e\,\gamma$	-2.3
$3, -1,\ \ 1$	141713	$-\frac{3}{4}\,e\,\gamma$	-2.7
$3,\ \ 1, -1$	142896	$-\frac{15}{4}\,e\,\gamma\,m$	-3.2
$4, -2, -1$	189129	$-2\,e^2\,\gamma$	-3.6
$4,\ \ 0, -1$	189930	$-\frac{3}{4}\,\gamma\,m^2$	-3.7

To discover inequalities of long period it is only necessary to compare the motions of this table with the motions of all possible arguments of the form $i_4\,g_4 - i'\,g'$.

§ 21.

ACTION OF MERCURY.

The adopted mass of Mercury, $\frac{1}{5\,000\,000}$, gives from the numbers of § 20

$$\mu_4\,a^3 = 0''.000\,231$$

If we substitute this coefficient in (110) and retain only the largest terms in each case, we shall have

$$\delta v = +0''.000\,233\;i\;\;\nu\;\{\mathrm{K}\sin(\mathrm{N}\pm\mathrm{U}) + \mathrm{K}'\cos(\mathrm{N}\pm\mathrm{U})\}$$

$$e\,\delta e = -0\;\;.000\,122\;i'\;\nu\;\{\mathrm{K}\sin(\mathrm{N}\pm\mathrm{U}) + \mathrm{K}'\cos(\mathrm{N}\pm\mathrm{U})\}$$

$$\gamma\,\delta\gamma = -0\;\;.000\,029\;i''\nu\;\{\mathrm{K}\sin(\mathrm{N}\pm\mathrm{U}) + \mathrm{K}'\cos(\mathrm{N}\pm\mathrm{U})\}$$

To facilitate an estimate of the order of magnitude of the coefficients of any of these inequalities we recall the values (107) of K and K′

$$\pm\mathrm{K} = \varkappa_1\,\mathrm{K}'_1 + \varkappa_2\,\mathrm{K}'_2 + \;.\;.\;.$$

$$\mathrm{K}' = \varkappa_1\,\mathrm{K}_1 + \varkappa_2\,\mathrm{K}_2 + \;.\;.\;.$$

The largest values of K_1, K_2 . . . K'_1, K'_2 . . . are of the order of magnitude unity for some of the arguments of lowest index, and decrease indefinitely with larger values of the indices.

By inspecting the values of $\varkappa_1$, $\varkappa_2$, etc., on p. 217, etc., it will be seen that the value unity is approximated to only in the terms corresponding to the following system of indices

$$i\ =0;\ 2$$

$$i'\ =0;\ 0$$

$$i''=0;\ 0$$

In the case of the first set of indices the values of δv, δe, and $\delta\gamma$ vanish entirely, and in the case of the second the largest terms can only be of the order of magnitude,

$$0''.0005\ \nu\,\mathrm{K}_i$$

and since K_i never much exceeds unity, this term can become sensible only for a very large value of ν. Now referring to (112) and (113) it will be seen that if ν is very large, the terms in $\varepsilon_c, \varepsilon_s$, etc., multiplied by ν^2 will, in general, be much larger than those multiplied by ν, and that in nearly every case the perturbations of ε will be larger than those of π or θ. If, then, there is any sensible inequality of long period arising from the action of Mercury it is to be sought in the terms

$$\varepsilon_c=(0''.000\,35\,i-0''.000\,005\,i'+0''.000\,001\,i'')\,\nu^2\,\mathrm{K}$$

$$\varepsilon_s=(0''.000\,35\,i-0\ \ .000\,005\,i'+0\ \ .000\,001\,i'')\,\nu^2\,\mathrm{K}'$$

In order that these coefficients may amount to $0''.1$ we must have, for $i=1$

when	$\mathrm{K}=1,$	$\nu>15$
"	$\mathrm{K}=.01,$	$\nu>150$
"	$\mathrm{K}=.0001,$	$\nu>1500$
etc.		etc.

The value of ν is inversely proportional to the coefficient of the time in the angle $\mathrm{N}\pm\mathrm{U}$. These angles are of the form

$$\mathrm{N}=i\,\varepsilon+i'\,\pi+i''\,\theta+i'''\,g'$$

$$\mathrm{U}=j'\,g'+j_4\,g_4$$

The angle $\mathrm{N}\pm\mathrm{U}$ will therefore comprise five separate variables, the angle of g' being common to both N and U. It follows that in seeking for large values of ν it is only necessary to consider those values of N in which $i'''=0$, and to find, for each of such values, whether there are any integral values of j' and j_4 which will make the coefficient of the time in $\mathrm{N}\pm\mathrm{U}$ very small. On the next page is found a complete list of the combinations of the indices i, i', and i'', which we have found to appear in the development of x^2, xy, etc., to terms of the third order, with the daily motion of the argument corresponding to each combination.

Opposite each set of indices is given the order of magnitude of the largest corre-

sponding value of $\varkappa$, which may be regarded as nearly a superior limit of any possible corresponding value of K or K′.

We have next to consider the probable order of magnitude of the coefficients K_1, K_2, etc I find, by induction, that these coefficients are of the same order of magnitude with those which we meet in the development of $\frac{1}{\rho_3}$, a result which might have been deduced *a priori*. Comparing the several multiples of the mean motions we find that within these limits those in the following table are all that nearly coincide with any one of the preceding values of $\frac{n}{\nu}$. With each combination of arguments is given a computation of the order of magnitude of the corresponding term in $\delta\varepsilon$, or, rather, a computation showing a value which the coefficient of the term is not likely to exceed. The third column shows the daily motion of the argument corresponding to the indices in the first two columns, or the value of

$$i\,n+i'\pi_1+i''\,\theta_1\pm(j'\,n'+j_4 n_4)$$

We then have, in the fourth column, the approximate logarithm of ν^2.

In the fifth is log $\varkappa_i$, concluded from the order of magnitude of the largest value of $\varkappa$ in the preceding table.

In the sixth is the largest probable value of log $\varkappa_i$, concluded by induction from the magnitude of the terms in the development of $\frac{1}{\rho^3}$.

In the seventh and eighth are the resulting largest probable value of ε_c or ε_s, concluded from the formulæ (113)

		Daily motion.	log ν^2	log $\varkappa_i$	log K_i	log ε_c	ε_c
i i' i''	j' j_4	″				+10	″
1 1 0	+ 1 + 3	90	5.5	−1.1	−2.	9.0	0.1
1 1 0	− 3 + 4	449	4.1	−1.1	−1.	8.6	0.04
1 1 −2	− 3 + 4	68	5.7	−4.3	−1.	7.0	0.001
2 −2 0	−15 + 10	33	6.4	−3 1	−5.	5.0	0.000
3 −1 0	−14 + 13	58	5.9	−1.6	−4.	6.9	0.001
3 0 0	−18 + 14	81	5.6	−4.3	−5.	2.9	0.000
1 1 −1	− 3 + 4	258	4.6	−1.8	−1.	8.4	0.02
1 1 1	+ 1 + 3	100	5.4	−3.6	−2.	6.4	0.000

The first three terms are the only ones in which there is any reasonable hope of finding a sensible coefficient, and they may therefore merit a closer examination, although their actual coefficients are probably smaller than those here given. The

following are the actual values of the highest terms in K, and the actual value of the coefficient in the development of $\frac{1}{\rho^3}$ for the combinations of arguments which form the highest terms in the final coefficient.

1. For $N = \epsilon + \pi$ For $U = g' + 3g_4$

$$\varkappa_1 = -\frac{3}{2}e = -.082 \qquad K_i = .0006 \pm$$

$$\varkappa_2 = \frac{3}{2}e = +.082$$

$$\varkappa_4 = -\frac{3}{2}e = -.082$$

$\varkappa_3$, $\varkappa_5$, and $\varkappa_6 = 0$

2. For $N = \epsilon + \pi - 2g'$ For $U = -g' + 3g_4$

$$\varkappa_1 = \varkappa_2 = -\frac{15}{8}em = -.0077 \qquad K_i = .016 \pm$$

$\varkappa_3$ to $\varkappa_6 = 0$

The coefficient corresponding to this value of K_i would be of the order of magnitude 0.″03. It seems to me without the limits of reasonable probability that the actual values of K_1 and K_2 should exceed the value here assigned by more than two or three times, and therefore very unlikely that the coefficient should amount to 0.″1. It is equally probable that the values of the following terms are less than those assigned. I conclude that the motion of the Moon is subject to no sensible inequality of long period arising from the direct action of Mercury.

§ 22.

ACTION OF VENUS.

With the adopted mass 1 ÷ 420 000 we find

$$\mu_4 a^3 = 0.''002\,746$$

By the substitution of this value the secular terms (114) become

$$\delta\epsilon = 0.''002\,746 \left\{ -2.023\ K'_0 - 1.011\ \frac{dK'_0}{dv} + 0.0005\ \frac{dK'_0}{e\,de} \right\} nt$$

$$\delta\pi = 0.''002\,746 \left\{ +0.0293\,K'_0 + 0.0147\,\frac{dK'_0}{dv} + 0.526\ \frac{dK'_0}{e\,de} \right\} nt$$

$$\delta\theta = 0.''002\,746 \left\{ -0.0074\,K'_0 - 0.0037\,\frac{dK'_0}{dv} + 0.125\ \frac{dK'_0}{\gamma\,d\gamma} \right\} nt$$

The largest values of K'_0 and its derivatives will be those which arise from the combination of the zero terms in K_1, K_2, etc., with the values of $\varkappa_1$, $\varkappa_2$, etc., corresponding to $N=0$, and these terms must be doubled, because the angles $N+U$ and $N-U$ will then both be zero. From the developments given on pp. 252–257 and 262, we find the following numerical values of K_1, K_2, etc., or the constant terms in the developments (103). Along side of them we place the corresponding values of $\varkappa_1$, $\varkappa_2$, etc., and their derivatives from the first terms in the analytical tables, pages 217-224.

When $U=0$

$$32\ K_1=+38.98 \qquad \varkappa_1=+0.49947;\ \frac{d\varkappa_1}{dw}=+.0021;$$

$$32\ K_2=+38.06 \qquad \varkappa_2=\varkappa_1 \qquad \frac{d\varkappa_1}{ede}=+1.52;\ \frac{d\varkappa_1}{\gamma d\gamma}=-1.998$$

$$32\ K_3=-77.10 \qquad \varkappa_3=+0\ 00403$$

$$32\ K_4=+57.51 \qquad \varkappa_4=0$$

$$32\ K_5=+19.14 \qquad \varkappa_5=0$$

$$32\ K_6=+\ 8\ 76 \qquad \varkappa_6=0$$

An important remark is to be made here. It will be seen that in these and many other terms we have

$$\varkappa_1=\varkappa_2;\quad \varkappa_4=\varkappa_5=\varkappa_6=0$$

The computation of these terms admits of being greatly simplified. If we take the sum of the first three equations (103) noticing that

$$(x'+x_4)^2+(y'+y_4)^2+(z'+z_4)^2=\rho^2$$

we find

$$K_1+K_2+K_3=0$$
$$K'_1+K'_2+K'_3=0$$

two equations which form a valuable check on the accuracy of the computation of K_1, K_2, and K_3. Now, in the case supposed, the value of K and K′ in the equations (107) becomes

$$\pm K=(K'_1+K'_2)\,\varkappa_1+K'_3\,\varkappa_3$$
$$K'=(K_1+K_2)\,\varkappa_1+K_3\,\varkappa_3$$

Substituting the value of $K'_1+K'_2$ and of K_1+K_2, given by the preceding equations, these quantities become

$$\begin{aligned} K&=\pm K'_3\,(\varkappa_3-\varkappa_1)\\ K'&=\ \ K_3\,(\varkappa_3-\varkappa_1)\end{aligned} \qquad (150)$$

Hence the only values of K_i and K'_i we need in this case are those corresponding to $i=3$, and these are the ones most easily computed, owing to the small number of terms in z_4.

In another extended class of cases we have

$$\varkappa_1 = -\varkappa_2 = \pm \varkappa_4; \quad \varkappa_2 = \varkappa_5 = \varkappa_6 = 0$$

Then (107) gives

$$\begin{aligned} K &= \pm(K'_1 - K'_2 \pm K_4)\,\varkappa_1 \\ K' &= \quad (K_1 - K_2 \pm K'_4)\,\varkappa_1 \end{aligned} \tag{151}$$

In (150) and (151) the upper sign of the double ones without the parentheses corresponds to the combination $N+U$, and the lower to $N-U$.

The terms now under consideration belong to the former class. Forming the values of K' and its derivatives by (150) and substituting in the preceding expressions we find, doubling the magnitude of the term,

$$\tfrac{1}{2}\,\delta\varepsilon = -0.''00663nt$$

$$\tfrac{1}{2}\,\delta\pi = +0.''00539nt$$

$$\tfrac{1}{2}\,\delta\theta = -0.''00165nt$$

Taking the Julian year as the unit of time we shall have $n=84.00$, which will give

$$\delta\varepsilon = -1.''113t$$

$$\delta\pi = +0.''905t$$

$$\delta\theta = -0.''277t$$

Now, taking up the periodic terms, the equations (110) become, by substituting the values of $\mu_4 a^3$,

$$\delta v = \{0.''002\,78i - 0.00004i'\} \quad \{\nu\,K \sin(N \pm U) + \nu\,K' \cos(N \pm U)\}$$

$$e\,\delta e = -0.''001\,444i' \quad \{\nu\,K \sin(N \pm U) + \nu\,K' \cos(N \pm U)\}$$

$$\gamma\,\delta\gamma = -0.''000\,343i'' \quad \{\nu\,K \sin(N \pm U) + \nu\,K' \cos(N \pm U)\}$$

while the coefficients (113) become

$$\varepsilon_c = (\;0.''004\,166i - 0.''000\,060i' + 0.000\,014i'')\,\nu^2 K$$

$$+0.''005\,56\,\nu\,K + 0.''002\,78\,\nu\frac{dK}{dv}$$

$$\varepsilon_s = (-0.''004\,166i + 0.''000\,060i' - 0.''000\,014i'')\,\nu^2 K'$$

$$-0.''005\,56\,\nu\,K' - 0.''007\,28\,\nu\,\frac{dK'}{d\varpi}$$

$$\pi_c = (-0.''000\,060i - 0.''000\,027i' - 0.''000\,034i'')\,\nu^2 K$$

$$-0.''000\,082\,\nu\,K - 0.''000\,040\,\nu\,\frac{dK}{d\varpi} - 0.''001\,44\,\nu\,\frac{dK}{e\,de}$$

$$\pi_s = (\;0.''000\,060i + 0.''000\,027i' + 0.''000\,034i'')\,\nu^2 K'$$

$$+0.''000\,082\,\nu\,K' + 0.''000\,040\,\nu\,\frac{dK'}{d\varpi} + 0.''001\,44\,\nu\,\frac{dK'}{e\,de}$$

$$\theta_c = (\;0.''000\,016i - 0.''000\,034i' + 0.''000\,005i'')\,\nu^2 K$$

$$+0.''000\,021\,\nu\,K + 0.''000\,010\,\nu\,\frac{dK}{d\varpi} - 0.''000\,34\,\nu\,\frac{dK}{\gamma\,d\gamma}$$

$$\theta_s = (-0.''000\,016i + 0.''000\,034i' - 0.''000\,005i'')\,\nu^2 K'$$

$$-0.''000\,021\,\nu\,K - 0.''000\,010\,\nu\,\frac{dK'}{d\varpi} + 0.''000\,34\,\nu\,\frac{dK'}{\gamma\,d\gamma}$$

The magnitude of these coefficients will depend mainly upon that of the factors ν and K, or, since the latter is made up of the two factors K_i and $\varkappa_i$, there will be three factors in all, the order of magnitude of which is to be considered. Since the largest value which K can attain is only two or three units, we see at a glance that no term can become sensible unless ν is a large number. Now, there are two classes of terms in which ν may be large, in other words terms, the period of which may be many times that of the Moon, namely:

(1) Those in which i, i', and i'' all vanish, so that the term contains only the mean longitude of the Earth and planet. But in this case the coefficient of $\nu^2 K$ will also vanish, so that the perturbation will contain only the first power of ν.

(2) Those in which one or more of the indices i, i', and i'' occur, so that the term contains the square of ν. In all these terms K is necessarily extremely minute, for if the term contains i, the coefficient of the Moon's mean longitude, the coefficient of t in the angle $N \pm U$, or

$$il + i'\pi + i''\theta \pm U,$$

can become very minute only for very high multiples of the mean longitudes of the planet and Earth in U, owing to the magnitude of the mean motion of the Moon compared with that of the Earth and planet. If, on the other hand, i does not occur in the term, the condition of long period may be fulfilled by the lower multiples of U, when K_1, K_2, etc., will be considerable. But in this case two very minute factors will enter in, namely, $\varkappa_1$, $\varkappa_2$, etc., which are very minute when $i=0$, and the coefficient will be only $0.''000060i'$ or $0.''000014i''$.

We begin by giving all the values of K_1, K_2, etc., which have been actually com-

puted, each of them being multiplied by 32. With them are given for comparison, at the bottom of the column, the corresponding coefficient in the development of $\frac{32}{\rho^3}$.

U	$-g'+g_4$		$-2g'+2g_4$		$-3g'+2g_4$	
i	$32\ K_i$	$32\ K'_i$	$32\ K_i$	$32\ K'_i$	$32\ K_i$	$32\ K'_i$
1	$+60.25$	-34.5	$+50.56$	-49.23	$+1.75$	-2.09
2	$+58.92$	-33.3	$+8.26$	-47.25	$+4.19$	-4.56
3	-118.98	$+66.7$	-58.96	$+96.37$	-6.21	$+6.44$
4	-1.89	$+0.76$	-1.57	$+1.32$	$+0.69$	$+1.53$
5	$+31.61$	-21.62	$+14.21$	-34.16	$+2.08$	-2.19
6	$+18.96$	-2.02	$+19.39$	$+7.80$	$+1.27$	-0.53
	$+183.06$	-102.61	$+90.99$	-148.73	$+9.66$	-10.00

In all these lower terms, that part of K_3 which results from the development of $\frac{3}{2}\frac{z_4^2}{\rho^5}$ is much smaller than that which results from the development of $\frac{1}{2}\frac{1}{\rho^3}$ so that we might have rejected the former without sensible error. Rejecting them in the term of which the argument is $-5g'+3g_4$ we shall have for this term

$$32\ K_3 = -0.95$$

$$32\ K'_3 \quad +1.03$$

In the term $-3g'+3g$ we shall have

$$32\ K_3 = -3.57$$

$$32\ K'_3 = +93.39$$

On the same supposition we should have for the terms $-13g'+8g_4$,

$$32\ K_3 = +.0021$$

$$32\ K'_3 = +.0016$$

and for $U = -11g'+8g_4$,

$$32\ K_3 = +0.127$$

$$32\ K'_3 = -0.003$$

but owing to the minuteness of these we can not be sure that they express more than the order of magnitude of K_3 and K'_3.

These terms comprise all those independent of i, i', and i'', to which correspond very large values of either K or ν. They are to be combined with the first four terms in

the development of x^2, y^2, etc. The values of $\varkappa_1$, $\varkappa'_2$, etc., corresponding to the zero term of this development have been already given. The next largest term is the third, corresponding to $N = 2g'$, for which

$$\varkappa_1 = -\frac{19}{16}m^2 - \frac{97}{24}m^3 + \frac{75}{16}e^2 m + \frac{3}{4}\gamma^2 m$$

and in numbers

$$\varkappa_1 = -\varkappa_2 = \varkappa_4 = -.007\,17$$

$$\frac{d\varkappa_1}{d\nu} = -\frac{d\varkappa_2}{d\nu} = \frac{d\varkappa_4}{d\nu} = +.0258$$

$$\frac{d\varkappa_1}{de} = -\frac{d\varkappa_2}{de} = \frac{d\varkappa_4}{de} = +.039$$

$$\frac{d\varkappa_1}{d\gamma} = -\frac{d\varkappa_2}{d\gamma} = \frac{d\varkappa_4}{d\gamma} = +.0050$$

This term can give rise to a sensible result only by being combined with a large term in which the coefficient of the time is nearly equal to $2n'$. The two most favorable ones are

$$U = -3g' + 3g_4 \text{ which gives } N - U = 5g' - 3g_4$$

$$U = -11g' + 8g_4 \quad . \quad . \quad . \quad . \quad N - U = 13g' - 8g_4$$

both of which are of long period.

Our formulæ now gives the following computation of all the terms of the class under consideration which can become sensible.

U	$-g'+g_4$	$-2g'+2g_4$	$-3g'+3g_4$	$-3g'+2g_4$	$-5g'+3g_4$	$-13g'+8g_4$
N	0	0	0	0	0	0
K_3	-3.72	-1.84	-0.11	-0.19	-0.030	$-.00065$
K'_3	$+2.08$	$+3.01$	$+2.92$	$+0.20$	$+0.032$	$+.00050$
K	∓ 1.030	∓ 1.491	∓ 1.447	∓ 0.099	∓ 0.0159	$\mp .00025$
K′	$+1.843$	$+0.912$	$+0.054$	$+0.094$	$+0.0149$	$-.00032$
ν	± 21.37	± 10.68	± 7.12	± 53.25	∓ 108.3	$\pm 3196.$
ε_c	$-0.''1226$	$-0.''0887$	$-0.''0574$	$-0.''0293$	$+0.''0096$	$-0.''0045$
ε_s	∓ 0.2190	∓ 0.0542	∓ 0.0021	∓ 0.0278	± 0.0090	∓ 0.0056
π_c	$+0.097$	$+0.071$	$+0.05$			
π_s	± 0.175	± 0.043	0 00			

Where there are double signs the upper corresponds to the angle $+U$ and the lower to $-U$, and both are to be combined into a single term. Combining them, the results are

$$\begin{aligned}
\delta l = & -0.''438 \sin(-g'+g_4) - 0.''245 \cos(-g'+g_4) \\
& -0\ .108 \sin(-2g'+2g_4) - 0\ .177 \cos(-2g'+2g_4) \\
& -0\ .004 \sin(-3g'+3g_4) - 0\ .115 \cos(-3g'+3g_4) \\
& -0\ .056 \sin(-3g'+2g_4) - 0\ .059 \cos(-3g'+2g_4) \\
& +0\ .018 \sin(-5g'+3g_4) + 0\ .019 \cos(-5g'+3g_4) \\
& -0\ .011 \sin(-13g'+8g_4) - 0\ .009 \cos(-13g'+8g_4) \\
\delta \pi = & +0\ .35\ \sin(-g'+g_4) + 0\ .19\ \cos(-g'+g_4)
\end{aligned}$$

The combinations of $N=2g'$ with $U=-3g'+3g_4$ and $U=-11g'+8g_4$ give rise to terms having the same argument. They have not, however, been computed, because we have not $\varkappa_1 = \varkappa_2$, and therefore require $\varkappa_1$ and $\varkappa_2$, and it is easy to assure one's self that they are only of the order of magnitude with those of the same argument just given. In fact $\varkappa_1$, $\varkappa_2$, etc., are 70 times smaller, while K_1, K_2, etc., can not be much more than 70 times greater, and the results are not to be doubled as in the other terms. They can not, therefore, amount to more than one or two hundredths of a second.

The perturbations of θ are much smaller than those of π, and may therefore be neglected entirely.

We have now to consider the terms into which the square of ν enters. By combining the values of K_1, K_2, etc., for $U=0$, which have been already given, with the values of $\varkappa_1$, $\varkappa_2$, etc., in which i is not zero, we shall find a term corresponding to each value of N (pp. 217–224), and proportional to $\varkappa_1$, $\varkappa_2$, etc., the largest of which will be that corresponding to $2\,\varepsilon$, but none of which will amount to one-hundredth of a second. But the corresponding terms, where $i=0$, will be of long period, and therefore, owing to the large values of ν, may become sensible. The largest of these will correspond to $N=2\,\theta$ and $N=2\,\pi$, both of which belong to that class in which

$$\varkappa_1 = -\varkappa_2 = \varkappa_4;\quad \varkappa_3 = \varkappa_5 = \varkappa_6 = 0.$$

Their computation is as follows:

N	$2\,\theta$	$2\,\pi$
U	0	0
$K_1 - K_2 + K_4$	+ 1.85	+ 1.85
$\varkappa_1$	+ .0020	+ .0038
K'	+ .0037	+ .0070
ν^2	15450	3498
$2\,\varepsilon_s$	$-0.''003$	$+0.''006$

The action of Venus, therefore, gives rise to the terms of long period

$$\delta\,\varepsilon = -0.''003 \sin (2\,\theta - 2\,\pi')$$
$$+0.''006 \sin (2\,\pi - 2\,\pi')$$

We have finally to consider the terms of long period in which neither N nor U are zero. To find these terms we take the table of daily motions of the argument corresponding to each term of x^2, y^2, etc., and compare with the daily motion of the several arguments composed of the mean anomalies of Venus and the Sun, and select those of which the sum or difference nearly vanish. All these combinations which, as it seems to me, can merit examination are given in the following table, which includes also a computation of the probable order of magnitude of the coefficient for the perturbation of mean longitude. The first five columns of this table show the combinations of the several indices i, i', i'', j, j', which make the mean motion of the argument

$$i\,l + i'\,\pi + i''\,\theta + j\,(g_4 - g') + j'\,g'$$

very small.

The sixth column gives the mean daily motion of the argument in seconds. The first line, for example, shows that twice the mean motion of the Moon's node, plus five times the mean motion of the Earth, minus three times that of Venus, is only 56″, and that the corresponding inequality is therefore one of very long period.

The seventh column gives the value of ν, which really represents the period of the argument in sidereal revolutions of the Moon. Multiplying ν by .075, we shall have the period of the inequality in years.

The three next columns give, as in the case of Mercury, the probable order of magnitude of the logarithms of ν^2, $\varkappa_i$, and K_i, the last being deduced by induction from the law of development of $\frac{1}{\rho^3}$.

The last two columns give log ε_c and the probable limiting order of magnitude of ε_c itself, hence concluded. These quantities may also be regarded as the probable values of ε_s.

It will be seen that several terms have been included, the components of which are not found in the preceding developments. This has been done to include every term which could possibly become sensible in the course of a few centuries. The magnitudes assigned to $\varkappa_i$ and K_i in these cases are hardly more than rough guesses, and it must be understood that precision is not aimed at in any case.

i	i'	i''	j	j'	ω	ν	$lg\ \nu^2$	$lg\ \varkappa_1$	$lg\ K_1$	$lg\ \varepsilon_c$ +10	ε_c
					$''$						$''$
0	0	2	−3	+2	+56	847	5.9	−2.7	−1.5	....	.00
0	2	0	−2	+1	−89	533	5.6	−2.4	−0.8	....	.00
1	−3	0	−24	+2	+61	780	5.8	−4.6	−3.0	7.8	.00
1	−3	0	−16	−3	+76	624	5.6	−4.6	−2.7	7.5	.00
1	−3	−2	−21	0	+4	12000	8.2	−6. (?)	−1.4	8.4	.02
1	−1	0	−26	+3	−28	1695	6.5	−1.3	−4.0	8.8	.07
1	−1	0	−18	−2	−13	3560	7.1	−1.3	−2.0	11.7	25.
1	−1	0	−10	−7	+1.6	29600	9.0	−1.3	−5.0	10.3	2.
1	−1	−2	−23	+1	−84	560	5.5	−4.0	−2.	7.1	.00
1	−1	2	−21	0	+43	1100	6.1	−3.6	−1.5	8.6	.04
1	1	0	−20	−1	−102	465	5.3	−1.1	−2.5	9.4	.2
1	0	0	−23	+1	−65	730	5.7	−3.6	−2.0	7.7	.00
1	0	2	−18	−2	+6	8000	7.8	−5. (?)	−2.7	7.7	.005
2	0	0	−38	−3	−116	409	5.2	−0.3	−5.0	7.8	.01
2	0	2	−41	−1	−59	802	5.8	−3.9	−4.5	5.3	.00
2	−2	0	−44	+1	−41	1150	6.1	−3.1	−5.0	5.9	.00
2	−2	2	−55	+8	+0.7	67000	9.6	−5. (?)	−7.0	5.5	.00
0	0	1	2	−1	+700	68	3.7	−1.1	−0.5	7.3	.00
0	0	1	−3	+2	+247	192	4.6	−1.1	−0.5	8.2	.02
0	2	−1	−2	+1	+102	465	5.3	−3.3	−0.5	7.5	.00
1	1	−1	−20	−1	+88	540	5.5	−1.8	−2.5	8 8	.06
1	3	−1	−22	0	+0.4	120000	10.2	−6. (?)	−1.5	10.3	2.
0	2	2	3	−2	18	2630	6.8	−5. (?)	−0.5	8.9	.08

The only coefficients which from their magnitude merit farther examination are those corresponding to terms 7, 8, 9, 11, 12, 22, and 23. Of these numbers 9 and 22 are of such immensely long period, several thousand years, that their effect could not possibly be sensible in all the observations hitherto made on the Moon. The first can hardly be accurately computed by any method hitherto employed in celestial mechanics, because the secular variations of the perihelia and nodes of the Earth and Venus will sensibly affect the arguments. In fact, the rigorous development does not contain simply the term $3g'-10g_4$, but $3l'-10l_4$ plus many combinations of angles varying in consequence of the secular variations of the planetary orbits. In the present theory these angles are supposed constant, the theory is therefore not rigorous for very long intervals. The further consideration of these two terms will therefore be omitted entirely.

An examination of 7 shows that its real value is smaller than that here assigned, so that it also may be omitted. We therefore begin with the exact computation of the eighth term, which is HANSEN's first inequality. An examination of the values of K_1, K_2, etc., shows that there are really six terms in N, each of which, being combined with some value of U, will make

$$N\pm U=\varepsilon-\pi+16g'-18g_4$$

These terms, together with the numerical values of K_1, K_2, etc., are investigated as follows:

We find that the analytical expressions for x^2, y^2, z^2, $y\,z$, $z\,x$, and $x\,y$ contain the following terms, which, combined with the proper multiples of the mean anomalies of the Earth and Venus, will give rise to terms having the argument $\varepsilon-\pi+16\,g'-18\,g_4$

$$\frac{x^2}{a^2}=\left\{-e+\frac{1}{8}e^3+2\,e\,\gamma^2+\frac{13}{12}e\,m^2\right\}\cos(l-\pi)$$
$$-\frac{21}{8}e\,e'\,m\cos(l-\pi-g')$$
$$+\frac{87}{16}e\,e'\,m\cos(l-\pi+g')$$
$$+\left\{-\frac{45}{16}e\,m-\frac{773}{64}e\,m^2\right\}\cos(l-\pi+2\,g')$$
$$+\frac{55}{32}e\,m^2\cos(l-\pi-2\,g')$$
$$-\frac{105}{16}e\,e'\,m\cos(l-\pi+3\,g')$$

$$\frac{y^2}{a^1}=\left\{-e+\frac{1}{8}e^3+2\,e\,\gamma^2+\frac{13}{12}e\,m^2\right\}\cos(l-\pi)$$
$$-\frac{21}{8}e\,e'\,m\cos(l-\pi-g')$$
$$-\frac{3}{16}e\,e'\,m\cos(l-\pi+g')$$
$$+\left\{\frac{45}{16}e\,m+\frac{773}{64}e\,m^2\right\}\cos(l-\pi+2\,g')$$
$$-\frac{55}{32}e\,m^2\cos(l-\pi-2\,g)$$
$$+\frac{105}{16}e\,e'\,m\,(l-\pi+3\,g')$$

$$\frac{x\,y}{a^2}=\left\{-\frac{45}{16}e\,m-\frac{773}{64}e\,m^2\right\}\sin(l-\pi+2\,g')$$
$$-\frac{55}{32}e\,m^2\sin(l-\pi-2\,g')$$
$$+\frac{45}{16}e\,e'\,m\sin(l-\pi+g)$$
$$-\frac{105}{16}e\,e'\,m\sin(l-\pi+3\,g)$$

$$z^2=-4\,e\,\gamma^2\cos(l-\pi)$$

Reducing to numbers, these values become

$$\frac{x^2}{a^2}=-.05433\cos(\varepsilon-\pi)$$
$$-.00018\cos(\varepsilon-\pi-g')$$
$$+.00037\cos(\varepsilon-\pi+g')$$
$$-.01525\cos(\varepsilon-\pi+2g')$$
$$+.00053\cos(\varepsilon-\pi-2g')$$
$$-.00045\cos(\varepsilon-\pi+3g')$$

$$\frac{y^2}{a^2}=-.05433\cos(\varepsilon-\pi)$$
$$-.00018\cos(\varepsilon-\pi-g')$$
$$-.00001\cos(\varepsilon-\pi+g')$$
$$+.01525\cos(\varepsilon-\pi+2g')$$
$$-.00053\cos(\varepsilon-\pi-2g')$$
$$+.00045\cos(\varepsilon-\pi+3g')$$

$$\frac{xy}{a^2}=-.01525\sin(\varepsilon-\pi+2g')$$
$$-.00053\sin(\varepsilon-\pi-2g')$$
$$+.00020\sin(\varepsilon-\pi+g')$$
$$-.00045\sin(\varepsilon-\pi+3g')$$

$$z^2=-.00044\cos(\varepsilon-\pi)$$

We thus have the following values of $\varkappa_i$ for the six arguments N:

N	$\varkappa_1$	$\varkappa_2$	$\varkappa_3$	$\varkappa_4$	U
$\varepsilon-\pi-2g'$	$+.00053$	$-.00053$	0	$-.00053$	$-18g'+18g_4$
$\varepsilon-\pi-g'$	$-.00018$	$-.00018$	0	0	$-17g'+18g_4$
$-\pi$	$-.05433$	$-.05433$	$-.00044$	0	$-16g'+18g_4$
$\varepsilon-\pi+g'$	$+.00037$	$-.00001$	0	$+.00019$	$-15g'+18g_4$
$\varepsilon-\pi+2g'$	$-.01525$	$+.01525$	0	$-.01525$	$-14g'+18g_4$
$\varepsilon-\pi+3g'$	$-.00045$	$+.00045$	0	$-.00045$	$-13g'+18g_4$

From the numerical developments already given we obtain the following values of K_1, K_2, etc., the last value of U being omitted owing to the necessary minuteness of the term to which it gives rise:

U	32 K_1	32 K_2	32 K_3	32 Δ	32 K_4
$-18g'+18g_4$	-0.8052	-0.3925	$+1.2021$	$+.0044$	$+0.2972$
$-17\ \ +18$	.....	......	-0.0144	......	......
$-16\ \ +18$	-0.1095	-0.0253	$+0.12068$	$-.0141$	......
$-15\ \ +18$	-0.0039	$+0.0008$	$+0.0027$	$-.0004$	-0.0201
$-14\ \ +18$	-0.011	$+0.006$	......	$-.005$	-0.013
U	32 K'_1	32 K'_2	32 K'_3	32 Δ	32 K'_4
$-18g'+18g_4$	-0.0067	-0.2821	$+0.2864$	$-.0024$	$+0.3878$
$-17\ \ +18$	......	......	$+0.0055$	......	......
$-16\ \ +18$	$+0.0246$	$+0.0599$	-0.08774	$-.0032$	......
$-15\ \ +18$	-0.0093	$+0.0096$	$+0.0013$	$+.0016$	-0.0107
$-14\ \ +18$	$+0.008$	-0.007	-0.004	$-.003$	-0.017

We have found that the sum $K_1+K_2+K_3$ ought to vanish. The column $32\varDelta$ shows the sum of the computed numerical values, and hence, the sum of the errors with which the computations are affected. As has already been remarked, the difficulty of computing accurate values of K_1 and K_2 by the method of computation adopted is very great, because these quantities appear as small differences of very large numbers, while the same difficulty does not attend the computation of K_3. Hence, in the case of the most important term, that depending on the argument $-16g'+18g_4$, we may conclude with considerable probability that the outstanding errors are entirely in the quantities K_1 and K_2. The only other error of the least importance is that corresponding to the argument $-14g'+18g_4$. Here the value of $\varDelta$ is so great that we may conclude that the values of K_1, K_2, and K_3 are uncertain to almost their entire amount, and the chances are decidedly in favor of the hypothesis that their values are numerically too great.

Combining the terms by the formulæ (111) or (112) we find that the different combinations give rise to the following values of 32 K and 32 K':

N	U	32 K	32 K'
$\varepsilon-\pi-2g'$;	$-18g'+18g_4$	$-.000305$	$-.000424$
$\varepsilon-\pi-g'$;	$-17\quad+18$	$-.000001$	$-.000003$
$\varepsilon-\pi$;	$-16\quad+18$	$+.004728$	$+.006503$
$\varepsilon-\pi+g'$;	$-15\quad+18$	0	$-.000003$
$\varepsilon-\pi+2g'$;	$-14\quad+18$	$+.000427$(?)	$+.000518$(?)
Sum		$+.004849$	$+.006591$

For all these values of N we have

$$i=+1$$

$$i'=-1$$

$$i''=0$$

which, being substituted in the values of ε_c and ε_s, give

$$\varepsilon_c=\ 0.''004226\,\nu^2\,K+0.''00556\,\nu\left(K+\tfrac{1}{2}\frac{dK}{d\nu}\right)$$

$$\varepsilon_s=-0.''004226\,\nu^2\,K'-0.''00556\,\nu\left(K'+\tfrac{1}{2}\frac{dK'}{d\nu}\right)$$

The annual motion of the argument

$$N-U=\varepsilon-\pi+16g'-18g_4$$

taking for the motions of g' and g_4 the mean sidereal motions of the Earth and Venus is

$$-4750.''$$

which gives

$$\nu = 3648$$

We thus have

$$\varepsilon_c = +8.''52$$

$$\varepsilon_s = -11.58$$

or

$$\begin{aligned} \delta\varepsilon &= -11.''58 \sin(\varepsilon - \pi + 16g' - 18g_4) \\ &\quad + 8.\ 52 \cos(\varepsilon - \pi + 16g' - 18g_4) \\ &= \quad 14.\ 38 \sin(18g_4 - 16g' - g + 36^\circ 20') \end{aligned}$$

The adopted mass of Venus is about $\frac{1}{20}$ less than that used by HANSEN and DELAUNAY. In increasing the result by $\frac{1}{20}$ the coefficient will agree very nearly with that adopted by HANSEN in his *Tables de la Lune*, but will still be more than 1'' smaller than that found by DELAUNAY.

It will be seen that had we omitted the doubtful term in K and K′, the coefficient would have been one-twelfth, or about 1'' smaller, and it is not improbable that this term is numerically too large. If it were diminished the final value of the coefficient would diverge yet more widely from the values of HANSEN and of DELAUNAY.

It is a curious fact that when we consider terms depending on these elevated multiples of the mean longitudes, the terms of the second order become proportionally much larger than in the case of the terms usually considered in problems of celestial mechanics. In computing the term under consideration we have determined the direct action of the planet on the Moon, on the supposition that both the Earth and Venus move in their elliptic orbits of 1800. Since the entire action of Venus on the Moon is a quantity barely sensible, it might be considered that the change in this action due to the perturbations of a few seconds which the relative coordinates of these bodies suffer in consequence of their mutual action would necessarily be minute in the extreme, and entirely unworthy of consideration. But it must be remembered that any individual term of high order of the class under consideration really represents only a very minute portion of the total disturbing force of the planet—a minute residuum, in fact, which could produce no appreciable effect but for its period very nearly coinciding with that of some argument in the motion of the Moon. Now, we must not conclude without examination that the effect of a term in the perturbations would be insensible alongside a term of this character.

Let us inquire to what extent the values of K and K′ may be altered by the perturbations. And, first, let us consider the changes introduced by the perturbations of ρ^3. The value of ρ is determined by the equation

$$\rho^2 = r'^2 + r_4^2 - 2\, r r' \,(\cos v_4 \cos v' + \cos \gamma \sin v_4 \sin v')$$

v_4 and v' being the distances of Venus and of the Earth from the common node of their orbits, and γ the mutual inclination of the orbits. For our present purpose we may put $\cos \gamma = 1$, and may suppose the perturbations of v and v' to be the same as those of the longitudes in the orbits. These suppositions give, by putting $v_4 - v' = \mathrm{V}$,

$$\rho^2 = r'^2 + r_4^2 - 2\, r' r_4 \cos \mathrm{V}$$

and

$$\delta \frac{1}{\rho_3} = \frac{3\,(r_4 \cos \mathrm{V} - r')}{\rho^5}\, \delta r' + \frac{3\,(r' \cos \mathrm{V} - r_4)}{\rho^5}\, \delta r_4 - \frac{3\, r' r_4}{\rho^5} \sin \mathrm{V}\, \delta \mathrm{V}$$

According to LE VERRIER (*Annales del Observatoire de Paris, Tomes* IV, VI) the largest terms in $\delta r'$, δr_4, $\delta v'$, and δv_4 depending on the mutual action of Venus and the Earth are

$$\begin{aligned}
\delta r' = & - 1.''1 \cos (l_4 - l') \\
& + 3.''3 \cos 2\,(l_4 - l') \\
& + 0.''5 \cos 3\,(l_4 - l') \\
\delta r_4 = & \quad 0.''7 \cos (l_4 - l') \\
& + 3.''1 \cos 2\,(l_4 - l') \\
& - 2.''6 \cos 3\,(l_4 - l') \\
\delta v' = & + 4.''9 \sin (l_4 - l') \\
& - 5.''6 \sin 2\,(l_4 - l') \\
& - 0.''7 \sin 3\,(l_4 - l') \\
& + 2.''5 \cos (-2\, l_4 + 3\, l') \\
& + 1.''6 \cos (-3\, l_4 + 4\, l') \\
& + 1.''0 \sin (-3\, l_4 + 5\, l') \\
& + 0.''3 \cos (-3\, l_4 + 5\, l') \\
\delta v_4 = & - 4.''5 \sin (l_4 - l') \\
& - 10.''4 \sin 2\,(l_4 - l') \\
& + 6.''6 \sin 3\,(l_4 - l') \\
& - 3.''2 \cos (-2\, l_4 + 3\, l') \\
& \quad 0.''6 \cos (-3\, l_4 + 4\, l') \\
& - 1.''5 \cos (-4\, l_4 + 5\, l') \\
& - 1.''3 \sin (-3\, l_4 + 5\, l') \\
& - 0.''4 \cos (-3\, l_4 + 5\, l')
\end{aligned}$$

Whence

$$
\begin{aligned}
\delta V = & -9.''4 \sin \quad (l_4 - l') \\
& -4.''8 \sin 2\,(l_4 - l') \\
& +7.''3 \sin 3\,(l_4 - l') \\
& -5.''7 \cos (-2\, l_4 + 3\, l') \\
& -2.''2 \cos (-3\, l_4 + 4\, l') \\
& -1.''5 \cos (-4\, l_4 + 5\, l') \\
& -2.''3 \sin (-3\, l_4 + 5\, l') \\
& -0.''7 \cos (-3\, l_4 + 5\, l')
\end{aligned}
$$

These expressions are now to be substituted in the above expressions for $\delta \frac{1}{\rho^3}$. In making this substitution we may, in a first approximation, put

$$
\begin{aligned}
r' &= a' = 1 \\
r_4 &= a_4 = 0.72 \\
V &= l_4 - l'
\end{aligned}
$$

and for brevity, we shall put

$$l_4 - l' = L$$

We thus find

$$
\begin{aligned}
\rho^5\, \delta \frac{1}{\rho^3} = & +10.''0 \\
& +15.''2 \cos \quad L \\
& -38.''3 \cos 2\,L \\
& +\ 7.''1 \cos 3\,L \\
& -\ 2.''5 \cos (4\,L - 2\,l_4) \\
& +\ 2.''5 \cos (6\,L - 2\,l_4) \\
& -\ 6.''2 \sin (2\,L - \quad l_4) \\
& -\ 2.''4 \sin (3\,L - \quad l_4) \\
& +\ 4.''6 \sin (4\,L - \quad l_4) \\
& +\ 2.''4 \sin (5\,L - \quad l_4) \\
& +\ 1.''6 \sin (6\,L - \quad l_4) \\
& -\ 0.''8 \sin (4\,L - 2\,l_4) \\
& +\ 0.''8 \sin (6\,L - 2\,l_4)
\end{aligned}
$$

We must now substitute for L and l_4 their values in g' and g_4, namely,

$$
\begin{aligned}
L &= -g' + g_4 + 29.^\circ 1 \\
l_4 &= \quad\ \ g_4 + 129.^\circ 45
\end{aligned}
$$

when the expression in question becomes in units of the seventh place of decimals.

g'	g_4	$\rho_5\ \delta\ \frac{1}{\rho_3}$ cos	$\rho_5\ \delta\ \frac{1}{\rho_3}$ sin
0	0	+485	......
−1	1	+645	− 359
−2	1	+286	− 97
−2	2	−980	+1576
−3	2	+ 78	− 82
−3	3	+ 20	− 344
−4	2	+121	− 44
−4	3	− 53	+ 218
−5	4	+ 29	+ 116
−6	4	− 29	+ 126
−6	5	+ 53	+ 53

If we multiply this expression by that for $\frac{24}{\rho^5}$ given in pp. 252–257, and retain only the terms which depend on the argument $-16g'+18g_4$, we find the sum of those terms to be

Sum of positive products . . . $+.001\ 28\cos+.001\ 12\sin$

Sum of negative products . . . $-.001\ 29\cos-.001\ 15\sin$

$$\text{Term of } \delta\,\frac{24}{\rho^3}=-.000\,01\cos(-16\,g'+18\,g_4)-.000\,03\sin(-16\,g'+18\,g_4)$$

It appears that although the individual products are large enough sensibly to alter the expression for $\frac{1}{\rho^3}$, yet the sums of the different products so destroy each other that the entire term is quite insensible. This, however, may not be true of the values of $\frac{(x_4+x')^2}{\rho^5}$, etc. The computation of the inequalities in these quantities can not, however, be considered in the present paper.

A closer examination of all the other terms (Tabulated on p. 282) shows that their true values are generally smaller than those here assigned them. Their more exact computation may therefore be dispensed with.

§ 22.

ACTION OF MARS.

The adopted mass of this planet being $\frac{1}{3\,000\,000}$ we find by proceeding as in the case of Venus,

$$\mu_4\, a^3 = 0.''000\;384\;5$$

This value being substituted in the equations (114) the secular terms become

$$\delta\,\epsilon = 0.''000\;3845\left\{-2.023\;K'_0 - 1.011\,\frac{d\,K'_0}{d\,\nu} + .0005\,\frac{d\,K'_0}{e\,d\,e}\right\} n\,t$$

$$\delta\,\pi = 0.''000\;3845\left\{+.0293\;K'_0 + .0147\,\frac{d\,K'_0}{d\,\nu} + \;.526\,\frac{d\,K'_0}{e\,d\,e}\right\} n\,t$$

$$\delta\,\theta = 0.''000\;3845\left\{-.0074\;K'_0 - .0037\,\frac{d\,K'_0}{d\,\nu} + \;.125\,\frac{d\,K'_0}{\gamma\,d\,\gamma}\right\} n\,t$$

Among the terms of small multiples those multiplied by z_4^2 are comparatively small. If we omit them we shall have in the equations (103) for $U = 0$

$$32\;K_3 = -17.3$$

K_3 being, when we neglect z_4, the negative of one-half the constant term of $\frac{1}{\rho^3}$. Using the values of $\varkappa_3$ and $\varkappa_1$ found in the case of Venus, we find

$$32\;\;K'_0 = +\;\;8.6$$

$$32\,\frac{d\,K'_0}{e\,d\,e} = +26.4$$

Substituting in the above equations, we find, by putting $n = 84$, as in the case of Venus,

$$\delta\,\epsilon = -0.''035\,t$$

$$\delta\,\pi = +0.''028\,t$$

$$\delta\,\theta = -0.''008\,t$$

The equations (113) become by substituting the preceding value of $\mu_4\, a^3$

$$\epsilon_c = (\;\;.''000\;583\;i - .''000\;008\;i' + .''000\;002\;i'')\;\nu^2\,K$$

$$+\quad .''000\;778\;\nu\,K + .''000\;389\;\nu\,\frac{d K}{d\,\nu}$$

$$\epsilon_s = (-\;''000\;583\;i + .''000\;008\;i' - .''00\;002\;i'')\;\nu^2\,K'$$

$$-.''000\;788\;\nu\,K - .''000\;389\;\nu\,\frac{d K}{d\,\nu}$$

The corresponding coefficients for the perigee and node are too small to be worth computing.

Proceeding as in the case of Venus we begin with those terms in which i, i', and and i'' are all zero, so that the term contains the mean longitudes of only the Earth and Mars. In the case of these terms we have

$$K = \mp 0.495\ K'_3$$

$$K' = -0.495\ K_3$$

In the terms of small multiples we may take as an approximate value of K_3 the corresponding coefficient in the development of $-\frac{1}{2}\frac{1}{\rho^3}$.

We now select the following terms from the development of $\frac{1}{\rho^3}$ in the case of Mars and the Earth as the only ones in which either from the length of the period or the magnitude of the coefficient we may find sensible values of ε_c or ε_s.

U	K_3	K'_3	2 K	2 K'	ν	$2\varepsilon_c$	$2\varepsilon_s$
						$''$	$''$
$-g_4$	$-.340$	$-.034$	$\pm .034$	$+.337$	∓ 25.14	$-.001$	$+.007$
$-g_4+g'$	$+.553$	$+.728$	$\mp .721$	$-.547$	± 28.55	$-.014$	$+.011$
$-2g_4+g'$	$+.209$	$+.342$	$\mp .338$	$-.207$	∓ 210.9	$+.055$	$-.033$
$-2g_4+2g'$	$+.189$	$-.675$	$\pm .668$	$-.187$	± 14.28	$+.007$	$+.002$
$-3g_4+2g'$	$+.149$	$-.374$	$\pm .370$	$-.148$	± 33.02	$+.009$	$+.004$
$-3g_4+3g'$	$-.474$	$+.186$	$\mp .184$	$+.469$	± 9.52	$-.001$	$-.003$
$-4g_4+2g'$	$+.068$	$-.137$	$\pm .136$	$-.067$	∓ 105.5	$-.010$	$-.001$
$-5g_4+3g'$	$-.160$	$+.030$	$\mp .030$	$+.158$	± 39.14	$-.001$	$-.005$
$-6g_4+3g'$	$-.058$	$+.006$	$\mp .006$	$+.057$	∓ 70.30	0	$+.003$

The inequalities in question are therefore

$$-.''001 \cos g_4 - .''007 \sin g_4$$

$$-.''014 \cos (g_4 - g') - .''011 \sin (g_4 - g')$$

$$\text{etc.} \qquad\qquad \text{etc.}$$

We next consider the terms into the arguments of which the elements of the Moon's orbit enters. Proceeding as in the case of Venus we find the following terms of long period. The computation of the probable order of magnitude of the inequalities is conducted in the same way as in the case of Venus.

i	i'	i''	j	j_4	ω	$\log \nu^2$	$\log \varkappa_i$	$\log K_i$	$\log \varepsilon + 10$	ε
					$''$					$''$
0	0	2	−2	+4	68	5.7	−2.7	−0.4	7.2	0.02
0	2	0	−5	+9	42	6.1	−2.4	−2.0	6.9	.01
1	−3	0	−29	+30	69	5.7	−4.6	−3.2	4.7	.00
1	−3	0	−21	+15	18	6.8	−4.6	−3.4	5.6	.00
1	−1	0	−26	+24	57	5.8	−1.3	−3.0	8.3	.02
1	0	0	−32	+35	78	5.6	−3.6	−3.3	5.5	.00
1	1	0	−31	+33	96	5.4	−1.1	−3.3	7.0	.01
1	−1	2	−28	+28	125	5.2	−3.6	−3.0	5.4	.00
1	−1	−2	−24	+20	10	7.4	−3.6	−3.0	7.6	.00
1	1	−2	−29	+29	29	6.4	−4.3	−3.1	5.9	.00
0	0	1	−1	+2	34	6.2	−1.1	−0.5	9.0	0.10
0	2	−1	−4	+7	6	7.8	−3.3	−1.0	8.7	.07
1	−1	1	−27	+26	91	5.4	−2.3	−3.2	6.7	.00
1	−1	1	−35	+41	6	7.8	−2.3	−4.0	8.3	.02
1	−1	−1	−25	+22	23	6.6	−2.3	−3.5	7.6	.00
1	1	−1	−30	+31	64	5.7	−1.8	−3.2	7.4	.00
1	1	1	−23	+18	7	7.7	−3.6	−3.0	7.9	.01

The largest of these terms, the eleventh, is one in which it is reasonably certain that the value of K_i does not exceed the limit here assigned, because it contains the latitude of Mars as a factor. A coefficient of $0.''1$ could scarcely be detected in observations; the computation of the term is therefore dispensed with.

§ 24.

ACTION OF JUPITER.

With the adopted mass of Jupiter (BESSEL'S) we find

$$\mu_4 a^3 = 1''101$$

The substitution of this value in the expressions (112) and (114) give for the secular and periodic terms in the perturbations

$$\delta\varepsilon = \left\{ -2.''227\, K'_0 - 1.''113 \frac{d\, K'_0}{d\, \nu} \right\} n\, t$$

$$\delta\pi = \left\{ \quad 0.''032\, K'_0 + 0.''016 \frac{d\, K'_0}{d\, \nu} + 0.''579 \frac{d\, K'_0}{e\, d\, e} \right\} n\, t$$

$$\delta\theta = \left\{ -0''.008\, K'_0 - 0.''004 \frac{d\, K'_0}{d\, \nu} + 0.''138 \frac{d\, K'_0}{\gamma\, d\, \gamma} \right\} n\, t$$

$$\varepsilon_c = (1.''67 \quad i - 0.''024\, i' + 0.''006\, i'')\, \nu^2\, K$$

$$+ 2.''227\, \nu\, K + 1.''113\, \nu \frac{d\, K}{d\, \nu}$$

$$\varepsilon_s = (-1.''67 \quad i + 0.''024\, i' - 0.''006\, i'')\, \nu^2\, K'$$

$$-2''227\, \nu\, K' - 1.''113\ 2\ \nu \frac{d\, K'}{d\, \nu}$$

$$\pi_c = -0.''579\, \nu \frac{d\,K}{e\,d\,e}$$

$$\pi_s = \quad 0.\ 579\, \nu \frac{d\,K'}{e\,d\,e}$$

$$\theta_c = -0.''138 \frac{d\,K}{\gamma\,d\,\gamma}$$

$$\theta_s = \quad 0.\ 138 \frac{d\,K'}{\gamma\,d\,\gamma}$$

Owing to the distance of Jupiter the developments (103) are comparatively simple, and it is unnecessary to enter extensively into their numerical details. If we put

r_4, the radius vector of Jupiter.

V, the angular distance between Jupiter and the Earth, as seen from the Sun.

L, the difference of their mean longitudes,

we have

$$\rho^2 = r_4^2 \left(1 - 2 \frac{r'}{r_4} \cos V + \frac{r'^2}{r_4^2} \right)$$

$$\frac{1}{\rho^3} = \frac{1}{r_4^3} \left\{ 1 + 3 \frac{r'}{r_4} \cos V + \frac{15}{2} \frac{r'^2}{r_4^2} \cos^2 V - \frac{3}{2} \frac{r'^2}{r_4^2} + \text{etc.} \right\}$$

$$\frac{1}{r_4^3} = \frac{1}{a_4^3} \left\{ 1 + \frac{3}{2} e_4^2 + 3\, e_4 \cos g_4 + \frac{9}{2} e^2_4 \cos 2\, g_4 + \text{etc.} \right\}$$

Beginning with the constant term of $\frac{1}{\rho^3}$ we find that its value is, with sufficient approximation,

$$\frac{1}{a_4^3} \left(b_{\frac{3}{2}}^{(0)} + \frac{3}{2} e_2 \right)$$

$b_{\frac{3}{2}}^{(0)}$ being the constant term in the development of

$$(1 - 2\, \alpha \cos L + \alpha^2)^{-\frac{3}{2}}$$

The value of this constant is

$$b_{\frac{3}{2}}^{(0)} = 1.0883.$$

In the expressions (103) the terms of $\frac{z_4^2}{\rho^5}$ will be very small in comparison with those of $\frac{1}{\rho^3}$, we have therefore only to consider the development of the latter term in

forming the values of K_3 and K'_3. Beginning with the constant term, we have

$$K_3 = -.003\,876 \qquad K'_3 = 0$$

From this we have

$$K'_0 \qquad = +.001\,920$$

$$\frac{d\,K'_0}{d\,x} = K'_0 = +.000\,008$$

$$\frac{d\,K'_0}{e\,d\,e} = +.\ 00\,589$$

$$\frac{d\,K'_0}{\gamma\,d\,\gamma} = -.\ 00\,774$$

Proceeding as in the case of Venus we find, for the secular terms:

$$\delta\,l = -0.''720\,t$$

$$\delta\,\pi = +0.''573\,t$$

$$\delta\,\theta = -0.''179\,t$$

We have next to consider the periodic terms arising from the constant of $\frac{1}{\rho^3}$, which can readily be shown to be too small to be taken into account. In a first approximation to the corresponding values of K_i we may neglect x' and y' alongside of x_4 and y_4. We thus find

$$K_1 = +.0019$$

$$K_2 = +.0019$$

$$K_3 = -.0039$$

$$K_4 = 0$$

Referring to the tables, p. 217, we find that where i, or the coefficient of (l) is not zero, the largest value of $\varkappa$ is less than 0.5, so that the largest value of K or K′ is less than .001. In these terms the coefficient ν will not differ much from unity, the values of ε_c and ε_s will therefore never exceed 0.″002.

In the seven terms of the table referred to which follow the first, i vanishes, and ν is therefore large. But $\varkappa$ and the coefficient of i' and i'' are in these cases very small, and the largest terms in N are of the second class, in which $K_1 + K_2 = 0$; $K_3 = 0$, so that the principal term depends on $K_4\,\varkappa_4$. The constant term of K_4, which we have above put equal to zero, must be less than .0006, while the largest value of k_4 is less than .003. K or K′ must therefore be less than .000002, while ν is between 100 and 150. Substituting these values in ϵ_c and ϵ_s we find that the latter do not attain a sensible value. The constant term of $\frac{1}{\rho^3}$ therefore gives rise to no sensible periodic inequality.

Passing now to the periodic terms we shall begin with the combination of these terms with the zero, or first term of N. This term is of the first class (150), p. 275, so

that we only want the values of K_3 and, as already shown, may neglect the square of z_4, and so consider only the development of $\frac{1}{\rho^3}$. The largest periodic terms in this development are numerically, putting L for the difference of mean longitude of Jupiter and the Earth,

$$\frac{1}{\rho^3}=.004\ 38 \cos L+.001\ 08 \cos 2\ L+.001\ 03 \cos g_4$$

We have therefore the following combinations of the preceding values of U with the zero term in N:

U	K_3	K'_3	ν	ε_s
L	$-.00\,219$	$\pm.00\,100$	14.6	$\mp.''032$
2 L	$-.00\,054$	$\pm.00\,024$	7.4	$\mp.''004$
g_4	$-.00\,052$	$\pm.00\,023$	$+158.7$	$\mp.''080$

The resulting inequalities in the mean longitude are

$$\begin{aligned}\delta\varepsilon &= 0.''064 \sin\ (l_4-l') \\ &+0.''008 \sin 2\ (l_4-l') \\ &-0.''060 \sin g_4\end{aligned}$$

The combination of the terms in $\frac{1}{\rho^3}$ which depend on multiples of the mean anomaly of Jupiter with those which depend on the lunar perigee and node lead to terms of long period, but none of the coefficients are sensible. The only terms of U which, combined with those terms of N containing ε can be of long period, depend on such high powers both of α and of e_4 and e' that they can not be sensible.

Postscript (added August, 1894).—In the above reasoning it is not noticed that in the developments (103) the periodic term having much the largest coefficient depends on the argument twice the mean longitude of Jupiter. This term gives rise to the "Jovian evection," discovered by Mr. Neison, and computed with great precision by Mr. Hill, in Vol. III of the present series.

SECULAR VARIATIONS OF THE ORBITS

OF THE

FOUR INNER PLANETS.

CONTENTS.

§ I.

INTRODUCTORY.

Two methods are available for computing the secular variations of the elements of the planets, produced by their mutual action. One is founded on the solution of the well-known problem of GAUSS: To determine the attraction exerted at every point by an elliptic ring of matter extended along the orbit of a planet, having at each point a density inversely proportional to the linear velocity of the planet at that point. This method has been developed in several memoirs with great fullness, and has been extensively used by several recent investigators.*

The other method rests upon the development of the perturbative function in powers of the eccentricities and inclinations. The relative advantages of these methods probably correspond closely to those of the numerical and algebraic methods in perturbations generally. A numerical theory, designed to be extremely accurate only at a special epoch, can probably be most easily constructed by the GAUSSIAN method. Its extension through a period of centuries would, however, have to be made by a recomputation probably repeated several times at certain intervals with a view of reaching expressions for the variations of the elements by a series of successive approximations. On the other hand, by the analytic method, general expressions can be found which, by a simple substitution of the values of the elements from time to time, will serve for any epoch whatever. In most cases the computation by this method will be simpler than by that of GAUSS. The exceptions are the cases of two neighboring planets, and especially those in which the eccentricities of one of these planets is large, as is the case with Mercury. The labor of the second method will consist very largely in preparing the preliminary developments and computing the constants, which are functions of the ratio of the mean distances. As the greater part of this work was already done in connection with the planetary theories, I have deemed it advisable to adopt it in the computation of the secular variations.

An important remark is to be made respecting the extent to which the secular variations computed by either method are valid. Although, by computing in succession the coefficients of the several powers of the time, successive powers of the masses are taken into account in the approximations, it is nevertheless true that the whole theory is valid only to quantities of the first order as to the masses. Whether the GAUSSIAN

*In the original Memoir of GAUSS, Determinatio Attractionis, etc. (Opera, Vol. V), no application is made to the determination of secular variations. The first development having this end in view is that of CLAUSEN (CRELLE'S Journal, Vol. VI). In Astronomical Papers, Vol. I, G. W. HILL develops the method with practical precepts for making the computation. An extensive modification of HILL'S method, with the necessary tables, is given by CALLANDREAU, in Annales de l'Observatoire de Paris, Mémoirs, Tome XVI.

method or the other one be used, the fundamental hypothesis is that the disturbing planet moves in an elliptic orbit. If the variations of this orbit are taken account of, then the distinction between secular and periodic inequalities is to a certain extent obliterated. If we seek for a rigorous development in powers of the times of those parts of the expression for the varying elliptic elements which are independent of the mean longitudes of the planets, it is necessary to take into account those products of periodic terms into which the same multiples of the mean longitudes enter, because from some of those products the mean longitude will disappear. Thus we have, at the first step, secular variations of the second order as to the masses, which are, however, multiplied only by the first power of the time. The works of both HILL and LE VERRIER on the theory of Jupiter and Saturn show that a material change may be made in this way, in that the secular variations of the eccentricities and perihelia of those planets are increased by perhaps their tenth part by these terms of the second order.

The question has been raised by some astronomers whether the same may not be true for the four inner planets; whether, in fact, the increased motion of the perihelion of Mercury may not arise in this way. It may be well, therefore, to point out that such is not the case. The great distance of the larger planets, Jupiter and Saturn, from the four inner ones, and the small masses of the latter, prevent these terms from having any appreciable value in the case of the planets in question.

The fundamental general formulæ for the variations of the planetary elements are so well known that it might be unnecessary to consider their derivation. But the quantities which enter into the present theory, and the general principles of making the computations differ so much from those previously employed, that I have found it practically necessary to assume only the general canonical equations for the variations of the elements, and to show what formulæ flow from these equations when the special quantities used in the present theory are introduced.

The fundamental expression on which the whole work rests is the non-periodic part of the perturbative function. This expression is derived from the development given in Part I of the present volume, and can be carried to any required power of the eccentricities and mutual inclinations by means of the rules and methods there set forth.

The very important question of the convergency of the development presents itself at the outset. This question seems to me satisfactorily settled in the affirmative on the three following bases:

1. The demonstration of CAUCHY, that the development of the equation of the center in powers of the eccentricity is convergent up to a limit far exceeding the value of that element in the case of any planet of our system, and the inference that the same is true of the radius vector.

2. The expression of all the coefficients of the development as linear functions of convergent hypergeometric series.

3. The demonstration in Article III of Part I of the present volume, that the development in powers of the mutual inclination is convergent within limits far exceeding those found in the solar system, the cases of the mutual actions of the small planets, and possibly of Mars upon a few of them, being excepted.

Such being the case, the method of developing in series has the great advantages that the effect of any change in the adopted elements can be readily traced, the computation can be made for any required epoch, and any numerical error in the work can be easily found and corrected.

The question how far the development should be carried can be settled only by trial. I have deemed it advisable to extend it to terms of the eighth order in the eccentricities and mutual inclination of orbits, considering, however, only those terms of the eighth order likely to be most important, but including terms of the tenth order, mostly by induction, when required, stopping in each special case when, and only when, the terms are found by actual computation to be insensible.

§ 2.

FORMATION OF THE NON-PERIODIC PART OF THE PERTURBATIVE FUNCTION.

We have first to show how the non-periodic terms of the development may be found to any extent. In Part I, Exhibit III, of the present volume (pp. 23-27), I have given a symbolic development of R, in which the arguments are expressed as a function of the four quantities,

$$\lambda',\ \lambda,\ g',\ g$$

or of

$$w',\ w,\ g',\ g$$

The condition that a term shall be non-periodic is that when its argument is expressed in the last way the coefficients of g' and g shall vanish simultaneously for some one value of i. In the passage in question, pp. 23, 25, 26, it is shown that the general form of the argument for terms of class k is

$$(k-i)\,w+(k+i)\,w'+(k+j-i)\,g+(k+j'+i)\,g'$$

where k takes the values 0, 1, 2, . . . and j, j', and i all integral values, positive and negative, though in class 0 terms corresponding to negative values of i are merged with the corresponding terms for positive values. That the coefficients of g and g' may both vanish we must have, for any value of i,

$$j=\ \ i-k$$
$$j'=-i-k$$

And hence

for terms of class 0: $j=i$; $j'=-i$

" " " 1: $j=i-1$; $j'=-i-1$

" " " 2: $j=i-2$; $j'=-i-2$

. . . .

Moreover the largest terms of each class contain as a factor the coefficient $\sigma^{2k}\,e^{j}\,e'^{j'}$, the

positive values of j and j' being taken. The complete coefficient of each term corresponding to any one argument is of the form

$$e^n e'^{n'} (\overset{(k)}{P}{}^{n,n'}_{j,j'} + e^2 \overset{(k)}{P}{}^{n+2,n'}_{j,j'} + e'^2 \overset{(k)}{P}{}^{n,n'+2}_{j,j'} + \ldots)$$

where n and n' are the positive values of j and j'. From the values of j and j' just given it will be seen that in any class, k, the terms in question are at least of the order $2k$ in the eccentricities, while all the values of P contain σ^{2k} as a factor. Hence the order of magnitude of the lowest terms of the class in the eccentricities and mutual inclination is $4k$. Hence the terms of class 3 are at least of the 12th order, and it is quite unnecessary to go beyond class 2, which is of the 8th order and not likely to lead to any appreciable result.

We shall now write the symbolic expression of the terms thus found, complete to the sixth order, and including such of the eighth as are most likely to be important. We put for the non-periodic part of the perturbative function

$$R = \frac{m}{\Delta} = \frac{m}{a'} F$$

or

$$F = \text{non-periodic part of } \frac{a'}{\Delta}$$

so that F is a purely numerical quantity, independent of the adopted units of length, time, or mass.

In writing the value of F, I have changed the sign of the argument under the symbol cosine whenever it seemed convenient. The value of i from which each argument is formed is written in parentheses under the symbol P.

The selection of the eighth order terms to be first examined is guided by the following considerations: The slowness of the convergence, and hence the importance of the terms in question, increase with the ratio of the mean distances of the pair of planets in question. Hence the cases of pairs of neighboring planets are those for which these terms are first to be considered. Now, in each of these cases, one of the eccentricities is much larger than the other. The ratio of the derivative of the factor $e^4 e'^4$ to that of e^8, $e^7 e'$, e'^8, or ee'^7 for each of these pairs is approximately as follows:

Venus on Mercury . . .	$8e^7 : 4e^3 e'^4 =$	3 240 000 : 1
Mercury on Venus	$e^7 : 4e^4 e'^3 =$	6 800 : 1
Earth on Venus	$e'^7 : 4e^3 e'^4 =$	3.7 : 1
Mars on Earth	$e'^7 : 4e^3 e'^4 =$	41. : 1
Earth on Mars	$8e'^7 : 4e^4 e'^3 =$	1 815 : 1

From an approximate computation of the terms containing the largest of these factors, which may be e^8, e'^8, $e^7 e'$, or ee'^7, it may be inferred whether there is a possibility of them or any others becoming sensible.

$$
\begin{aligned}
\mathrm{F}=\tfrac{1}{2}\Big\{&\underset{(0)}{\mathrm{P}^{0,0}_{0,0}}+(e^2+e'^2)\underset{(0)}{\mathrm{P}^{2,0}_{0,0}}+e^4\underset{(0)}{\mathrm{P}^{4,0}_{0,0}}+e^2e'^2\underset{(0)}{\mathrm{P}^{2,2}_{0,0}}+e'^4\underset{(0)}{\mathrm{P}^{0,4}_{0,0}}+e^6\underset{(0)}{\mathrm{P}^{6,0}_{0,0}}+e^4e'^2\underset{(0)}{\mathrm{P}^{4,2}_{0,0}}\\
&+e^2e'^4\underset{(0)}{\mathrm{P}^{2,4}_{0,0}}+e'^6\underset{(0)}{\mathrm{P}^{0,6}_{0,0}}+e^8\underset{(0)}{\mathrm{P}^{8,0}_{0,0}}+\ \ldots\ +e'^8\underset{(0)}{\mathrm{P}^{0,8}_{0,0}}+\ \ldots\ \Big\}\\
+\Big\{&e\,e'\underset{(-1)}{\mathrm{P}^{1,1}_{-1,1}}+e^3e'\underset{(-1)}{\mathrm{P}^{3,1}_{-1,1}}+e\,e'^3\underset{(-1)}{\mathrm{P}^{1,3}_{-1,1}}+e^5e'\underset{(-1)}{\mathrm{P}^{5,1}_{-1,1}}+e^3e'^3\underset{(-1)}{\mathrm{P}^{3,3}_{-1,1}}+e\,e'^5\underset{(-1)}{\mathrm{P}^{1,5}_{-1,1}}\\
&+e^7e'\underset{(-1)}{\mathrm{P}^{7,1}_{-1,1}}+\ \ldots\ +e\,e'^7\underset{(-1)}{\mathrm{P}^{1,7}_{-1,1}}+\ \ldots\ \Big\}\cos(w-w')\\
+\Big\{&e^2e'^2\underset{(-2)}{\mathrm{P}^{2,2}_{-2,2}}+e^4e'^2\underset{(-2)}{\mathrm{P}^{4,2}_{-2,2}}+e^2e'^4\underset{(-2)}{\mathrm{P}^{2,4}_{-2,2}}+\ \ldots\ \Big\}\cos(2w-2w')\\
+\Big\{&e^3e'^3\underset{(-3)}{\mathrm{P}^{3,3}_{-3,3}}+\ \ldots\ \Big\}\cos(3w-3w')\\
+\Big\{&e^2\underset{(-1)}{\mathrm{P}'^{2,0}_{-2,0}}+e^4\underset{(-1)}{\mathrm{P}'^{4,0}_{-2,0}}+e^2e'^2\underset{(-1)}{\mathrm{P}'^{2,2}_{-2,0}}+\ \ldots\ \Big\}\cos 2w\\
+\Big\{&e\,e'\underset{(0)}{\mathrm{P}'^{1,1}_{-1,-1}}+e^3e'\underset{(0)}{\mathrm{P}'^{3,1}_{-1,-1}}+e\,e'^3\underset{(0)}{\mathrm{P}'^{1,3}_{-1,-1}}+\ \ldots\ \Big\}\cos(w+w')\\
+\Big\{&e'^2\underset{(1)}{\mathrm{P}'^{0,2}_{0,-2}}+e^2e'^2\underset{(1)}{\mathrm{P}'^{2,2}_{0,-2}}+e'^4\underset{(1)}{\mathrm{P}'^{0,4}_{0,-2}}+\ \ldots\ \Big\}\cos 2w'\\
+\Big\{&e^3e'\underset{(-2)}{\mathrm{P}'^{3,1}_{-3,1}}+\ \ldots\ \Big\}\cos(3w-w')\\
+\Big\{&e\,e'^3\underset{(2)}{\mathrm{P}'^{1,3}_{1,-3}}+\ \ldots\ \Big\}\cos(w-3w')\\
+\Big\{&e^4\underset{(-2)}{\mathrm{P}''^{4,0}_{-4,0}}+\ \ldots\ \Big\}\cos 4w\qquad+\Big\{e^3e'\underset{(-1)}{\mathrm{P}''^{3,1}_{-3,-1}}+\ \ldots\ \Big\}\cos(3w+w')\\
+\Big\{&e^2e'^2\underset{(0)}{\mathrm{P}''^{2,2}_{-2,-2}}+\ \ldots\ \Big\}\cos(2w+2w')+\Big\{e\,e'^3\underset{(1)}{\mathrm{P}''^{1,3}_{-1,-3}}+\ \ldots\ \Big\}\cos(w+3w')\\
+\Big\{&e'^4\underset{(2)}{\mathrm{P}''^{0,4}_{0,-4}}+\ \ldots\ \Big\}\cos 4w'
\end{aligned}
$$

The symbolic values of such of the coefficients $\mathrm{P}^{n,n'}_{j,j'}$, as I have found to be required in the present investigation, are as follow. For brevity, such a function as

$$n\,a'\,\mathrm{A}+n'\,a'\,\mathrm{DA}+n''\,a'\,\mathrm{D}^2\mathrm{A}+\ \ldots$$

is written in the form

$$(n+n'\,\mathrm{D}+n''\,\mathrm{D}^2+\ \ldots\ )\,a'\,\mathrm{A}$$

$$\underset{(0)}{\mathrm{P}^{0,0}_{0,0}}=a'\,\mathrm{A}_0$$

$$4\,\underset{(0)}{\mathrm{P}^{2,0}_{0,0}}=4\,\underset{(0)}{\mathrm{P}^{0,2}_{0,0}}=(\mathrm{D}+\mathrm{D}^2)\,a'\,\mathrm{A}_0$$

$$64\,\underset{(0)}{\mathrm{P}^{4,0}_{0,0}}=(2\,\mathrm{D}-\mathrm{D}^2-2\,\mathrm{D}^3+\mathrm{D}^4)\,a'\,\mathrm{A}_0$$

$$16\,\underset{(0)}{\mathrm{P}^{2,2}_{0,0}}=(\mathrm{D}^2+2\,\mathrm{D}^3+\mathrm{D}^4)\,a'\,\mathrm{A}_0$$

$$64\,\underset{(0)}{\mathrm{P}^{0,4}_{0,0}}=(6\,\mathrm{D}+11\,\mathrm{D}^2+6\,\mathrm{D}^3+\mathrm{D}^4)\,a'\,\mathrm{A}_0$$

$$2304\, \underset{(0)}{P^{6,0}_{0,0}} = (24\,D - 26\,D^2 - 15\,D^3 + 25\,D^4 - 9\,D^5 + D^6)\, a'\, A_0$$

$$256\, \underset{(0)}{P^{4,2}_{0,0}} = (2\,D^2 + D^3 - 3\,D^4 - D^5 + D^6)\, a'\, A_0$$

$$256\, \underset{(0)}{P^{2,4}_{0,0}} = (6\,D^2 + 17\,D^3 + 17\,D^4 + 7\,D^5 + D^6)\, a'\, A_0$$

$$2304\, \underset{(0)}{P^{6,6}_{0,0}} = (120\,D + 274\,D^2 + 225\,D^3 + 85\,D^4 + 15\,D^5 + D^6)\, a'\, A_0$$

$$147\,456\, \underset{(0)}{P^{8,0}_{0,0}} = (720\,D - 1044\,D^2 - 140\,D^3 + 889\,D^4 - 560\,D^5 + 154\,D^6 - 20\,D^7 + D^8)\, a'\, A_0$$

$$9216\, \underset{(0)}{P^{2,6}_{0,0}} = (120\,D^2 + 394\,D^3 + 499\,D^4 + 310\,D^5 + 100\,D^6 + 16\,D^7 + D^8)\, a'\, A_0$$

$$9216\, \underset{(0)}{P^{6,2}_{0,0}} = (24\,D^2 - 2\,D^3 - 41\,D^4 + 10\,D^5 + 16\,D^6 - 8\,D^7 + D^8)\, a'\, A_0$$

$$147\,456\, \underset{(0)}{P^{0,8}_{0,0}} = (5040\,D + 13\,068\,D^2 + 13\,132\,D^3 + 6769\,D^4 + 1960\,D^5 + 322\,D^6 + 28\,D^7 + D^8)\, a'\, A_0$$

$$4\, \underset{(-1)}{P^{1,1}_{-1,1}} = (2 - D - D^2)\, a'\, A_1$$

$$32\, \underset{(-1)}{P^{3,1}_{-1,1}} = (-2\,D + 3\,D^2 - D^4)\, a'\, A_1$$

$$32\, \underset{(-1)}{P^{1,3}_{-1,1}} = (4 + 4\,D - 3\,D^2 - 4\,D^3 - D^4)\, a'\, A_1$$

$$768\, \underset{(-1)}{P^{5,1}_{-1,1}} = (-12\,D + 28\,D^2 - 17\,D^3 - 3\,D^4 + 5\,D^5 - D^6)\, a'\, A_1$$

$$256\, \underset{(-1)}{P^{3,3}_{-1,1}} = (-4\,D + 7\,D^3 + D^4 - 3\,D^5 - D^6)\, a'\, A_1$$

$$768\, \underset{(-1)}{P^{1,5}_{-1,1}} = (48 + 76\,D - 4\,D^2 - 65\,D^3 - 43\,D^4 - 11\,D^5 - D^6)\, a'\, A_1$$

$$36\,864\, \underset{(-1)}{P^{7,1}_{-1,1}} = (-240\,D + 668\,D^2 - 604\,D^3 + 121\,D^4 + 110\,D^5 - 68\,D^6 + 14\,D^7 - D^8)\, a'\, A_1$$

$$36\,864\, \underset{(-1)}{P^{1,7}_{-1,1}} = (1440 + 2808\,D + 764\,D^2 - 1918\,D^3 - 2009\,D^4 - 868\,D^5 - 194\,D^6 - 22\,D^7 - D^8)\, a'\, A_1$$

$$64\, \underset{(-2)}{P^{2,2}_{-2,2}} = (12 - 8\,D - 7\,D^2 + 2\,D^3 + D^4)\, a'\, A_2$$

$$768\, \underset{(-2)}{P^{4,2}_{-2,2}} = (24 - 52\,D + 22\,D^2 + 17\,D^3 - 11\,D^4 - D^5 + D^6)\, a'\, A_2$$

$$768\, \underset{(-2)}{P^{2,4}_{-2,2}} = (72 + 12\,D - 70\,D^2 - 31\,D^3 + 9\,D^4 + 7\,D^5 + D^6)\, a'\, A_2$$

$$24\,576\, \underset{(-2)}{P^{6,2}_{-2,2}} = (288 - 792\,D + 652\,D^2 - 2\,D^3 - 229\,D^4 + 82\,D^5 + 8\,D^6 - 8\,D^7 + D^8)\, a'\, A_2$$

$$24\,576\, \underset{(-2)}{P^{2,6}_{-2,2}} = (1440 + 888\,D - 1220\,D^2 - 1238\,D^3 - 169\,D^4 + 190\,D^5 + 92\,D^6 + 16\,D^7 + D^8)\, a'\, A_2$$

$$2304\, \underset{(-3)}{P^{3,3}_{-3,3}} = (144 - 108\,D - 88\,D^2 + 39\,D^3 + 17\,D^4 - 3\,D^5 - D^6)\, a'\, A_3$$

$$8\, \underset{(-1)}{P'^{2,0}_{-2,0}} = (6 + 5\,D + D^2)\, \sigma^2 a'\, B_1$$

$$96\,\underset{(-1)}{P'}{}^{4,\;0}_{-2,\;0}=(12-8D-7D^2+2D^3+D^4)\,\sigma^2 a' B_1$$

$$32\,\underset{(-1)}{P'}{}^{2,\;2}_{-2,\;0}=(6D+11D^2+6D^3+D^4)\,\sigma^2 a' B_1$$

$$3072\,\underset{(-1)}{P'}{}^{6,\;0}_{-2,\;0}=(144-180D-16D^2+65D^3-9D^4-5D^5+D^6)\,\sigma^2 a' B_1$$

$$384\,\underset{(-1)}{P'}{}^{4,\;2}_{-2,\;0}=(12D+4D^2-15D^3-5D^4+3D^5+D^6)\,\sigma^2 a' B_1$$

$$4\,\underset{(0)}{P'}{}^{1,\;1}_{-1,-1}=(2-D-D^2)\,\sigma^2 a' B_0$$

$$32\,\underset{(0)}{P'}{}^{3,\;1}_{-1,-1}=(-2D+3D^2-D^4)\,\sigma^2 a' B_0$$

$$32\,\underset{(0)}{P'}{}^{1,\;3}_{-1,-1}=(4+4D-3D^2-4D^3-D^5)\,\sigma^2 a' B_0$$

$$768\,\underset{(0)}{P'}{}^{5,\;1}_{-1,-1}=(-12D+28D^2-17D^3-3D^4+5D^5-D^6)\,\sigma^2 a' B_0$$

$$768\,\underset{(0)}{P'}{}^{1,\;5}_{-1,-1}=(48+76D-4D^2-65D^3-43D^4-11D^5-D^6)\,\sigma^2 a' B_0$$

$$8\,\underset{(1)}{P'}{}^{0,\;2}_{0,-2}=(2-3D+D^2)\,\sigma^2 a' B_1$$

$$32\,\underset{(1)}{P'}{}^{2,\;2}_{0,-2}=(2D-D^2-2D^3+D^4)\,\sigma^2 a' B_1$$

$$96\,\underset{(1)}{P'}{}^{0,\;4}_{0,-2}=(12-8D-7D^2+2D^3+D^4)\,\sigma^2 a' B_1$$

$$512\,\underset{(1)}{P'}{}^{4,\;2}_{0,-2}=(4D-8D^2+D^3+7D^4-5D^5+D^6)\,\sigma^2 a' B_1$$

$$3072\,\underset{(1)}{P'}{}^{0,\;6}_{0,-2}=(240-52D-200D^2-31D^3+31D^4+11D^5+D^6)\,\sigma^2 a' B_1$$

$$96\,\underset{(-2)}{P'}{}^{3,\;1}_{-3,\;1}=(24+2D-17D^2-8D^3-D^4)\,\sigma^2 a' B_2$$

$$1536\,\underset{(-2)}{P'}{}^{5,\;1}_{-3,\;1}=(144-108D-88D^2+39D^3+17D^4-3D^5-D^6)\,\sigma^2 a' B_2$$

$$96\,\underset{(2)}{P'}{}^{1,\;3}_{1,-3}=(12-16D+D^2+4D^3-D^4)\,\sigma^2 a' B_2$$

$$384\,\underset{(-2)}{P''}{}^{4,\;0}_{-4,\;0}=(120+154D+71D^2+14D^3+D^4)\,\sigma^4 a' C_2$$

$$96\,\underset{(-1)}{P''}{}^{3,\;1}_{-3,-1}=(24+2D-17D^2-8D^3-D^4)\,\sigma^4 a' C_1$$

$$64\,\underset{(0)}{P''}{}^{2,\;2}_{-2,-2}=(12-8D-7D^2+2D^3+D^4)\,\sigma^4 a' C_0$$

$$96\,\underset{(1)}{P''}{}^{1,\;3}_{-1,-3}=(12-16D+D^2+4D^3-D^4)\,\sigma^4 a' C_1$$

$$384\,\underset{(2)}{P''}{}^{0,\;4}_{0,-4}=(24-50D+35D^2-10D^3+D^4)\,\sigma^4 a' C_2$$

The expressions to which the preceding operations are applied are derived from those found on p. 27 of this volume, and are as follow:

$$a' A_0 = b_1^{(0)} - \sigma^2 \alpha b_3^{(1)} + \frac{3}{4}\sigma^4 (2\alpha^2 b_5^{(0)} + \alpha^2 b_5^{(2)}) - \frac{5}{8}\sigma^6 (9\alpha^3 b_7^{(1)} + \alpha^3 b_7^{(3)})$$

$$+ \frac{35}{64}\sigma^8 (18\alpha^4 b_9^{(0)} + 16\alpha^4 b_9^{(2)} + \alpha^4 b_9^{(4)}) - \text{etc.}$$

$$a' A_1 = b_1^{(1)} - \frac{1}{2}\sigma^2 (\alpha b_3^{(0)} + \alpha b_3^{(2)}) + \frac{3}{8}\sigma^4 (5\alpha^2 b_5^{(1)} + \alpha^2 b_5^{(3)})$$

$$- \frac{5}{16}\sigma^6 (9\alpha^3 b_7^{(0)} + 10\alpha^3 b_7^{(2)} + \alpha^3 b_7^{(4)}) + \text{etc.}$$

$$a' A_2 = b_1^{(2)} - \frac{1}{2}\sigma^2 (\alpha b_3^{(1)} + \alpha b_3^{(3)}) + \frac{3}{8}\sigma^4 (\alpha^2 b_5^{(0)} + 4\alpha^2 b_5^{(2)} + \alpha^2 b_5^{(4)}) - \text{etc.}$$

$$a' A_3 = b_1^{(3)} - \frac{1}{2}\sigma^2 (\alpha b_3^{(2)} + \alpha b_3^{(4)}) + \text{etc.}$$

$$a' B_0 = \frac{1}{2}\alpha b_3^{(0)} - \frac{3}{2}\sigma^2 \alpha^2 b_5^{(1)} + \frac{15}{16}\sigma^4 (3\alpha^3 b_7^{(0)} + 2\alpha^3 b_7^{(2)}) - \text{etc.}$$

$$a' B_1 = \frac{1}{2}\alpha b_3^{(1)} - \frac{3}{4}\sigma^2 (\alpha^2 b_5^{(0)} + \alpha^2 b_5^{(2)}) + \frac{15}{16}\sigma^4 (4\alpha^3 b_7^{(1)} + \alpha^3 b_7^{(3)}) - \text{etc.}$$

$$a' B_2 = \frac{1}{2}\alpha b_3^{(2)} - \frac{3}{4}\sigma^2 (\alpha^2 b_5^{(1)} + \alpha^2 b_5^{(3)}) + \frac{15}{16}\sigma^4 (\alpha^3 b_7^{(0)} + 3\alpha^3 b_7^{(2)} + \alpha^3 b_7^{(4)}) - \text{etc.}$$

$$a' C_0 = \frac{3}{8}\alpha^2 b_5^{(0)} - \frac{15}{8}\sigma^2 \alpha^3 b_7^{(1)} + \text{etc.}$$

$$a' C_1 = \frac{3}{8}\alpha^2 b_5^{(1)} - \frac{15}{16}\sigma^2 (\alpha^3 b_7^{(0)} + \alpha^3 b_7^{(2)}) + \text{etc.}$$

$$a' C_2 = \frac{3}{8}\alpha^2 b_5^{(2)} - \frac{15}{16}\sigma^2 (\alpha^3 b_7^{(1)} + \alpha^3 b_7^{(3)}) + \text{etc.}$$

The symbolic operators D indicate derivatives as to the logarithm of α, the formation of which is discussed in preceding papers of this series, especially Vol. III, and Part I, p. 6, of the present volume. It has hitherto been more common to express this class of functions by the derivatives of A, B, etc., as to α simply. By the following relations between the two classes of derivatives the one form of derivative may be reduced to the other.

I. *Derivatives D operating on $b_n^{(i)}$ in terms of $D_a\, b_n^{(i)}$.*

$$D = \alpha D_a$$
$$D^2 = \alpha D_a + \alpha^2 D_a^2$$
$$D^3 = \alpha D_a + 3\alpha^2 D_a^2 + \alpha^3 D_a^3$$
$$D^4 = \alpha D_a + 7\alpha^2 D_a^2 + 6\alpha^3 D_a^3 + \alpha^4 D_a^4$$
$$D^5 = \alpha D_a + 15\alpha^2 D_a^2 + 25\alpha^3 D_a^3 + 10\alpha^4 D_a^4 + \alpha^5 D_a^5$$
$$D^6 = \alpha D_a + 31\alpha^2 D_a^2 + 90\alpha^3 D_a^3 + 65\alpha^4 D_a^4 + 15\alpha^5 D_a^5 + \alpha^6 D_a^6$$
$$D^7 = \alpha D_a + 63\alpha^2 D_a^2 + 301\alpha^3 D_a^3 + 350\alpha^4 D_a^4 + 140\alpha^5 D_a^5 + 21\alpha^6 D_a^6 + \alpha^7 D_a^7$$
$$D^8 = \alpha D_a + 127\alpha^2 D_a^2 + 966\alpha^3 D_a^3 + 1701\alpha^4 D_a^4 + 1050\alpha^5 D_a^5 + 266\alpha^6 D_a^6 + 28\alpha^7 D_a^7 + \alpha^8 D_a^8$$

II. *Derivatives D operating on $\alpha\, b_n^{(i)}$*

$$D\,(\alpha b_n^{(i)}) = \alpha b_n^{(i)} + \alpha^2 D_a b_n^{(i)}$$
$$D^2(\alpha b_n^{(i)}) = \alpha b_n^{(i)} + 3\alpha^2 D_a b_n^{(i)} + \alpha^3 D_a^2 b_n^{(i)}$$
$$D^3(\alpha b_n^{(i)}) = \alpha b_n^{(i)} + 7\alpha^2 D_a b_n^{(i)} + 6\alpha^3 D_a^2 b_n^{(i)} + \alpha^4 D_a^3 b_n^{(i)}$$
$$D^4(\alpha b_n^{(i)}) = \alpha b_n^{(i)} + 15\alpha^2 D_a b_n^{(i)} + 25\alpha^3 D_a^2 b_n^{(i)} + 10\alpha^4 D_a^3 b_n^{(i)} + \alpha^5 D_a^4 b_n^{(i)}$$

etc. etc. etc. etc etc.

the coefficients which give D^i being the same as those which give D^{i+1} in I.

III. *Derivatives D operating on $\alpha^2\, b_n^{(i)}$.*

$$D\,(\alpha^2 b_n^{(i)}) = 2\alpha^2 b_n^{(i)} + \alpha^3 D_a b_n^{(i)}$$
$$D^2(\alpha^2 b_n^{(i)}) = 4\alpha^2 b_n^{(i)} + 5\alpha^3 D_a b_n^{(i)} + \alpha^4 D_a^2 b_n^{(i)}$$
$$D^3(\alpha^2 b_n^{(i)}) = 8\alpha^2 b_n^{(i)} + 19\alpha^3 D_a b_n^{(i)} + 9\alpha^4 D_a^2 b_n^{(i)} + \alpha^5 D_a^3 b_n^{(i)}$$
$$D^4(\alpha^2 b_n^{(i)}) = 16\alpha^2 b_n^{(i)} + 65\alpha^3 D_a b_n^{(i)} + 55\alpha^4 D_a^2 b_n^{(i)} + 14\alpha^5 D_a^3 b_n^{(i)} + \alpha^6 D_a^4 b_n^{(i)}$$

§ 3.

FORMATION AND EXPRESSION OF THE REQUIRED FUNCTIONS OF THE RATIO OF THE MEAN DISTANCES.

In the preceding expressions the function $b_n^{(i)}$ is defined as the coefficient of $\cos i\,L$ in the development

$$\left(1 - 2\alpha\cos L + \alpha^2\right)^{-\frac{n}{2}} = \frac{1}{2}\sum_{i=-\infty}^{i=+\infty} b_n^{(i)} \cos i\,L$$

and may be developed in powers of α^2 in the form

$$\frac{1}{2} b_n^{(i)} = \frac{n(n+2)\ \ .\ \ .\ \ .\ \ (n+2i-2)}{2.4.6\ \ .\ \ .\ \ .\ \ 2i}\,\alpha^i\, F\left(\frac{n}{2}, \frac{n}{2}+i, i+1, \alpha^2\right)$$

To form the derivatives D of these functions I have, in preceding papers,* introduced the auxiliary function

$$\frac{1}{2} b_n^{(i,j)} = n(n+2) \; . \; . \; . \; (n+2j-2) \frac{n(n+2) \; . \; . \; . \; (n+2i+2j-2)}{2.4.6 \; . \; . \; . \; . \; 2(i+j)}$$

$$\times \alpha^{i+2j} F\left(\frac{n}{2}+j, \frac{n}{2}+i+j, i+j+1, \alpha^2\right)$$

It will be noticed that, when $j=0$, $b_n^{(i,j)}$ reduces to $b_n^{(i)}$, the latter function is, therefore, a special case of the former one.

In all the quantities we have to form each of these functions is always multiplied by the factor $\alpha^{\frac{n-1}{2}}$. We shall therefore put, hereafter

$$c_n^{(i,j)} = \alpha^{\frac{n-1}{2}} b_n^{(i,j)},$$

or, for special values of n

$$c_1^{(i,j)} = b_1^{(i,j)}$$

$$c_3^{(i,j)} = \alpha b_3^{(i,j)}$$

$$c_5^{(i,j)} = \alpha^2 b_5^{(i,j)}$$

etc., etc.

We shall also express the term $a'A$, $a'B$, etc., which is independent of σ, the coefficient of σ^2, of σ^4, etc., by the notation

$$a'A_i(0), \quad a'A_i(1), \quad a'A_i(2), \quad \text{etc.},$$

so that the complete value of this quantity will be

$$a'A_i = a'A_i(0) - \sigma^2 a'A_i(1) + \sigma^4 a'A_i(2) - \; . \; . \; .$$

The functions of α required in the work will then be the following

$$a'A_0(0) = b^{(0)} = c_1^{(0)}$$

$$a'A_0(1) = c_3^{(1)}$$

$$a'A_0(2) = \frac{3}{2} c_5^{(0)} + \frac{3}{4} c_5^{(2)}$$

$$a'A_0(3) = \frac{45}{8} c_7^{(1)} + \frac{5}{8} c_7^{(3)}$$

$$a'A_0(4) = \frac{315}{32} c_9^{(0)} + \frac{35}{4} c_9^{(2)} + \frac{35}{64} c_9^{(4)}$$

$$a'A_1(0) = b_1^{(1)} = c_1^{(1)}$$

* Vol. III, Part I; Vol. V, Part I. In Vol. III, Part V, the headings of the columns which give the numerical values of $b_3^{(i,j)}$ and $b_5^{(i,j)}$ should have the respective factors α and α^2, the actual quantities being $c_3^{(i,j)}$ and $c_5^{(i,j)}$.

$$a'\, A_1(1)=\frac{1}{2}\, c_3^{(0)}+\frac{1}{2}\, c_3^{(2)}$$

$$a'\, A_1(2)=\frac{15}{8} c_5^{(1)}+\frac{3}{8}\, c_5^{(3)}$$

$$a'\, A_1(3)=\frac{45}{16} c_7^{(0)}+\frac{25}{8}\, c_7^{(2)}+\frac{5}{16}\, c_7^{(4)}$$

$$a'\, A_2(0)=b_1^{(2)}=c_1^{(2)}$$

$$a'\, A_2(1)=\frac{1}{2}\, c_3^{(1)}+\frac{1}{2}\, c_3^{(3)}$$

$$a'\, A_2(2)=\frac{3}{8}\, c_5^{(0)}+\frac{3}{2}\, c_5^{(2)}+\frac{3}{8}\, c_5^{(4)}$$

$$a'\, A_3(0)=b_1^{(3)}=c_1^{(3)}$$

$$a'\, A_3(1)=\frac{1}{2}\, c_3^{(2)}+\frac{1}{2}\, c_3^{(4)}$$

$$a'\, B_0(0)=\frac{1}{2}\, c_3^{(0)}$$

$$a'\, B_0(1)=\frac{3}{2}\, c_5^{(1)}$$

$$a'\, B_0(2)=\frac{45}{16} c_7^{(0)}+\frac{15}{8}\, c_7^{(2)}$$

$$a'\, B_1(0)=\frac{1}{2}\, c_3^{(1)}$$

$$a'\, B_1(1)=\frac{3}{4} c_5^{(0)}+\frac{3}{4} c_5^{(2)}$$

$$a'\, B_1(2)=\frac{15}{4} c_7^{(1)}+\frac{15}{16} c_7^{(3)}$$

$$a'\, B_2(0)=\frac{1}{2}\, c_3^{(2)}$$

$$a'\, B_2(1)=\frac{3}{4} c_5^{(1)}+\frac{3}{4} c_5^{(3)}$$

$$a' \, B_2(2) = \frac{15}{16} c_7^{(0)} + \frac{45}{16} c_7^{(2)} + \frac{15}{16} c_7^{(4)}$$

$$a' \, C_0(0) = \frac{3}{8} c_5^{(0)}$$

$$a' \, C_0(1) = \frac{15}{8} c_7^{(1)}$$

$$a' \, C_1(0) = \frac{3}{8} c_5^{(1)}$$

$$a' \, C_1(1) = \frac{15}{16} c_7^{(0)} + \frac{15}{16} c_7^{(2)}$$

$$a' \, C_2(0) = \frac{3}{8} c_5^{(2)}$$

$$a' \, C_2(1) = \frac{15}{16} c_7^{(1)} + \frac{15}{16} c_7^{(3)}$$

The logarithmic derivatives D of these functions are also required. Each of these is a linear function of the $c_n^{(i,j)}$'s, whose coefficients are entire functions of i. For convenient reference the expressions for such of these derivatives as enter into our formulæ are here given in a condensed shape. To form each expression it is only necessary to write each symbol found at the head of a column in any one line after any symbol D, and after the respective numbers in the line with the D For example, we have from the first set of derivatives

$$D \, c_1^{(1)} = c_1^{(1)} + c_1^{(1,1)}$$

$$D^2 c_1^{(1)} = c_1^{(1)} + 4 c_1^{(1,1)} + c_1^{(1,}$$

etc. etc.

$$D \, c_3^{(0)} = c_3^{(0)} + c_3^{(0,1)}$$

etc. etc.

From the second set

$$D^4 c_1^{(0)} = 8 c_1^{(0.1)} + 28 c_1^{(0,2)} + 12 c_1^{(0,3)} + c_1^{(0,4)}$$

$$D^3 c_3^{(1)} = 8 c_3^{(1)} + 28 c_3^{(1,1)} + 12 c_3^{(1,2)} + c_3^{(1,3)}$$

Derivatives of $c_1^{(1)}$ and $c_3^{(0)}$.

$c_1^{(1)}$ $c_3^{(0)}$	$c_1^{(1)}$ $c_3^{(0)}$	$c_1^{(1,1)}$ $c_3^{(0,1)}$	$c_1^{(1,2)}$ $c_3^{(0,2)}$	$c_1^{(1,3)}$ $c_3^{(0,3)}$	$c_1^{(1,4)}$ $c_3^{(0,4)}$	$c_1^{(1,5)}$ $c_3^{(0,5)}$	$c_1^{(1,6)}$ $c_3^{(0,6)}$
$D =$	1	1					
$D^2 =$	1	4	1				
$D^3 =$	1	13	9	1			
$D^4 =$	1	40	58	16	1		
$D^5 =$	1	121	330	170	25	1	
$D^6 =$	1	364	1771	1520	395	36	1

Derivatives of $c_1^{(0)}$, $c_1^{(2)}$, $c_3^{(1)}$, and $c_5^{(0)}$.

$c_1^{(2)}$ $c_3^{(1)}$ $c_5^{(0)}$ $c_1^{(0)}$	$c_1^{(2)}$ $c_3^{(1)}$ $c_5^{(0)}$ $c_1^{(0,1)}$	$c_1^{(2,1)}$ $c_3^{(1,1)}$ $c_5^{(0,1)}$ $c_1^{(0,2)}$	$c_1^{(2,2)}$ $c_3^{(1,2)}$ $c_5^{(0,2)}$ $c_1^{(0,3)}$	$c_1^{(2,3)}$ $c_3^{(1,3)}$ $c_5^{(0,3)}$ $c_1^{(0,4)}$	$c_1^{(2,4)}$ $c_3^{(1,4)}$ $c_5^{(0,4)}$ $c_1^{(0,5)}$	$c_1^{(2,5)}$ $c_3^{(1,5)}$ $c_5^{(0,5)}$ $c_1^{(0,6)}$	$c_1^{(2,6)}$ $c_3^{(1,6)}$ $c_5^{(0,6)}$ $c_1^{(0,7)}$	$c_1^{(2,7)}$ $c_3^{(1,7)}$ $c_5^{(0,7)}$ $c_1^{(0,8)}$
$D =$	1							
$D,\ D^2 =$	2	1						
$D^2,\ D^3 =$	4	6	1					
$D^3,\ D^4 =$	8	28	12	1				
$D^4,\ D^5 =$	16	120	100	20	1			
$D^5,\ D^6 =$	32	496	720	260	30	1		
$D^6,\ D^7 =$	64	2016	4816	2800	560	42	1	
$D^7,\ D^3 =$	128	8128	30 912	27 216	8400	1064	56	1

Derivatives of $c_1^{(3)}$, $c_3^{(2)}$, $c_5^{(1)}$, and $c_7^{(0)}$.

$c_1^{(3)}$ $c_3^{(2)}$ $c_5^{(1)}$ $c_7^{(0)}$	$c_1^{(3)}$ $c_3^{(2)}$ $c_5^{(1)}$ $c_7^{(0)}$	$c_1^{(3,1)}$ $c_3^{(2,1)}$ $c_5^{(1,1)}$ $c_7^{(0,1)}$	$c_1^{(3,2)}$ $c_3^{(2,2)}$ $c_5^{(1,2)}$ $c_7^{(0,2)}$	$c_1^{(3,3)}$ $c_3^{(2,3)}$ $c_5^{(1,3)}$ $c_7^{(0,3)}$	$c_1^{(3,4)}$ $c_3^{(2,4)}$ $c_5^{(1,4)}$ $c_7^{(0,4)}$	$c_1^{(3,5)}$ $c_3^{(2,5)}$ $c_5^{(1,5)}$ $c_7^{(0,5)}$	$c_1^{(3,6)}$ $c_3^{(2,6)}$ $c_5^{(1,6)}$ $c_7^{(0,6)}$
$D =$	3	1					
$D^2 =$	9	8	1				
$D^3 =$	27	49	15	1			
$D^4 =$	81	272	154	24	1		
$D^5 =$	243	1441	1350	370	35	1	
$D^6 =$	729	7448	10 891	4680	755	48	1

Derivatives of $c_5^{(2)}$, $c_7^{(1)}$, and $c_9^{(0)}$.

$c_5^{(2)}$ $c_7^{(1)}$ $c_9^{(0)}$	$c_5^{(2)}$ $c_7^{(1)}$ $c_9^{(0)}$	$c_5^{(2,1)}$ $c_7^{(1,1)}$ $c_9^{(0,1)}$	$c_5^{(2,2)}$ $c_7^{(1,2)}$ $c_9^{(0,2)}$	$c_5^{(2,3)}$ $c_7^{(1,3)}$ $c_9^{(0,3)}$	$c_5^{(2,4)}$ $c_7^{(1,4)}$ $c_9^{(0,4)}$	$c_5^{(2,5)}$ $c_7^{(1,5)}$ $c_9^{(0,5)}$	$c_5^{(2,6)}$ $c_7^{(1,6)}$ $c_9^{(0,6)}$
$D =$	4	1					
$D^2 =$	16	10	1				
$D^3 =$	64	76	18	1			
$D^4 =$	256	520	220	28	1		
$D^5 =$	1024	3376	2280	500	40	1	
$D^6 =$	4096	21 280	21 616	7280	980	54	1

Derivatives of $c_5^{(3)}$, $c_7^{(2)}$, and $c_9^{(1)}$.

$c_5^{(3)}$ $c_7^{(2)}$ $c_9^{(1)}$	$c_5^{(3)}$ $c_7^{(2)}$ $c_9^{(1)}$	$c_5^{(3,1)}$ $c_7^{(2,1)}$ $c_9^{(1,1)}$	$c_5^{(3,2)}$ $c_7^{(2,2)}$ $c_9^{(1,2)}$	$c_5^{(3,3)}$ $c_7^{(2,3)}$ $c_9^{(1,3)}$	$c_5^{(3,4)}$ $c_7^{(2,4)}$ $c_9^{(1,4)}$
$D =$	5	1			
$D^2 =$	25	12	1		
$D^3 =$	125	109	21	1	
$D^4 =$	625	888	298	32	1

Derivatives of $c_7^{(3)}$ and $c_9^{(2)}$.

$c_7^{(3)}$ $c_9^{(2)}$	$c_7^{(3)}$ $c_9^{(2)}$	$c_7^{(3,1)}$ $c_9^{(2,1)}$	$c_7^{(3,2)}$ $c_9^{(2,2)}$	$b_7^{(3,3)}$ $b_9^{(2,3)}$	$b_7^{(3,4)}$ $b_9^{(2,4)}$
$D =$	6	1			
$D^2 =$	36	14	1		
$D^3 =$	216	148	24	1	
$D^4 =$	1296	1400	388	36	1

The law of formation is now obvious, and gives

$$
\begin{aligned}
D\, c_9^{(3)} &= 7\, c_9^{(3)} + c_9^{(3,1)} \\
D^2\, c_9^{(3)} &= 49\, c_9^{(3)} + 16\, c_9^{(3,1)} + c_9^{(3,2)} \\
D\, c_9^{(4)} &= 8\, c_9^{(4)} + c_9^{(4,1)} \\
D^2\, c_9^{(4)} &= 64\, c_9^{(4)} + 18\, c_9^{(4,1)} + c_9^{(4,2)}
\end{aligned}
$$

while the expressions for the derivatives of $c_7^{(4)}$ and $c_7^{(5)}$ have the same coefficients as those of $c_9^{(3)}$ and $c_9^{(4)}$, respectively.

We might substitute these expressions as they stand in the values of P, P′, and P″, already given, and thus express each of these quantities as a linear function of the $c_n^{(i,j)}$'s, thus leading to their explicit expressions as linear functions of certain GAUSSIAN functions. Thus, for so much of the first term of F, as is independent of σ, we have

$$4\,P^{2,0}_{0,0}(0)=4\,P^{0,2}_{0,0}=3\,b_1^{(0,1)}+b_1^{(0,2)}$$
$$64\,P^{4,0}_{0,0}(0)=15\,b_1^{(0,2)}+10\,b_1^{(0,3)}+b_1^{(0,4)}$$
$$16\,P^{2,2}_{0,0}(0)=18\,b_1^{(0,1)}+41\,b_1^{(0,2)}+14\,b_1^{(0,3)}+b_1^{(0,4)}$$
$$64\,P^{0,4}_{0,0}(0)=60\,b_1^{(0,1)}+75\,b_1^{(0,2)}+18\,b_1^{(0,3)}+b_1^{(0,4)}$$
$$2304\,P^{6,0}_{0,0}(0)=105\,b_1^{(0,3)}+105\,b_1^{(0,4)}+21\,b_1^{(0,5)}+b_1^{(0,6)}$$
$$256\,P^{4,2}_{0,0}(0)=300\,b_1^{(0,2)}+585\,b_1^{(0,3)}+237\,b_1^{(0,4)}+29\,b_1^{(0,5)}+b_1^{(0,6)}$$
$$256\,P^{2,4}_{0,0}(0)=360\,b_1^{(0,1)}+1920\,b_1^{(0,2)}+1641\,b_1^{(0,3)}+417\,b_1^{(0,4)}+37\,b_1^{(0,5)}+b_1^{(0,6)}$$
$$2304\,P^{0,6}_{0,0}(0)=2520\,b_1^{(0,1)}+6300\,b_1^{(0,2)}+3465\,b_1^{(0,3)}+645\,b_1^{(0,4)}+45\,b_1^{(0,5)}+b_1^{(0,6)}$$
$$147\,456\,P^{8,0}_{0,0}(0)=945\,b_1^{(0,4)}+1260\,b_1^{(0,5)}+378\,b_1^{(0,6)}+36\,b_1^{(0,7)}+b_1^{(0,8)}$$
$$147\,456\,P^{0,8}_{0,0}(0)=181\,440\,b_1^{(0,1)}+740\,880\,b_1^{(0,2)}+687\,960\,b_1^{(0,3)}+235\,305\,b_1^{(0,4)}$$
$$+35\,700\,b_1^{(0,5)}+2562\,b_1^{(0,6)}+84\,b_1^{(0,7)}+b_1^{(0,8)}$$

For the coefficients of σ^2 we have

$$4\,P^{2,0}_{0,0}(1)=6\,\alpha\,b_3^{(1)}+7\,\alpha\,b_3^{(1,1)}+\alpha\,b_3^{(1,2)}$$

. . . .

For the purposes of numerical computation it is, however, simpler to form the numerical values of the derivatives D, D^2, etc., and to use these values in the computation of the quantities P, P′, etc. The values of the functions $c_n^{(i,j)}$ required for this purpose are found in Vol. III, Part VI, for the action of all the planets as far as Jupiter, inclusive. But these values extend only to terms of the fourth order in the mutual inclinations of the two orbits, and of the sixth in the eccentricities. For the required quantities to terms of the eighth order we need also the quantities

$$D_7 c_1^{(0)};\ D^8 c_1^{(0)};\ D^6 c_3^{(1)};\ D^4 c_5^{(0)};\ D^4 c_5^{(2)}$$

$$(1, D, D^2)\,(c_7^{(1)}, c_7^{(3)});\ c_9^{(0)};\ c_9^{(2)};\ c_9^{(4)}$$

The functions $c_n^{(i,j)}$ which are required for these derivatives are those having corresponding values of i and n, and values of j extending in each case from 0 to the highest exponent of D. The numerical values up to $c_1^{(i,6)}$, $c_3^{(i,6)}$ and $c_5^{(i,5)}$ are found in Vol. III.* The additional ones now necessary may be computed by the methods

* It should be noted that, through inadvertence, the headings of the columns in which these numbers are found do not contain the factors $\alpha^{\frac{n-1}{2}}$ on pp. 410, 433, etc. In column $b_3^{(i,j)}$ the quantities actually given are the values of $\alpha\,b_3^{(i,j)}$ which we at present call $c_3^{(i,j)}$ and under $b_5^{(i,j)}$ are given the values of $\alpha^2\,b_5^{(i,j)}$ which we now call $c_5^{(i,j)}$

developed in Part I of that volume, or by some of the known relations of GAUSS among contiguous hypergeometric series. For $n=1$, 3 or 5 and higher values of j, we require only the relations cited in Vol. III, p. 73. For $n=1$ they are

$$(1-\alpha^2)\, b_1^{(0,7)} = 121\, \alpha^2\, b_1^{(0,5)} - 12\,(1-2\,\alpha^2)\, b_1^{(0,6)}$$

$$(1-\alpha^2)\, b_1^{(0,8)} = 169\, \alpha^2\, b_1^{(0,6)} - 14\,(1-2\,\alpha^2)\, b_1^{(0,7)}$$

The quantities for $n=7$ and $n=9$ may be obtained by the GAUSSIAN relations, thus: If from the general formula

$$b_n^{(i,j)} = 2\,n\,(n+2)\,(n+4)\;.\;\;.\;\;.\;(n+2\,j-2)\left[\frac{n+2\,i+2\,j-2}{i+j}\right]\alpha^{i+2j}$$

$$\times \mathrm{F}\left(\frac{n}{2}+j,\frac{n}{2}+i+j,i+j+1,\alpha^2\right)$$

we form the expressions for the three quantities

$$b_{n+2}^{(i,j)};\quad b_n^{(i,j+1)};\quad b_n^{(i+1,j)}$$

we find that, using the GAUSSIAN notation, and to avoid confusion, putting x for what in our own notation is called α, the expressions may be written in the form

$$b_{n+2}^{(i,j)} = \mathrm{P}_1\, x^{i+2j} \cdot \mathrm{F}\,(\alpha,\beta,\gamma-1,x^2)$$

$$b_n^{(i,j+1)} = \mathrm{P}_2\, x^{i+2j+2} \cdot \mathrm{F}\,(\alpha,\beta,\gamma,x^2)$$

$$b_n^{(i+1,j)} = \mathrm{P}_3\, x^{i+2j+1}\, \mathrm{F}\,(\alpha-1,\beta,\gamma,x^2)$$

where we have

$$\alpha = \frac{n}{2}+j+1$$

$$\beta = \frac{n}{2}+i+j+1$$

$$\gamma = i+j+2$$

while P_1, P_2, and P_3 are factorials so related that

$$n^2\,\mathrm{P}_1 = 2\,(i+j+1)\,\mathrm{P}_2 = 2\,(n+2\,j)\,(i+j+1)\,\mathrm{P}_3.$$

Between the three functions F exists the linear relation

$$\left(\alpha-1-(\gamma-\beta-1)x^2\right)\mathrm{F}_2 + (\gamma-\alpha)\,\mathrm{F}_3 - (\gamma-1)\,(1-x^2)\,\mathrm{F}_1 = 0$$

F_1, F_2, and F_3 being the three functions taken in order. Substituting for α, β, γ and x their values in our own notation, this relation may be put into the form

$$(i+j+1)(1-\alpha^2)F_1=\left(\frac{n}{2}+j+\frac{n}{2}\alpha^2\right)F_2+\left(i+1-\frac{n}{2}\right)F_3$$

By (*c*) the expressions for F_1, F_2, and F_3 in terms of the b's may be written

$$P\,F_1=n^2\alpha^2 b_{n+2}^{(i,j)}$$

$$P\,F_2=2\,(i+j+1)\,b_n^{(i,j+1)}$$

$$P\,F_3=2\,(i+j+1)(n+2j)\,\alpha\, b_n^{(i+1,j)}$$

P being a quantity unnecessary to write, because it will divide out of the equation. The above relation thus becomes

$$n^2\alpha^2(1-\alpha^2)\,b_{n+2}^{(i,j)}=(n+2j+n\alpha^2)\,b_n^{(i,j+1)}+(n+2j)(2i+2-n)\,\alpha\, b_n^{(i+1,i)}$$

Hence, the coefficients actually required may be obtained by the general formulæ

$$\alpha^{\frac{n+1)}{2}}\,b_{n+2}^{(i,j)}=\frac{n+2j+n\alpha^2}{n^2\alpha(1-\alpha^2)}\,\alpha^{\frac{n-1}{2}}\,b_n^{(i,j+1)}+\frac{(n+2j)(2i+2-n)}{n^2(1-\alpha^2)}\,\alpha^{\frac{n-1}{2}}\,b_n^{(i+1,j)}$$

which, using the new notation, may be written

$$c_{n+2}^{(i,j)}=\frac{n+2j+n\alpha^2}{n^2\alpha(1-\alpha^2)}\,c_n^{(i,j+1)}+\frac{(n+2j)(2i+2-n)}{n^2(1-\alpha^2)}\,c_n^{(i+1,j)}$$

To guard against error several of the quantities in question have also been computed by the general method of Vol. III. The results are shown in tabular form in § 6.

§ 4.

DEVELOPMENT OF THE FUNCTIONS P, P′ AND P″, IN POWERS OF α.

The quantities $P_{j,\,j'}^{n,\,n'}$, etc., are functions of α, the ratio of the mean distances, and can each be developed in a series proceeding according to the powers of α, which will always be convergent when $\alpha < 1$, though the first terms of the series may often seem to diverge. The method of development is as follows:

Firstly. We have the well-known expressions of each $b_n^{(i)}$ as the product of a hypergeometric series, whose fourth term is α^2, by a simple factorial, dependent on n, i and α

Secondly. By these expressions the values of $a'\,A_i$, $a'\,B_i$, etc., become at once developable in powers of α by simple substitution in those expressions. Let

$$n_k\,\alpha^k$$

be any term of this development. We shall then have for any of the expressions $a'\,A_i$, $a'\,B_i$, etc., the series

$$\Sigma\, n_k\,\alpha^k$$

where the values of k will be either

$$0, 2, 4, \quad . \quad . \quad . \quad \infty$$

or

$$1, 3, 5, \quad . \quad . \quad . \quad \infty$$

The laws of the series are such that those for $a'\,A_i$ and $a'\,C_i$ will contain only even values of k for even values of i, and odd values of k for odd values of i, while in the case of $a'\,B_i$ this is reversed. Now let us put

K, any one of the quantities $a'\,A_i$, $a'\,B_i$, etc.

P, any one of the coefficients P, P′, or P″.

Let

$$\varphi\,(\mathrm{D})\,\mathrm{K}$$

be the expression for any P as a linear function of K and its derivatives, found in § 3. Since

$$\mathrm{D}^n\,\alpha^k = k^n\,\alpha^k$$

it follows that from any such series as

$$\mathrm{K} = \Sigma\, n_k\,\alpha^k$$

we shall have, as the series for P,

$$\mathrm{P} = \Sigma\, n_k\,\varphi\,(k)\,\alpha^k$$

The values of $a'\,A_i$, etc., are shown in the following exhibit. Under each value of α^k is shown the coefficient by which it must be multiplied to form the quantity whose symbolic expression is found at the left. If we represent these coefficients by n_k^0, n'_k, etc., the complete value of any K will be

$$\mathrm{K} = \Sigma_k\,(n_k^0 - n'_k\,\sigma^2 + n''_k\,\sigma^4 - \text{etc.} \quad . \quad .)\,\alpha^k$$

For example, the series for $a'\,A_0$ are

$$a'A_0(0) = \frac{1}{2}\,\alpha^2 + \frac{9}{2^5}\,\alpha^4 + \frac{25}{2^7}\,\alpha^6 + \quad . \quad . \quad .$$

$$a'A_0(1) = 3\,\alpha^2 + \frac{45}{2^3}\,\alpha^4 + \frac{525}{2^6}\,\alpha^6 + \quad . \quad . \quad .$$

$$a'A_0(2) = 3\,\alpha^2 + \frac{405}{2^4}\,\alpha^4 + \frac{2625}{2^5}\,\alpha^6 + \quad . \quad . \quad .$$

$$. \qquad . \qquad . \qquad .$$

$$. \qquad . \qquad . \qquad .$$

We then have

$$a'A_0 = a'A_0(0) - \sigma^2\,a'A_0(1) + \sigma^4\,a'A_0(2) - \quad . \quad . \quad .$$

To form the value of any P, P′, or P″ found at the left we suppose it thrown into the form

$$P = P(0) - \sigma^2 P(1) + \sigma^4 P(2) + \quad . \quad . \quad .$$

and the values of P (o) etc. are formed by multiplying each factor found in the same line with it by the factor above it in the line having the same index. We have, for example,

$$P^{2,0}_{0,0}(0) = \frac{3}{2} \cdot \frac{1}{2}\alpha^2 + \frac{5}{1} \cdot \frac{9}{2^5}\alpha^4 + \frac{21}{2} \cdot \frac{25}{2^7}\alpha^6 + \quad . \quad .$$

$$P^{2,0}_{0,0}(1) = \frac{3}{2} \cdot \frac{3}{1}\alpha^2 + \frac{5}{1} \cdot \frac{45}{2^8}\alpha^4 + \frac{21}{2} \cdot \frac{525}{2^6}\alpha^6 + \quad . \quad .$$

$$P^{2,4}_{0,0}(2) = \frac{45}{2^4} \cdot \frac{3}{1}\alpha^2 + \frac{525}{2^3} \cdot \frac{405}{2^4}\alpha^4 + \quad . \quad .$$

etc. etc. etc.

Values of the coefficients $a'A$, $a'B$, etc., developed in powers of α, and of the factors $\varphi(k)$

	α^2	α^4	α^6	α^8	α^{10}	α^{12}
$a' A_0(0) =$	$1:2$	$9:2^5$	$25:2^7$	$1225:2^{13}$	$3969:2^{15}$	$53\,361:2^{19}$
$a' A_0(1) =$	$3:1$	$45:2^3$	$525:2^6$	$11\,025:2^{10}$	$218\,295:2^{14}$	
$a' A_0(2) =$	$3:1$	$405:2^4$	$2625:2^5$	$385\,875:2^{11}$	$5\,893\,965:2^{14}$	
$a' A_0(3) =$	0	$315:2^3$	$2625:2^3$	$1\,414\,875:2^{10}$	$8\,513\,505:2^{11}$	

Factors for—

$P^{2,0}_{0,0} = P^{0,2}_{0,0}$;	$3:2$	$5:1$	$21:2$	$18:1$	$55:2$	$39:1$
$P^{4,0}_{0,0}$;	0	$15:2^3$	$105:2^3$	$189:2^2$	$495:2^2$	$2145:2^3$
$P^{2,2}_{0,0}$;	$9:2^2$	$25:1$	$441:2^2$	$324:1$	$3025:2^2$	$1521:1$
$P^{0,4}_{0,0}$;	$15:2^3$	$105:2^3$	$189:2^2$	$495:2^2$	$2145:2^3$	$4095:2^3$
$P^{6,0}_{0,0}$;	0	0	$35:2^4$	$105:2^2$	$1155:2^3$	
$P^{4,2}_{0,0}$;	0	$75:2^3$	$2205:2^4$	$1701:2$	$27\,225:2^3$	
$P^{2,4}_{0,0}$;	$45:2^4$	$525:2^3$	$3969:2^3$	$4455:2$	$117\,975:2^4$	
$P^{0,6}_{0,0}$;	$35:2^4$	$105:2^2$	$1155:2^3$	$2145:2^2$	$25\,025:2^4$	

	α	α^3	α^5	α^7	α^9	α^{11}
$a' A_1(0) =$	$1:1$	$3:2^3$	$15:2^6$	$175:2^{10}$	$2205:2^{14}$	$14\,553:2^{17}$
$a' A_1(1) =$	$1:1$	$33:2^3$	$435:2^6$	$9625:2^{10}$	$196\,245:2^{14}$	$1\,906\,443:2^{17}$
$a' A_1(2) =$	0	$75:2^3$	$735:2^4$	$127\,575:2^{10}$	$266\,805:2^{10}$	$20\,495\,475:2^{17}$
$a' A_1(3) =$	0	$45:2^3$	$945:2^3$	$695\,625:2^{10}$	$4\,923\,765:2^{11}$	$854\,188\,335:2^{17}$

Factors for—

$P^{1,1}_{-1,1};$	0	$-5:2$	$-7:1$	$-27:2$	$-22:1$	$-65:2$
$P^{3,1}_{-1,1};$	0	$-15:2^3$	$-35:2$	$-567:2^3$	$-198:1$	$-3575:2^3$
$P^{1,3}_{-1,1};$	0	$-25:2^2$	$-147:2^2$	$-243:2$	$-605:2$	$-2535:2^2$
$P^{5,1}_{-1,1};$	0	0	$-35:2^3$	$-945:2^4$	$-693:2$	
$P^{3,3}_{-1,1};$	0	$-75:2^4$	$-735:2^3$	$-5103:2^3$	$-5445:2$	
$P^{1,5}_{-1,1};$	0	$-175:2^4$	$-441:2^2$	$-4455:2^3$	$-7865:2^2$	

	α^2	α^4	α^6	α^8	α^{10}
$a' A_2(0) =$	$3:2^2$	$5:2^4$	$105:2^9$	$315:2^{11}$	$8085:2^{16}$
$a' A_2(1) =$	$3:2$	$5:1$	$1995:2^8$	$5355:2^9$	$428\,505:2^{15}$
$a' A_2(2) =$	$3:2^2$	$285:2^4$	$36\,015:2^9$	$353\,745:2^{11}$	$22\,290\,345:2^{16}$

Factors for—

$P^{2,2}_{-2,2};$	0	$63:2^4$	$45:2$	$1155:2^4$	$351:2$
$P^{4,2}_{-2,2};$	0	$63:2^5$	$75:2$	$8085:2^5$	$1053:1$
$P^{2,4}_{-2,2};$	0	$441:2^5$	$135:1$	$21\,175:2^5$	$4563:2$

	α	α^3	α^5	α^7	α^9	α^{11}
$a' A_3(0) =$	0	$5:2^3$	$35:2^7$	$189:2^{10}$	$1155:2^{13}$	$15\,015:2^{17}$
$a' A_3(1) =$	0	$15:2^3$	$735:2^7$	$8883:2^{10}$	$93\,555:2^{13}$	$1\,846\,845:2^{17}$

Factors for—

$P^{3,3}_{-3,3};$	0	0	$-21:2^2$	$-825:2^4$	$-1001:2^2$	$-6825:2^3$

	α	α^3	α^5	α^7	α^9	α^{11}
$a' B_0(0)=$	$1:1$	$9:2^2$	$225:2^6$	$1225:2^8$	$99\,225:2^{14}$	$480\,249:2^{16}$
$a' B_0(1)=$	0	$15:2$	$525:2^4$	$11\,025:2^7$	$363\,825:2^{11}$	$10\,405\,395:2^{15}$
$a' B_0(2)=$	0	$45:2^3$	$1575:2^4$	$275\,625:2^9$	$7\,640\,325:2^{12}$	$655\,539\,885:2^{17}$
Factors for—						
$P'^{1,\,1}_{-1-1}$;	0	$-5:2$	$-7:1$	$-27:2$	$-22:1$	$-65:2$
$P'^{3,\,1}_{-1-1}$;	0	$-15:2^3$	$-35:2$	$-567:2^3$	$-198:1$	$-3575:2^3$
$P'^{1,\,3}_{-1-1}$;	0	$-25:2^2$	$-147:2^2$	$-243:2$	$-605:2$	$-2535:2^2$

	α^2	α^4	α^6	α^8	α^{10}
$a' B_1(0)=$	$3:2$	$45:2^4$	$525:2^7$	$11\,025:2^{11}$	$218\,295:2^{15}$
$a' B_1(1)=$	$3:2$	$255:2^4$	$6825:2^7$	$253\,575:2^{11}$	$7\,785\,855:2^{15}$
$a' B_1(2)=$	0	$105:2^2$	$29\,925:2^7$	$2\,061\,675:2^{11}$	$50\,135\,085:2^{14}$
Factors for—					
$P'^{2,\,0}_{-2,\,0}$;	$5:2$	$21:2^2$	$9:1$	$55:2^2$	$39:2$
$P'^{4,\,0}_{-2,\,0}$;	0	$21:2^3$	$15:1$	$385:2^3$	$117:1$
$P'^{2,\,2}_{-2,\,0}$;	$15:2^2$	$105:2^2$	$189:2$	$495:2$	$2145:2^2$
$P'^{0,\,2}_{0,\,-2}$;	0	$3:2^2$	$5:2$	$21:2^2$	$9:1$
$P'^{2,\,2}_{0,\,-2}$;	0	$15:2^2$	$105:2^2$	$189:2$	$495:2$
$P'^{0,\,4}_{0,\,-2}$;	0	$21:2^3$	$15:1$	$385:2^3$	$117:1$

	α^3	α^5		α^9	α^{11}
$a' B_2(0)=$	$15:2^3$	$105:2^5$	$4725:2^{10}$	$24\,255:2^{12}$	$945\,945:2^{17}$
$a' B_2(1)=$	$15:2^2$	$105:2^2$	$39\,375:2^9$	$169\,785:2^{10}$	$19\,864\,845:2^{16}$
$a' B_2(2)=$	$15:2^3$	$4305:2^6$	$455\,175:2^{10}$	$13\,607\,055:2^{13}$	$606\,350\,745:2^{17}$
Factors for—					
$P'^{3,1}_{-3,1}$;	$-35:2^3$	$-21:1$	$-495:2^3$	$-143:1$	$-2275:2^3$
$P'^{1,\,3}_{1,-3}$	0	$-7:2^2$	$-45:2^2$	$-77:2$	$-195:2$

	α^2	α^4	α^6	α^8	α^{10}
$a'C_0(0)=$	$3:2^2$	$75:2^4$	$3675:2^8$	$33\,075:2^{10}$	$4\,002\,075:2^{16}$
$a'C_0(1)=$	0	$105:2^3$	$6615:2^6$	$218\,295:2^9$	$10\,405\,395:2^{13}$
Factors for—					
$P''^{2,\ 2}_{-2,-2};$	0	$63:2^4$	$45:2$	$1155:2^4$	$351:2$

	α^3	α^5	α^7	α^9
$a'C_1(0)=$	$15:2^3$	$525:2^6$	$11\,025:2^9$	$363\,825:2^{13}$
$a'C_1(1)=$	$15:2^3$	$2415:2^6$	$108\,045:2^9$	$6\,039\,495:2^{13}$
Factors for—				
$P''^{3,\ 1}_{-3,-1};$	$-35:2^3$	$-21:1$	$-495:2^3$	$-143:1$
$P''^{1,\ 3}_{-1,-3};$	0	$-7:2^2$	$-45:2^2$	$-77:2$

	α^2	α^4	α^6	α^8	α^{10}
$a'C_2(0)=$	0	$105:2^5$	$1575:2^7$	$121\,275:2^{12}$	$945\,945:2^{14}$
$a'C_2(1)=$	0	$105:2^4$	$315:2^2$	$751\,905:2^{11}$	$4\,729\,725:2^{12}$
Factors for—					
$P''^{4,0}_{-4,0};$	$35:2^4$	$63:2^3$	$165:2^3$	$715:2^4$	$1365:2^4$
$P''^{0,\ 4}_{0,-4};$	0	0	$5:2^4$	$35:2^4$	$63:2^3$

Hence are derived the following developments of the functions of α alone, in which the coefficients are logarithmic:

$$P^{0,0}_{0,0}(0)=\ 2+[9.69\,897]\alpha^2+[9.449\,093]\alpha^4+[9.29\,073]\alpha^6+[9.1747]\alpha^8+[9.0833]\alpha^{10}+[9.008]\alpha^{12}$$

$$P^{2,0}_{0,0}(0)=\ [9.875\,061]\alpha^2+[0.148\,062]\alpha^4+[0.311\,919]\alpha^6+[0.43\,002]\alpha^8+[0.5226]\alpha^{10}+[0.599]\alpha^{12}$$

$$P^{4,0}_{0,0}(0)=\ [9.7221]\alpha^4+[0.409]\alpha^6+[0.849]\alpha^8+[1.176]\alpha^{10}+[1.436]\alpha^{12}$$

$$P^{2,2}_{0,0}(0)=\ [0.05\,115]\alpha^2+[0.8470]\alpha^4+[1.3331]\alpha^6+[1.685]\alpha^8+[1.962]\alpha^{10}+[2.190]\alpha^{12}$$

$$P^{0,4}_{0,0}(0)=\ [9.9719]\alpha^2+[0.5672]\alpha^4+[0.965]\alpha^6+[1.267]\alpha^8+[1.511]\alpha^{10}+[2.709]\alpha^{12}$$

$$P^{6,0}_{0,0}(0)=\ [9.63]\alpha^6+[0.59]\alpha^8+[1.24]\alpha^{10}$$

$$P^{4,2}_{0,0}(0)=\ [0.42]\alpha^4+[1.43]\alpha^6+[2.10]\alpha^8+[2.62]\alpha^{10}$$

$$P^{2,4}_{0,0}(0) = [0.148]\,\alpha^2+[1.266]\,\alpha^4+[1.986]\,\alpha^6+[2.52]\,\alpha^8+[2.95]\,\alpha^{10}$$

$$P^{0,6}_{0,0}(0) = [0.039]\,\alpha^2+[0.87]\,\alpha^4+[1.45]\,\alpha^6+[1.90]\,\alpha^8+[2.27]\,\alpha^{10}$$

$$P^{0,0}_{0,0}(1) =+[0.477\ 121]\,\alpha^2+[0.750\ 122]\,\alpha^4+[0.91\ 398]\,\alpha^6+[1.0321]\,\alpha^8+[1.12]\,\alpha^{10}$$

$$P^{2,0}_{0,0}(1) =+[0.653\ 212]\,\alpha^2+[1.449\ 092]\,\alpha^4+[1.935\ 168]\,\alpha^6+[2.28\ 735]\,\alpha^8+[2.564]\,\alpha^{10}$$

$$P^{4,0}_{0,0}(1) =+[1.0231]\,\alpha^4+[2.0321]\,\alpha^6+[2.706]\,\alpha^8+[3.21]\,\alpha^{10}$$

$$P^{2,2}_{0,0}(1) =+[0.82\ 930]\,\alpha^2+[2.14\ 806]\,\alpha^4+[2.95\ 636]\,\alpha^6+[3.5427]\,\alpha^8+[4.00]\,\alpha^{10}$$

$$P^{0,4}_{0,0}(1) =+[0.75\ 012]\,\alpha^2+[1.86\ 822]\,\alpha^4+[2.5884]\,\alpha^6+[3.1246]\,\alpha^8+[3.55]\,\alpha^{10}$$

$$P^{6,0}_{0,0}(1) =+[1.25]\,\alpha^6+[2.45]\,\alpha^8+[3.2]\,\alpha^{10}$$

$$P^{4,2}_{0,0}(1) =+[1.722]\,\alpha^4+[3.053]\,\alpha^6+[3.962]\,\alpha^8+[4.6]\,\alpha^{10}$$

$$P^{2,4}_{0,0}(1) =+[0.926]\,\alpha^2+[2.5671]\,\alpha^4+[3.6095]\,\alpha^6+[4.380]\,\alpha^8+[5.0]\,\alpha^{10}$$

$$P^{0,6}_{0,0}(1) =+[0.817]\,\alpha^2+[2.169]\,\alpha^4+[3.073]\,\alpha^6+[3.761]\,\alpha^8+[4.3]\,\alpha^{10}$$

$$P^{0,0}_{0,0}(2) = [0.477\ 121]\,\alpha^2+[1.403\ 334]\,\alpha^4+[1.913\ 979]\,\alpha^6+[2.275\ 117]\,\alpha^8+[2.56]\,\alpha^{10}$$

$$P^{2,0}_{0,0}(2) = [0.653\ 213]\,\alpha^2+[2.102\ 305]\,\alpha^4+[2.935\ 168]\,\alpha^6+[3.530\ 388]\,\alpha^8+[4.00]\,\alpha^{10}$$

$$P^{4,0}_{0,0}(2) = [1.67\ 634]\,\alpha^4+[3.03\ 208]\,\alpha^6+[3.94\ 952]\,\alpha^8+[4.7]\,\alpha^{10}$$

$$P^{2,2}_{0,0}(2) = [0.82\ 930]\,\alpha^2+[2.801\ 276]\,\alpha^4+[3.956\ 358]\,\alpha^6+[4.785\ 661]\,\alpha^8+[5.44]\,\alpha^{10}$$

$$P^{0,4}_{0,0}(2) = [0.75\ 012]\,\alpha^2+[2.521\ 434]\,\alpha^4+[3.58\ 836]\,\alpha^6+[4.36\ 766]\,\alpha^8+[5.0]\,\alpha^{10}$$

$$P^{6,0}_{0,0}(2) = [2.25]\,\alpha^6+[3.69]\,\alpha^8$$

$$P^{4,2}_{0,0}(2) = [2.375]\,\alpha^4+[4.0532]\,\alpha^6+[5.2048]\,\alpha^8+[6.]\,\alpha^{10}$$

$$P^{2,4}_{0,0}(2) = [0.926]\,\alpha^2+[3.2203]\,\alpha^4+[4.6095]\,\alpha^6+[5.6229]\,\alpha^8+[6.]\,\alpha^{10}$$

$$P^{0,6}_{0,0}(2) = [0.817]\,\alpha^2+[2.8224]\,\alpha^4+[4.0735]\,\alpha^6+[5.0045]\,\alpha^8+[6]\,\alpha^{10}$$

$$P^{1,1}_{-1,1}(0)=-[9.971\ 971]\,\alpha^3-[0.21\ 501]\,\alpha^5-[0.3631]\,\alpha^7-[0.4714]\,\alpha^9-[0.557]\,\alpha^{11}$$

$$P^{3,1}_{-1,1}(0)=-[9.8470]\,\alpha^3-[0.6130]\,\alpha^5-[1.083]\,\alpha^7-[1.426]\,\alpha^9$$

$$P^{1,3}_{1,1}(0)=-[0.3699]\,\alpha^3-[0.935]\,\alpha^5-[1.317]\,\alpha^7-[1.610]\,\alpha^9$$

$$P^{5,1}_{-1,1}(0)=-[0.01]\,\alpha^5-[1.00]\,\alpha^7$$

$$P^{3,3}_{-1,1}(0)=-[0.24]\,\alpha^3-[1.33]\,\alpha^5-[2.04]\,\alpha^7-[2.56]\,\alpha^9$$

$$P^{1,5}_{-1,1}(0)=-[0.61]\,\alpha^3-[1.41]\,\alpha^5-[1.98]\,\alpha^7$$

$$P^{1,1}_{-1,1}(1)=-[1.013\ 364]\,\alpha^3-[1.677\ 407]\,\alpha^5-[2.103\ 434]\,\alpha^7-[2.421]\,\alpha^9$$

$$P^{3,1}_{-1,1}(1) = -[0.88\,842]\alpha^3 - [2.07\,535]\alpha^5 - [2.8236]\alpha^7 - [3.375]\alpha^9$$

$$P^{1,3}_{-1,1}(1) = -[1.41\,130]\alpha^3 - [2.39\,757]\alpha^5 - [3.0577]\alpha^7$$

$$P^{5,1}_{-1,1}(1) = -[1.47]\alpha^5 - [2.74]\alpha^7$$

$$P^{3,3}_{-1,1}(1) = -[1.286]\alpha^3 - [2.795]\alpha^5 - [3.778]\alpha^7$$

$$P^{1,5}_{-1,1}(1) = -[1.654]\alpha^3 - [2.874]\alpha^5 - [3.719]\alpha^7$$

$$P^{1,1}_{-1,1}(2) = -[1.369\,911]\alpha^3 - [2.507\,265]\alpha^5 - [3.225\,799]\alpha^7 - [4.183]\alpha^9$$

$$P^{3,1}_{-1,1}(2) = -[1.24\,497]\alpha^3 - [2.90\,520]\alpha^5 - [3.94\,596]\alpha^7$$

$$P^{1,3}_{-1,1}(2) = -[1.767\,851]\alpha^3 - [3.227\,425]\alpha^5 - [4.180\,042]\alpha^7$$

$$P^{5,1}_{-1,1}(2) = -[2.303]\alpha^5 - [3.866]\alpha^7$$

$$P^{3,3}_{-1,1}(2) = -[1.643]\alpha^3 - [3.6254]\alpha^5 - [4.9002]\alpha^7$$

$$P^{1,5}_{-1,1}(2) = -[2.0109]\alpha^3 - [3.7045]\alpha^5 - [4.8412]\alpha^7$$

$$P^{2,2}_{-2,2}(0) = \quad [0.090]\alpha^4 + [0.664]\alpha^6 + [1.045]\alpha^8$$

$$P^{4,2}_{-2,2}(0) = \quad [9.79]\alpha^4 + [0.89]\alpha^6 + [1.59]\alpha^8$$

$$P^{2,4}_{-2,2}(0) = \quad [0.63]\alpha^4 + [1.44]\alpha^6 + [2.01]\alpha^8$$

$$P^{2,2}_{-2,2}(1) = +[1.2942]\alpha^4 + [2.2438]\alpha^6 + [2.8780]\alpha^8$$

$$P^{4,2}_{-2,2}(1) = +[0.99]\alpha^4 + [2.47]\alpha^6 + [3.42]\alpha^8$$

$$P^{2,4}_{-2,2}(1) = +[1.838]\alpha^4 + [3.022]\alpha^6 + [3.840]\alpha^8$$

$$P^{3,3}_{-3,3}(0) = +[0.15]\alpha^5 + [0.98]\alpha^7$$

$$P'^{2,0}_{-2,0}(0) = \quad [0.574\,031]\alpha^2 + [1.169\,252]\alpha^4 + [1.567\,192]\alpha^6 + [1.86\,935]\alpha^8$$

$$P'^{4,0}_{-2,0}(0) = \quad [0.8682]\alpha^4 + [1.7890]\alpha^6 + [2.413]\alpha^8$$

$$P'^{2,2}_{-2,0}(0) = \quad [0.75\,012]\alpha^2 + [1.86\,822]\alpha^4 + [2.5884]\alpha^6 + [3.1246]\alpha^8$$

$$P'^{2,0}_{-2,0}(1) = +[0.574\,031]\alpha^2 + [1.922\,580]\alpha^4 + [2.681\,135]\alpha^6 + [3.231\,079]\alpha^8$$

$$P'^{4,0}_{-2,0}(1) = +[1.62\,155]\alpha^4 + [2.90\,298]\alpha^6 + [3.77\,515]\alpha^8$$

$$P'^{2,2}_{-2,0}(1) = +[0.75\,012]\alpha^2 + [2.621\,550]\alpha^4 + [3.702\,324]\alpha^6 + [4.48\,635]\alpha^8$$

$$P'^{1,1}_{-1-1}(0) = -[0.750\,122]\alpha^3 - [1.391\,100]\alpha^5 - [1.81\,023]\alpha^7 - [2.1247]\alpha^9$$

$$P'^{3,1}_{-1-1}(0) = -[06252]\alpha^3 - [1.7890]\alpha^5 - [2.5304]\alpha^7$$

$$P'^{1,3}_{-1-1}(0) = -[1.14\,806]\alpha^3 - [2.11\,126]\alpha^5 - [2.7645]\alpha^7$$

$$P'^{\,1,\,1}_{-1-1}(1) = -[1.273\,001]\,\alpha^3 - [2.361\,137]\,\alpha^5 - [3.065\,502]\,\alpha^7 - [3.5920]\,\alpha^9$$

$$P'^{\,3,\,1}_{-1-1}(1) = -[1.14\,806]\,\alpha^3 - [2.75\,908]\,\alpha^5 - [3.78\,566]\,\alpha^7$$

$$P'^{\,1,\,3}_{-1-1}(1) = -[1.67\,094]\,\alpha^3 - [3.081\,297]\,\alpha^5 - [4.01\,974]\,\alpha^7$$

$$P'^{0,\,2}_{0-2}\ (0) = \ [0.32\,415]\,\alpha^4 + [1.01\,089]\,\alpha^6 + [1.4512]\,\alpha^8$$

$$P'^{2,\,2}_{0-2}\ (0) = \ [1.0231]\,\alpha^4 + [2.0321]\,\alpha^6 + [2.7065]\,\alpha^8$$

$$P'^{0,\,4}_{0-2}\ (0) = \ [0.8682]\,\alpha^4 + [1.7891]\,\alpha^6 + [2.413]\,\alpha^8$$

$$P'^{0,\,2}_{0-2}\ (1) = \ [1.077\,481]\,\alpha^4 + [2.124\,833]\,\alpha^6 + [2.812\,936]\,\alpha^8$$

$$P'^{2,\,2}_{0-2}\ (1) = \ [1.77\,650]\,\alpha^4 + [3.14\,600]\,\alpha^6 + [4.06\,820]\,\alpha^8$$

$$P'^{0,\,4}_{0-2}\ (1) = \ [1\ 62\,160]\,\alpha^4 + [2.90\,300]\,\alpha^6 + [3.77\,510]\,\alpha^8$$

$$P'^{\,3,1}_{-3,1}\ (0) = -[0.91\,398]\,\alpha^3 - [1.8383]\,\alpha^5 - [2.4556]\,\alpha^7$$

$$P'^{1,\,3}_{1-3}\ (0) = -[0.759]\,\alpha^5 - [1.715]\,\alpha^7$$

$$P'^{\,3,1}_{-3,1}(1) = -[1.21\,501]\,\alpha^3 - [2.74\,135]\,\alpha^5 - [3.6775]\,\alpha^7$$

$$P'^{1,\,3}_{1-3}\ (1) = -[1.6622]\,\alpha^5 - [2.9372]\,\alpha^7$$

§ 5.

DERIVATION OF THE GENERAL EQUATIONS FOR THE SECULAR VARIATIONS.

We have now to put the expressions for the secular variations in the form which corresponds to the development of F, given in the preceding pages. We start from the well-known canonical equations for the variations of the six elements of an orbit, which may be written in the form

$$\begin{aligned} \frac{d\,g_0}{d\,t} &= -\frac{d\,R_0}{d\,c_1}; & \frac{d\,c_1}{d\,t} &= \frac{d\,R_0}{d\,g} \\ \frac{d\,\omega}{d\,t} &= -\frac{d\,R_0}{d\,c_2}; & \frac{d\,c_2}{d\,t} &= \frac{d\,R_0}{d\,\omega} \\ \frac{d\,\theta}{d\,t} &= -\frac{d\,R_0}{d\,c_3}; & \frac{d\,c_3}{d\,t} &= \frac{d\,R_0}{d\,\theta} \end{aligned} \tag{1}$$

Here θ, ω, and g are, respectively, the longitude of the node of the orbit on any fixed plane of reference, the distance from this node to the perihelion, and the mean anomaly. By g_0 is meant the value of g, the mean anomaly, at the epoch of reference. The canonical elements, c_1, c_2, and c_3, are

$$\begin{aligned} c_1 &= a^2 n \\ c_2 &= a^2 n \cos \varphi = c_1 \cos \varphi \\ c_3 &= a^2 n \cos \varphi \cos i = c_2 \cos i \end{aligned} \tag{2}$$

φ being the angle of eccentricity ($e = \sin \varphi$).

From these equations may readily be derived the equations which express the variation of each of the six elements in terms of the derivatives of R as to these elements. But since R, in the form we have used it, is not an explicit function of the elements in question, we shall find it convenient not to use this form, but to pass at once to the derivatives of R or F as to the quantities of which it is an explicit function, namely,

$$e,\ w, \text{ and } \sigma, \text{ or } \gamma$$

The first pair of canonical equations do not enter into our theory at all, because R_0 does not contain g, and the secular variation of g_0 is not required for our purpose. Hence, as is well known, c_1 has no secular variation within the limits of our present theory, being a function of the mean distance and the sum of the masses of the Sun and Earth, formed by eliminating n from the first equation (2) by the equation

$$a^3 n^2 = M + m$$

Differentiating the values of c_2 and c_3, just given, and substituting them in the two corresponding canonical equations, we readily find,

$$\begin{aligned} \frac{c_1 d\cdot\cos\varphi}{dt} &= \frac{dR_0}{d\omega} \\ \frac{c_2 d\cdot\cos i}{dt} &= \frac{dR_0}{d\theta} - \cos i \frac{dR_0}{d\omega} \end{aligned} \tag{3}$$

In (1) R_0 is regarded as a function of c_1, c_2, and c_3. Since these quantities do not appear explicitly in R_0 we have to transform the latter into a function of e and i by (2). We find from (2) regarding e, (or φ) and i as functions of c_2 and c_3

$$\frac{de}{dc_2} = -\frac{\cot\varphi}{c_1}; \qquad \frac{de}{dc_3} = 0;$$

$$\frac{di}{dc_2} = \frac{\cot i}{c_2}; \qquad \frac{di}{dc_3} = -\frac{\operatorname{cosec} i}{c_2}$$

We thus find for the equations in ω and θ

$$\begin{aligned} \frac{d\omega}{dt} &= \frac{\cot\varphi}{c_1}\frac{dR_0}{de} - \frac{\cot i}{c_2}\frac{dR_0}{di}; \\ \frac{d\theta}{dt} &= \frac{\operatorname{cosec} i}{c_2}\frac{dR_0}{di} \end{aligned} \tag{4}$$

In the equations (3) and (4) R is regarded as a function of the six elements

$$a,\ e,\ i,\ \theta,\ \omega,\ g,$$

of which i, θ, and ω do not explicitly appear in the development. To find in the simplest form the partial derivatives with respect to these elements we consider the original form of the function. That part of R from which alone non-periodic varia-

tions can arise, is equal to the mass of the disturbing planet divided by its distance from the disturbed one. Thus, in its original form, we have

$$R_v = f.\,(r,\ r',\ \text{v},\ \text{v}',\ \gamma),$$

where I put R_v to show that R is, in forming the partial derivatives, considered as a function of these five quantities. As neither r nor r' contain the elements i, θ, and ω, of whose variations we are in quest, we may write simply

$$R_v = f.\,(\text{v},\ \text{v}';\ \gamma)$$

We note that v and v′ are the angular distances of the planets from the common nodes of their orbits, for which node I take, at present, that at which the disturbing planet passes to the north of the plane of the orbit of the disturbed one. I shall call this the node G.

The elements i, ω, and θ may now be introduced in the following way: Consider the spherical triangle whose vertices are the two nodes on the ecliptic and the node G, and whose sides are the arcs from the nodes on the ecliptic to the node G in the direction of motion, which arcs we call ψ and ψ', and the arc of the ecliptic between the nodes. The six parts of this triangle will then be

$$\text{sides:}\quad \theta' - \theta; \qquad \psi; \qquad \psi'$$

$$\text{opposite angles:}\quad \gamma; \qquad 180^\circ - i'; \quad i$$

The parts of this triangle supposed to be given are $\theta' - \theta$, $180^\circ - i'$ and i, and the other three are supposed to be expressed as functions of these three.

In the development we have put

w, w'; the distances from the node G to the respective perihelia.

E; the equation of the center. We thus have

$$\left.\begin{aligned} \text{v} &= w + g + E \\ \text{v}' &= w' + g' + E' \\ w &= \omega - \psi \\ w' &= \omega' - \psi' \\ \psi;\ \psi';\ \gamma &= f\cdot(i,\ i',\ \theta' - \theta) \end{aligned}\right\} \tag{5}$$

By the substitutions expressed in these equations, v, v′, and γ become functions of the five elements i, θ, e, w and g, which appear in the equations (3) and (4).

The derivatives of ψ, ψ' and γ as to i and θ are found from the spherical triangle to which they belong:

$$\begin{aligned} \frac{d\gamma}{di} &= -\cos\psi \\ \frac{d\psi}{di} &= \cot\gamma \sin\psi \end{aligned} \tag{6}$$

$$\left.\begin{aligned}
\frac{d\psi'}{d\,i} &= \quad \operatorname{cosec}\gamma \sin\psi \\
\frac{d\,\gamma}{d\,\theta} &= -\sin i \sin\psi \\
\frac{d\,\psi}{d\,\theta} &= -\sin i' \operatorname{cosec}\gamma\cos\psi' = -\cos i - \sin i \cot\gamma\cos\psi \\
\frac{d\,\psi'}{d\,\theta} &= -\sin i \operatorname{cosec}\gamma\cos\psi
\end{aligned}\right\} \qquad (6)$$

We then have from (5) considering i, θ and ω as independent variables,

$$d\,\mathrm{v} = d\,w = d\,\omega - d\,\psi$$

$$d\,\mathrm{v}' = d\,w' = d\,\omega' - d\,\psi'$$

Hence, from (6)

$$\frac{d\,\mathrm{v}}{d\,i} = -\cot\gamma\sin\psi$$

$$\frac{d\,\mathrm{v}'}{d\,i} = -\operatorname{cosec}\gamma\sin\psi$$

$$\frac{d\,\mathrm{v}}{d\theta} = \cos i + \sin i \cot\gamma\cos\psi$$

$$\frac{d\,\mathrm{v}'}{d\,\theta} = \sin i \operatorname{cosec}\gamma\cos\psi$$

$$\frac{d\,\mathrm{v}}{d\omega} = 1$$

$$\frac{d\,\mathrm{v}'}{d\,\omega} = 0$$

to which we may add for completeness

$$\frac{d\,\gamma}{d\,\omega} = 0$$

We thus have all the derivatives necessary to form

$$\frac{d\,\mathrm{R}}{d\,i} = \frac{d\,\mathrm{R}_v}{d\,\mathrm{v}}\frac{d\,\mathrm{v}}{d\,i} + \frac{d\,\mathrm{R}_v}{d\,\mathrm{v}'}\frac{d\,\mathrm{v}'}{d\,i} + \frac{d\,\mathrm{R}_v}{d\,\gamma}\frac{d\,\gamma}{d\,i}$$

with the corresponding derivatives as to θ and ω. Simple substitution gives

$$\frac{dR}{di} = -\sin\psi\left(\cot\gamma\frac{dR_v}{dv} + \operatorname{cosec}\gamma\frac{dR_v}{dv'}\right) - \cos\psi\frac{dR_v}{d\gamma}$$

$$\frac{dR}{d\theta} = \cos i\frac{dR_v}{dv} + \sin i\cos\psi\left(\cot\gamma\frac{dR_v}{d-} + \operatorname{cosec}\gamma\frac{dR_v}{dv'}\right) - \sin i\sin\psi\frac{dR_v}{d\gamma} \qquad (7)$$

$$\frac{dR}{d\omega} = \frac{dR_v}{dv}$$

In the development R_v is changed into R by the substitutions

$$v = w + g + E = w + f\cdot(e, g)$$

$$v' = w' + g' + E' = w' + f\cdot(e', g')$$

It follows that we have

$$\frac{dR_v}{dv} = \frac{dR}{dw} = \frac{m'}{a_1}\frac{dF}{dw},$$

$$\frac{dR_v}{dv'} = \frac{dR}{dw'} = \frac{m'}{a_1}\frac{dF}{dw'} \qquad (8)$$

which values being substituted in (7) give the required quantities in terms of the partial derivatives of F with respect to the quantities which explicitly appear in it.

Hereafter we use the symbol a_1 to represent the mean distance of the outer planet, whether disturbing or disturbed, for which we used a' in §§ 2–4. Making the substitution (8) in (7) and (7) in (1) and (3) we find from the latter by easy reductions and by introducing

$$\mu = a^3 n^2 = M + m$$

$$\frac{di}{dt} = \frac{m'}{\mu}\frac{a}{a_1} n \sec\varphi\left\{\sin\psi\frac{dF}{d\gamma} - \cos\psi\left(\cot\gamma\frac{dF}{dw} + \operatorname{cosec}\gamma\frac{dF}{dw'}\right)\right\}$$

$$\sin i\frac{d\theta}{dt} = -\frac{m'}{\mu}\frac{a}{a_1} n \sec\varphi\left\{\cos\psi\frac{dF}{d\gamma} + \sin\psi\left(\cot\gamma\frac{dF}{dw} + \operatorname{cosec}\gamma\frac{dF}{dw'}\right)\right\}$$

$$e\frac{de}{dt} = -\frac{m'}{\mu}\frac{a}{a_1} n\cos\varphi\frac{dF}{dw}$$

$$\frac{d\omega}{dt} = -\cos i\frac{d\theta}{dt} + \frac{m'}{\mu}\frac{a}{a_1} n\cot\varphi\frac{dF}{de} \qquad (9)$$

If we put π for the longitude of the perihelion, as that term is commonly defined, we have

$$\pi = \theta + \omega$$

But I have preferred to use at present, instead of π, an angle π_1, defined as the distance of the perihelion from a fixed departure point in the plane of the orbit. We then have

$$\frac{d\pi_1}{dt} = \frac{d\omega}{dt} + \cos i\frac{d\theta}{dt}. \qquad (9')$$

§ 6.

SPECIAL FORMULÆ FOR THE SECULAR VARIATIONS.

(α). *Node and inclination.*—Let any term of F, as developed in § 2, be

$$F = h \cos (j' w' + j w) = h \cos N$$

h being a function of e and of $\sigma = \sin \frac{1}{2} \gamma$. We shall have, from this term alone

$$\frac{d F}{d \gamma} = \frac{1}{2} \cos \frac{1}{2} \gamma \frac{d h}{d \sigma} \cos N$$

$$\frac{d F}{d w} = - j h \sin N$$

$$\frac{d F}{d w'} = - j' h \sin N$$

Taking the outer planet as the disturbing one, we shall put

$$M' = \frac{m'}{\mu} \frac{a}{a_1} n$$

$$M'_1 = M' \sec \varphi \qquad (10)$$

$$M'_2 = M' \cos \varphi$$

$$G = j \cot \gamma + j' \operatorname{cosec} \gamma = \frac{1}{2} (j' + j) \cot \frac{1}{2} \gamma + \frac{1}{2} (j' - j) \tan \frac{1}{2} \gamma$$

The last quantity may also take the form

$$G = \frac{(j' + j) \cos \frac{1}{2} \gamma}{2 \sigma} + \frac{1}{2} (j' - j) \sigma \sec \frac{1}{2} \gamma$$

or

$$\sigma G = \frac{1}{2} (j' + j) \cos \frac{1}{2} \gamma + \frac{1}{2} (j' - j) \sigma^2 \sec \frac{1}{2} \gamma$$

We thus have

$$\cot \gamma \frac{d F}{d w} + \operatorname{cosec} \gamma \frac{d F}{d w'} = - G h \sin N$$

The differential equations (9) in i and θ now reduce to

$$\frac{d i}{d t} = M'_1 \left(G h \sin N \cos \psi + \frac{1}{2} \cos \frac{1}{2} \gamma \frac{d h}{d \sigma} \cos N \sin \psi \right)$$

$$\sin i \frac{d \theta}{d t} = M'_1 \left(G h \sin N \sin \psi - \frac{1}{2} \cos \frac{1}{2} \gamma \frac{d h}{d \sigma} \cos N \cos \psi \right)$$

Of the quantities in the second members of these equations all but G, h and N have the same value for all the terms of F. Hence, making a summation of the quantities depending on G, h and N, which summation we represent by Σ, we find that the numerical computation may be made in the following form

Compute k and K from

$$k \sin K = \Sigma G\, h \sin N$$

$$k \cos K = -\frac{1}{2} \cos \frac{1}{2} \gamma\, \Sigma \frac{d\, h}{d\, \sigma} \cos N \qquad (11)$$

then

$$\frac{d\, i}{d\, t} = M'_1\, k \sin (K - \psi)$$

$$\sin i \frac{d\, \theta}{d\, t} = M'_1\, k \cos (K - \psi) \qquad (12)$$

So far our formulæ have been constructed as if the planet whose elements are accented were the disturbing one. But, in considering the mutual action of any pair, the same quantities appear, for the most part, in both actions, so that it will be convenient to apply the accents to the same elements in both cases, namely, those which belong to the outer planet. The preceding formulæ will then apply unchanged to the action of the outer planet on the inner one, and we shall now modify them for the action of the inner planet on the outer one.

As we have defined the node G, its position would be reversed by reversing the actions of the two planets, ψ becoming $\psi' \pm 180°$ and ψ' becoming $\psi \pm 180°$. Hence, in order to use the same values of ψ and ψ' in both cases, they must be changed by 180°

Instead of (10) we shall then use

$$M = \frac{m}{\mu} \frac{a'}{a_1} n'$$

$$M_1 = M \sec \varphi' \qquad (13)$$

$$M_2 = M \cos \varphi'.$$

Here a' and a_1 refer to the same planet; their ratio would therefore be unity, but for the small correction we apply to a' on account of the constant term of the mean distance due to the action of all the other planets.

The value of h and, therefore, of its derivative as to σ, and also N, remain unchanged, but j and j' are interchanged. Thus we have, instead of G

$$G' = \frac{1}{2} (j' + j) \cot \frac{1}{2} \gamma - \frac{1}{2} (j' - j) \tan \frac{1}{2} \gamma$$

$$= G - (j' - j) \tan \frac{1}{2} \gamma$$

Thus, instead of (11), we shall have,

$$k' \sin K' = k \sin K - \tan \frac{1}{2} \gamma\, \Sigma (j' - j)\, h \sin N$$

$$k' \cos K' = k \cos K = -\frac{1}{2} \cos \frac{1}{2} \gamma\, \Sigma \frac{d\, h}{d\, \sigma} \cos N \qquad (14)$$

and, instead of (12),

$$\frac{d\,i'}{d\,t}=-\mathrm{M}_1\,k'\sin(\mathrm{K}'-\psi')$$

$$\sin i'\frac{d\,\theta'}{d\,t}=-\mathrm{M}_1\,k'\cos(\mathrm{K}'-\psi') \tag{15}$$

It is worthy of note that the quantities k, k', K and K′, in (12) and (14) are dependent only on the relative positions of the orbits and other elements independent of the plane of reference. Hence the secular variations can be referred to any other fixed plane than that first selected, by substituting suitable values of ψ and ψ' in (12) and (15). In fact, k and K alone represent the actual motions of the orbital independently of any plane of reference in a way which will be shown hereafter.

(β). *Eccentricity and perihelion.*—The variations of these elements follow at once from the equations (9) and (9′)

$$\frac{d\,e}{d\,t}=\frac{m'}{\mu}\frac{a}{a_1}\,n\cos\varphi\frac{d\,\mathrm{F}}{d\,e}=-\mathrm{M}'_2\frac{1}{e}\frac{d\,\mathrm{F}}{d\,w}$$

$$e\frac{d\pi'}{d\,t}=\quad\mathrm{M}'_2\frac{d\,\mathrm{F}}{d\,e}$$

The e which is found as a divisor in the first equation always disappears by entering as a factor into $\frac{d\,\mathrm{F}}{d\,w}$. It is retained as a factor of the variation of π_1 in the second equation, because the motion of π_1 increases without limit as e approaches zero, while the actual perturbations of the coordinates do not so increase, but contain $e\delta\pi_1$ as a factor. Calling as before

$$h\cos\mathrm{N}=h\cos(j\,w+j'\,w')$$

any term of F, we have

$$\frac{1}{e}\frac{d\,\mathrm{F}}{d\,w}=-\Sigma\frac{j\;h}{e}\sin\mathrm{N}$$

$$\frac{d\,\mathrm{F}}{d\,e}=\quad\Sigma\frac{d\,h}{d\,e}\cos\mathrm{N}$$

For an inner planet the required quantities will then be given by the formulæ

$$\frac{d\,e}{d\,t}=\mathrm{M}'_2\,\Sigma\frac{j\,h}{e}\sin\mathrm{N}$$

$$e\frac{d\,\pi_1}{d\,t}=\mathrm{M}'_2\,\Sigma\frac{d\,h}{d\,e}\cos\mathrm{N} \tag{16}$$

For the reverse action we have

$$\frac{d\,e'}{d\,t} = M_2\,\Sigma \frac{j'\,h}{e'} \sin N$$

$$e\frac{d\,\pi'_1}{d\,t} = M_2\,\Sigma \frac{d\,h}{d\,e'} \cos N \qquad (17)$$

M_2 and M'_2 having the values given in (13) and (10).

§ 7.

REFERENCE OF VARIATIONS TO THE MOVING ECLIPTIC.

By the preceding method we may derive the numerical values of the secular variations of the orbits referred to an arbitrary fixed plane, say that of the ecliptic of 1850, and an arbitrary fixed point in that plane. We have employed the formulæ in computing these quantities for the three fundamental epochs, 1600, 1850, and 2100. The secular variation of the longitude of the perihelion, as determined by the formulæ, is the motion of the perihelion from a fixed departure point in the orbit.

These several quantities are not, however, those practically used in astronomy. Latitudes and longitudes are ordinarily referred either to the moving ecliptic and equinox of the date or to a fixed ecliptic and equinox not very distant from the date.

It is therefore essential to investigate the variations when referred to a moving plane and a moving point on that plane, the motion being determined without any reference to fixed points, but to points determined in each particular case by the conditions at the moment. We are therefore required to find expressions which have no reference to absolutely fixed planes, and it will be better, in order to derive these expressions, to proceed so that we may ultimately dispense with the consideration of absolute points or lines of reference, and of coordinates referred to them, deriving differentially the equations for the phenomena of motion and change at each instant, and passing from one epoch to another by a process of mechanical integration.

We begin with the consideration of the motion of the orbital planes. This motion is conceived in each case to take place around a line through the center of the Sun, in the plane of the orbit, which line we may consider to be the instantaneous rotation axis of the orbit. The instantaneous motion is completely determined by the position of this axis and the rate of rotation around it at the moment. By processes too simple to need development we pass from the preceding formulæ for the variation of the inclination and longitude of the node to the following method of determining the motion in question.

In the plane P of the moving orbit take an arbitrary point of reference, or departure point Q, from which longitudes in the orbit are to be measured.

Let N_1, N_2, . . . be the ascending nodes of the orbits of the several disturbing planets upon the orbital plane P, and let us put

ψ_1, ψ_2, . . the longitudes or departures of the points N_1, N_2, . . . measured from Q in the direction of the planets' orbital motion.

We then compute the quantities $\varkappa$ and ν from the equations

$$\begin{aligned} \varkappa \sin \nu &= \Sigma \, M_1 \, k \cos (K - \psi_i) \\ \varkappa \cos \nu &= \Sigma \, M_1 \, k \sin (K - \psi_i) \end{aligned} \tag{18}$$

where M_1 is to contain the mass of the disturbing planet, whether outer or inner, while ψ_i and n refer to the disturbed planet, and the quantities k and K represent either those defined in (11) or in (14). Then ν will be the longitude, measured from the point Q, of the rotation axis of the orbit P, and $\varkappa$ will be its instantaneous rate of rotation, taken positively in such a direction that ν will at each moment be the longitude of the ascending node of the moving plane upon its position a moment previous.

The point Q is entirely arbitrary. We may either assume, once for all, a position which shall change only in consequence of the motion of the orbit; we may take the ascending node at each instant; or we may measure back from the ascending node a distance equal to the longitude of the node at that instant and take the point thus reached as Q.

Taking for the point Q the longitude of the node on the ecliptic at the instant, we see that, in the case of the action of an outer on an inner planet, the quantities ψ_i just defined will be the same as ψ defined in § 6, while in the action of an inner on an outer planet, ψ_i will be the same as $\psi' \pm 180°$. The quantities $\varkappa \sin \nu$ and $\varkappa \cos \nu$ will then be equal to the secular variations found from (12) and (15), ν being the distance from the ascending node to the instantaneous rotation axis. If we use longitudes in the orbits as commonly defined, the values of ψ will be defined as follows in the special cases of Mercury, Venus, and Mars. We put, in each case, θ for the longitude of the node of the disturbed planet upon the ecliptic. We shall then have

In the case of Mercury:	Action of Venus,	$\psi_1 = \psi + \theta$	
	Action of Earth,	$\psi_2 = \psi + \theta$	
	. . . . . .		
In the case of Venus:	Action of Mercury,	$\psi_1 = \psi' + \theta + 180°$	
	Action of Earth,	$\psi_2 = \psi + \theta$	(19)
	. . . . . .		
In the case of Mars:	Action of Mercury,	$\psi_1 = \psi' + \theta + 180°$	
	Action of Venus,	$\psi_2 = \psi' + \theta + 180°$	
	Action of Earth,	$\psi_3 = \psi' + \theta + 180°$	
	Action of Jupiter,	$\psi_4 = \psi + \theta$	
	. . . .		

Motion of the perihelion referred to a fixed ecliptic.—In astronomical usage the longitude of the perihelion is commonly defined as equal to the longitude of the node plus the distance from the node to the perihelion, or

$$\pi = \theta + \omega$$

Moreover, the mean longitude of a planet is taken as equal to the longitude of the node plus the distance from the node to the mean position of the planet, or

$$l = \theta + u$$

u being the mean argument of latitude.

The motions of π and l, as thus defined, will be different from their motions from a departure point in the orbit, the difference being allowed for by a correction to the latter, given by the equation

$$\delta\, D_t\, \pi = \delta\, D_t\, l = (1 - \cos i)\, D_t\, \theta \tag{20}$$

which is to be added to the motion of the perihelion already found to obtain the motion of $\theta + w$ referred to a fixed ecliptic.

Motion of the ecliptic.—In the case of the ecliptic the points N_1, N_2, etc., as previously defined, are the ascending nodes of the several orbits upon its plane. Taking the equinox as the point Q, the motion of the plane at any instant will be defined by the quantities $\varkappa''$ and ν'', derived from the equations

$$\begin{aligned} \varkappa'' \sin \nu'' &= \Sigma\, M_1\, k \cos (K - \theta_i) \\ \varkappa'' \cos \nu'' &= \Sigma\, M_1\, k \sin (K - \theta_i) \end{aligned} \tag{21}$$

Thus ν'' will be, as before, the longitude of the instantaneous rotation axis of the ecliptic measured from the equinox, while $\varkappa''$ will be its instantaneous motion around that axis.

Effect of motion of the ecliptic on the longitudes of the perihelion and node.—We readily find that the instantaneous motion of the longitude of the node and of the inclination of any planet at any time in consequence of the motion of the ecliptic will be given by the equations

$$\begin{aligned} \tan i\, D'_t\, \theta &= -\varkappa'' \sin (\nu'' - \theta) - \varkappa'' \sin \nu'' \tan i \cot \varepsilon. \\ D'_t\, i &= -\varkappa'' \cos (\nu'' - \theta) \end{aligned} \tag{22}$$

In the first of these equations the last term arises from the planetary precession. Omitting it

$$\sin i\, D'_t\, \theta = -\varkappa'' \cos i \sin (\nu'' - \theta) \tag{23}$$

These motions are to be added to the secular variations previously found, in order to obtain the entire motion of the plane of the orbit relatively to the moving ecliptic.

A further correction will also be required to the mean motion of the planet, and the motion of the longitude of its perihelion, given by the equations

$$\delta'\, D_t\, \pi = \delta'\, D_t\, l = (1 - \sec i)\, D'_t\, \theta = \tan \frac{1}{2} i\, \varkappa'' \sin (\nu'' - \theta) \tag{24}$$

§ 8.

NUMERICAL VALUES OF MASSES AND OTHER FUNDAMENTAL CONSTANTS.

α. Constants depending on the masses and mean distances.

The adopted masses of the several planets and the logarithms of the constant coefficients M, M_1 and M_2, in the case of each mutual action of a pair of planets, are shown in the following table. The quantities actually given in the three lines of each set are:

$$M = \frac{m}{\mu} n'$$

for the action of an inner on an outer planet,

$$M' = \frac{m'}{\mu} \frac{a}{a'} n$$

for the action of an outer on an inner planet, and the products of each M or M′ by the values of sec φ and cos φ for the disturbed planet.

The units of arc and time are respectively 1″ and one Julian century.

Action of—	$\mu \div m$	Mercury.	Venus.	Earth.	Mars.
Mercury	7 500 000	. . . . . .	1. 4485	1. 2375	0. 9631
			1. 4485	1. 2376	0. 9650
			1. 4485	1. 2374	0. 9612
Venus	410 000	2. 846 56	. . . . . .	2. 499 82	2. 225 47
		2. 855 94	. . . . . .	2. 499 88	2. 227 37
		2. 837 18	. . . . . .	2. 499 76	2. 223 57
Earth and Moon . .	327 000	2. 804 13	2. 668 38	. . . . . .	2. 323 70
		2. 813 51	2. 668 39	. . . . . .	2. 325 60
		2. 794 73	2. 668 37	. . . . . .	2. 321 80
Mars	3 093 500	1. 645 33	1. 509 58	1. 439 24	. . . . . .
		1. 654 71	1. 509 59	1. 439 30	. . . . . .
		1. 635 95	1. 509 57	1. 439 18	. . . . . .
Jupiter	1 047. 88	4. 582 141	4. 446 384	4. 376 052	4. 284 603
		4. 591 521	4. 446 394	4. 376 113	4. 286 500
		4. 572 761	4. 446 374	4. 375 991	4. 282 706
Saturn	3 501. 6	3. 794 74	3. 658 99	3. 588 65	3. 497 20
		3. 804 12	3. 659 00	3. 588 71	3. 499 10
		3. 785 36	3. 658 98	3. 588 59	3. 495 30
Uranus	22 756	2. 678 47	2. 542 72	2. 472 38	2. 380 94
		2. 687 85	2. 542 73	2. 472 44	2. 382 84
		2. 669 09	2. 542 71	2. 472 32	2. 379 04
Neptune	19 540	2. 549 63	2. 413 88	2. 343 54	2. 252 09
		2. 559 01	2. 413 89	2. 343 60	2. 253 99
		2. 540 25	2. 413 87	2. 343 48	2. 250 19

β. Elements, etc., referred to the ecliptic and equinox of 1850.

	Inclinations.			Longitudes of nodes.		
	1600	1850	2100	1600	1850	2100
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″
Mercury	7 1 1.94	7 0 7.71	6 59 14.20	46 51 52.1	46 33 8.6	46 14 20.2
Venus	3 23 44.2	3 23 34.83	3 23 27.24	76 1 45.5	75 19 52.2	74 37 55.5
Earth	0 1 57.85	0 0 0	0 1 57.43	354 10 32.6	. . . .	172 57 24.0
Mars	1 52 16.23	1 51 2.28	1 49 49.41	49 7 24.4	48 23 53.0	47 39 23.3
Jupiter	1 19 1.2	1 18 42.10	1 18 24.4	98 30 30	98 56 19.8	99 22 52
Saturn	2 29 15	2 29 40.19	2 30 2	112 59 16	112 20 49.0	111 42 14
Uranus	0 46 20.54	0 46 20.54	0 46 20.54	73 14 8.0	73 14 8.0	73 14 8.0
Neptune	1 47 1.68	1 47 1.68	1 47 1.68	130 7 31.8	130 7 31.8	130 7 31.8

	Eccentricities.			Longitudes of perihelia.		
	1600	1850	2100	1600	1850	2100
				° ′ ″	° ′ ″	° ′ ″
Mercury	.205 553 39	.205 604 78	.205 656 17	74 43 9.78	75 7 13.78	75 31 17.78
Venus	.006 959 18	.006 843 31	.006 727 44	129 25 2.16	129 27 14.36	129 29 31.11
Earth	.016 876 47	.016 771 10	.016 668 99	99 33 51.13	100 21 21.34	101 9 27.02
Mars	.093 034 72	.093 261 13	.093 487 54	332 11 25.21	333 17 53.49	334 24 12.10
Jupiter	.047 835	.048 242 77	.048 668	11 22 36	11 56 9.33	12 26 36
Saturn	.056 913	.056 056 88	.055 201	88 42 48	90 6 46.22	91 30 54
Uranus	.046 9236	.046 9236	.046 9236	168 15 6.7	168 15 6.7	168 15 6.7
Neptune	.008 4962	.008 4962	.008 4962	43 17 30.3	43 17 30.3	43 17 30.3

	$\psi = G - \Omega$			$\psi' = G - \Omega'$		
	1600	1850	2100	1600	1850	2100
Mercury and	° ′	° ′	° ′	° ′	° ′	° ′
Venus	157 46.13	157 54.60	158 3.01	128 42.32	129 13.86	129 45.32
Earth	180 12.82	180 0.00	179 47.14	232 54.05	Arbitrary	53 4.17
Mars	179 10.69	179 20.19	179 29.78	176 55.42	177 29.67	178 4.90
Jupiter	170 31.43	170 28.17	170 24.81	118 56.60	118 8.80	117 20.11
Saturn	159 13.52	159 7.36	159 1.73	93 14.49	93 28.03	93 42.16
Uranus	176 53.35	176 50.91	176 48.47	150 32.34	150 11.19	149 49.95
Neptune	165 22.62	165 21.43	165 20.29	82 13.48	81 53.55	81 33.59
Venus and						
Earth	180 32.87	180 0.00	179 27.31	262 24.03	Arbitrary	81 7.89
Mars	206 7.35	205 41.16	205 15.82	233 0.20	232 35.66	232 12.88
Jupiter	166 59.00	166 30.93	166 3.47	144 31.16	142 55.40	141 19.50
Saturn	133 15.48	133 0.20	132 47.11	96 20.63	96 1.92	95 45.48
Uranus	180 49.33	180 37.05	180 24.72	183 36.89	182 42.74	181 48.48
Neptune	148 23.79	148 20.22	148 17.36	94 20.58	93 35.15	92 50.36

	$\psi = G - \Omega$			$\psi' = G - \Omega'$		
	1600	1850	2100	1600	1850	2100
Earth and	° ′		° ′	° ′	° ′	° ′
Mars	55 46.58	Arbitrary	233 52.52	0 49.74	0 0.00	359 10.50
Jupiter	105 42.21	Arbitrary	285 2.60	1 22.28	0 0.00	358 37.11
Saturn	119 28.09	Arbitrary	298 5.29	0 39.40	0 0.00	359 20.42
Uranus	81 27.71	Arbitrary	257 54.73	2 24.14	0 0.00	357 37.99
Neptune	136 40.26	Arbitrary	316 26.83	0 43.30	0 0.00	359 16.68
Mars and		° ′				
Jupiter	135 23.73	135 6.90	134 51.31	86 1.62	84 35.44	83 8.81
Saturn	109 7.66	108 36.00	108 6.38	45 17.99	44 41.24	44 5.69
Uranus	164 51.56	164 14.44	163 36.49	140 45.14	139 24.51	138 2.06
Neptune	132 5.41	132 4.04	132 3.79	51 7.01	50 22.11	49 37.35

	σ			$\log \sigma$		
	1600	1850	2100	1600	1850	2100
Mercury and						
Venus	.038 173 11	.037 910 76	.037 648 74	8.581 7575	8.578 7625	8.575 7505
Earth	.061 025 90	.061 067 23	.061 108 00	8.785 5142	8.785 8082	8.786 0981
Mars	.044 909 67	.044 951 61	.044 993 76	8.652 3399	8.652 7452	8.653 1523
Jupiter	.054 819 00	.054 842 04	.054 867 39	8.738 9311	8.739 1136	8.739 3143
Saturn	.056 033 62	.055 783 08	.055 533 93	8.748 4487	8.746 5025	8.744 5584
Uranus	.055 250 43	.055 137 50	.055 026 60	8.742 3357	8.741 4471	8.740 5727
Neptune	.061 340 06	.061 296 93	.061 255 77	8.787 7442	8.787 4387	8.787 1470
Venus and						
Earth	.029 588 84	.029 605 26	.029 629 30	8.471 1279	8.471 3688	8.471 7214
Mars	.016 782 17	.016 876 27	.016 976 77	8.224 8482	8.227 2764	8.229 8551
Jupiter	.019 511 94	.019 660 84	.019 816 48	8.290 3005	8.293 6020	8.297 0266
Saturn	.017 918 15	.017 917 66	.017 920 89	8.253 2932	8.253 2813	8.253 3595
Uranus	.022 900 45	.022 873 23	.022 851 56	8.359 8439	8.359 3275	8.358 9159
Neptune	.024 064 24	.024 234 04	.024 407 94	8.381 3721	8.384 4258	8.387 5311
Earth and						
Mars	.016 166 00	.016 149 12	.016 138 64	8.208 6025	8.208 1489	8.207 8671
Jupiter	.011 566 77	.011 446 45	.011 326 35	8.063 2120	8.058 6706	8.054 0899
Saturn	.021 844 91	.021 766 88	.021 684 30	8.339 3502	8.337 7961	8.336 1453
Uranus	.006 691 83	.006 740 17	.006 794 03	7.825 5450	7.828 6707	7.832 1273
Neptune	.015 772 51	.015 565 97	.015 358 45	8.197 9009	8.192 1760	8.186 3475
Mars and						
Jupiter	.012 424 01	.012 523 50	.012 628 71	8.094 2618	8.097 7257	8.101 3590
Saturn	.020 624 72	.020 632 62	.020 640 99	8.314 3881	8.314 5544	8.314 7304
Uranus	.010 542 57	.010 424 55	.010 312 37	8.022 9464	8.018 0572	8.013 3584
Neptune	.020 719 63	.020 752 05	.020 787 99	8.316 3821	8.317 0611	8.317 8126

γ. *Ratios α of the mean distances and their logarithms.*

	Mercury.	Venus.	Earth.	Mars.
Venus	0.535 1606	. . . . . .	. . . . . .	. . . . . .
	9.728 4841	. . . . . .	. . . . . .	. . . . . .
Earth	0.387 0985	0.723 3315	. . . . . .	. . . . . .
	9.587 8215	9.859 3374	. . . . . .	. . . . . .
Mars	0.254 0532	0.474 7242	0.656 3023	. . . . . .
	9.404 9247	9.676 4413	9.817 1039	. . . . . .
Jupiter	0.074 4020	0.139 0275	0.192 2042	0.292 8598
	8.871 5845	9.143 1006	9.283 7628	9.466 6598
Saturn	0.040 5813	0.075 8301	0.104 8344	0.159 7353
	8.608 3254	8.879 8416	9.020 5037	9.203 4008
Uranus	0.020 1787	0.037 7058	0.052 1279	0.079 4269
	8.304 8923	8.576 4084	8.717 0705	8.899 9676
Neptune	0.012 8875	0.024 0815	0.033 2925	0.050 7275
	8.110 1680	8.381 6841	8.522 3463	8.705 2433

δ. *Gaussian coefficients and their derivatives.*

MERCURY AND VENUS.

Values of $c_1^{(i,j)} = b_1^{(i,j)}$

i	$c_1^{(i)}$	$c_1^{(i,1)}$	$c_1^{(i,2)}$	$c_1^{(i,3)}$	$c_1^{(i,4)}$	$c_1^{(i,5)}$	$c_1^{(i,6)}$	$c_1^{(i,7)}$	$c_1^{(i,8)}$
0	2.172 16	0.417 53	0.371 86	0.617 67	1.512 37	4.903 73	19.8082	95.84	540.3
1	0.605 71	0.174 49	0.171 39	0.297 22	0.745 44	2.454 14	10.0180	[48]	[270]
2	0.246 60	0.079 45	0.081 91	0.145 86	0.371 93	1.238 42	5.097 00	. . . . .	. . . . .
3	0.110 78	0.037 68	0.040 00	0.072 54	0.187 24	0.628 94	2.605 76	. . . . .	. . . . .
4	0.052 11	0.018 31	0.019 82	0.036 42	0.094 91	0.321 06	1.337 42	. . . . .	. . . . .

Values of $c_3^{(i,j)} = \alpha\, b_3^{(i,j)}$

i	$c_3^{(i)}$	$c_3^{(i,1)}$	$c_3^{(i,2)}$	$c_3^{(i,3)}$	$c_3^{(i,4)}$	$c_3^{(i,5)}$	$c_3^{(i,6)}$
0	2.255 25	3.933 59	9.751 11	31.7711	128.644	623.234	3516.6
1	1.624 45	2.476 96	5.836 06	18.5152	73.7489	353.339	1977.8
2	1.043 83	1.496 94	3.420 46	10.6539	41.9215	199.111	1107.3
3	0.638 87	0.883 02	1.976 35	6.0735	23.6732	111.653	617.58
4	0.379 15	0.512 56	1.130 20	3.437 53	13.2973	62.358	343.35

NOTE.—Numbers in brackets are derived in whole or in part by induction.

Values of $c_5^{(i,j)}$

i	$c_5^{(i)}$	$c_5^{(i,1)}$	$c_5^{(i,2)}$	$c_5^{(i,3)}$	$c_5^{(i,4)}$	$c_5^{(i,5)}$	$c_5^{(i,6)}$
0	3.6560	14.686	66.252	342.93	2023.5	13 459.0	[94] M
1	3.2666	11.107	46.047	226.41	1290.3	8 373.4	[58] M
2	2.5728	7.8886	30.811	145.75	808.16	5 139.3	[33] M
3	1.8754	5.3617	20.036	91.960	498.77	3 118.9	.....
4	1.2978	3.5278	12.744	57.122	304.09	1 874.6	.....

Values of $c_7^{(i,j)}$

i	$c_7^{(i)}$	$c_7^{(i,1)}$	$c_7^{(i,2)}$	$c_7^{(i,3)}$	$c_7^{(i,4)}$
0	7.1473	45.440	305.02	2216.1	[17] M
1	6.7616	37.573	231.85	1590.3	[11] M
2	5.8402	29.316	169.36	1109.1	[8] M
3	4.7035	21.8485	119.770	755.15	[5] M
4	3.5892	15.6975	82.5066	504.19	.....
5	2.6259	10.949	55.680	330.72	.....

Values of $c_9^{(i,j)}$

i	$c_9^{(i)}$	$c_9^{(i,1)}$	$c_9^{(i,2)}$
0	15.098	131.027	1175.46
1	14.5733	113.716	945.40
2	13.1655	93.978	733.10
3	11.2323	74.477	550.76
4	9.1308	58.107	402.95
...	.....	.....	.....

Hence are derived the following values of $c_n^{(i)}$ and their derivatives:

i	$c_1^{(i)}$	$D\,c_1^{(i)}$	$D^2\,c_1^{(i)}$	$D^3\,c_1^{(i)}$	$D^4\,c_1^{(i)}$	$D^5\,c_1^{(i)}$	$D^6\,c_1^{(i)}$	$D^7\,c_1^{(i)}$	$D^8\,c_1^{(i)}$
0	2.172 16	0.417 530	1.206 918	4.518 94	22.6767	148.222	1202.7	11.66 M	131.5 M
1	0.605 71	0.780 20	1.475 05	4.713 79	23.0269	149.896	1212.2	.....	.....
2	0.246 60	0.572 64	1.545 00	5.326 1	24.9595	156.59	1254.2	.....	.....
3	0.110 78	0.370 02	1.338 48	5.510 1	27.311	169.25	1310.7	.....	.....

i	$c_3^{(i)}$	$D\,c_3^{(i)}$	$D^2\,c_3^{(i)}$	$D^3\,c_3^{(i)}$	$D^4\,c_3^{(i)}$	$D^5\,c_3^{(i)}$	$D^6\,c_3^{(i)}$
0	2.255 25	6.1888	27.741	172.92	1362.1	12.94 M	143.5 M
1	1.624 45	5.7259	27.196	170.90	1350.9	12.86 M	143.2 M
2	1.043 8	4, 6284	24.790	163.49	1316.1	12.64 M	141.3 M
3	0.638	3.435	21.016	149.6	1251.0	12.24 M	.....

i	$c_5^{(i)}$	$D\,c_5^{(i)}$	$D^2\,c_5^{(i)}$	$D^3\,c_5^{(i)}$	$D^4\,c_5^{(i)}$	$D^5\,c_5^{(i)}$	$D^6\,c_5^{(i)}$
0	3.6560	22.000	169.0	1578	173 28	218.4 M	3101 M
1	3.2666	20.907	164.3	1549	170 99	216.3 M	3080 M
2	2.5728	18.18	150.9	1464	164 28	217.1 M	2699 M
3	1.88	14.74	131.3	1332	153 46	.....	.....

Note.—Numbers in brackets are reached wholly or partly by induction.

i	$c_7^{(i)}$	$Dc_7^{(i)}$	$D^2c_7^{(i)}$	$D^3c_7^{(i)}$	$D^4c_7^{(i)}$
0	7.1473	66.882	732.9	9211	130 M
1	6.7616	64.619	715.8	9052	128 M
2	5.8402	58.517	667.2	8591	124 M
3	4.7035	50.070	595.0	7879	112 M
4	3.5892	40.82	509.5	6993	. . . . .

i	$c_9^{(i)}$	$Dc_9^{(i)}$	$D^2c_9^{(i)}$
0	15.098	191.4	2727
1	14.573	186.6	2674
2	13.166	173.0	2523
3	11.232	153.1	2293
4	9.131	131.2	2033

MERCURY AND EARTH.

i	$c_1^{(i)}$	$Dc_1^{(i)}$	$D^2c_1^{(i)}$	$D^3c_1^{(i)}$	$D^4c_1^{(i)}$	$D^5c_1^{(i)}$	$D^6c_1^{(i)}$
0	2.081 98	0.179 76	0.430 33	1.1957	4.0986	17.753	95.24
1	0.411 14	0.464 38	0.647 30	1.3300	4.1918	18.062	96.85
2	0.120 18	0.257 71	0.592 74	1.5487	4.9311	20.018	102.84
3	0.038 90	0.122 62	0.400 2	1.3840	5.244	22.654	115.31

i	$c_3^{(i)}$	$Dc_3^{(i)}$	$D^2c_3^{(i)}$	$D^3c_3^{(i)}$	$D^4c_3^{(i)}$
0	1.111 68	1.977 30	5.5218	22.254	114.88
1	0.610 09	1.626 07	5.2944	21.852	113.00
2	0.289 4	1.048 5	4.2272	19.594	106.50
3	0.129 4	0.595 3	2.9234	15.657	93.273

i	$c_5^{(i)}$	$Dc_5^{(i)}$	$D^2c_5^{(i)}$
0	0.769	3.016	14.98
1	0.586	2.647	14.01
2	0.362	1.932	11.45
3	0.200	1.249	8.38

MERCURY AND MARS.

i	$c_1^{(i)}$	$Dc_1^{(i)}$	$D^2c_1^{(i)}$	$D^3c_1^{(i)}$	$D^4c_1^{(i)}$	$D^5c_1^{(i)}$	$D^6c_1^{(i)}$
0	2.033 5	0.069 57	0.1499	0.3460	0.8965	2.741	10.156
1	0.260 5	0.273 8	0.3162	0.4556	0.9396	2.760	10.367
2	0.049 77	0.102 4	0.2166	0.4840	1.1917	3.414	11.867
3	0.010 55	0.032 29	0.10014	0.3177	1.0459	3.649	13.87

i	$c_3^{(i)}$	$Dc_3^{(i)}$	$D^2c_3^{(i)}$	$D^3c_3^{(i)}$	$D^4c_3^{(i)}$
0	0.5900	0.7718	1.3952	3.700	13.127
1	0.2195	0.4959	1.2425	3.638	12.897
2	0.0691	0.2241	0.7627	2.789	11.248

i	$c_5^{(i)}$	$Dc_5^{(i)}$	$D^2c_5^{(i)}$
0	0.1932	0.542	1.837
1	0.1096	0.395	1.560
2	0.0468	0.212	1.009

NOTE.—The symbol M means 1000.

MERCURY AND JUPITER.

i	$c_1^{(i)}$	$D c_1^{(i)}$	$D^2 c_1^{(i)}$	$D^3 c_1^{(i)}$	$D^4 c_1^{(i)}$	$D^5 c_1^{(i)}$	$D^6 c_1^{(i)}$
0	2. 002 78	0. 005 570	0. 011 211	0. 022 701	0. 046 54	0. 097 66	0. 214 05
1	0. 074 56	0. 074 87	0. 075 81	0. 078 64	0. 087 25	0. 113 64	0. 195 57
2	0. 004 16	0. 008 34	0. 016 76	0. 033 84	0. 068 93	0. 142 95	0. 306 68
3	0. 000 26	0. 000 78	0. 002 33	0. 007 03	0. 021 25	0. 064 57	0. 197 76

i	$c_3^{(i)}$	$D c_3^{(i)}$	$D^2 c_3^{(i)}$	$D^3 c_3^{(i)}$	$D^4 c_3^{(i)}$
0	0. 1507	0. 1544	0. 1659	0. 2009	0. 3092
1	0. 01678	0. 0339	0. 0692	0. 1442	0. 3117
2	0. 0016	0. 0047	0. 0143	0. 0436	0. 1348
3	0. 00014	0. 00054	0. 00219	. . . . .	. . . . .

i	$c_5^{(i)}$	$D c_5^{(i)}$	$D^2 c_5^{(i)}$
0	0. 0115	0. 0237	0. 0507
1	0. 0021	0. 0064	0. 0198
2	0. 00027	0. 001[illegible]	0. 0045
. . .	. . . . .	. . . . .	. . . . .

MERCURY AND SATURN.

i	$c_1^{(i)}$	$D c_1^{(i)}$	$D^2 c_1^{(i)}$	$D^3 c_1^{(i)}$	$D^4 c_1^{(i)}$
0	2. 000 82	0. 001 65	0. 003 31	0. 006 65	0. 013 41
1	0. 040 61	0. 040 66	0. 040 81	0. 041 26	0. 042 62
2	0. 001 24	0. 002 47	0. 004 95	0. 009 93	0. 019 98

i	$c_3^{(i)}$	$D c_3^{(i)}$	$D^2 c_3^{(i)}$
0	0. 081 46	0. 082 06	0. 083 88
1	0. 004 96	0. 009 95	0. 020 01
2	0. 000 25	0. 000 75	0. 002 27

VENUS AND EARTH.

Values of $c_1^{(i,j)}$

i	$c_1^{(i)}$	$c_1^{(i,1)}$	$c_1^{(i,2)}$	$c_1^{(i,3)}$	$c_1^{(i,4)}$	$c_1^{(i,5)}$	$c_1^{(i,6)}$
0	2. 386 37	1. 188 98	2. 850 19	12. 852 44	85. 698 96	757. 823	8355. 16
1	0. 942 41	0. 701 34	1. 836 35	8. 586 64	58. 371 62	522. 340	5805. 21
2	0. 527 58	0. 442 02	1. 212 68	5. 807 54	40. 045 53	361. 721	4046. 72
3	0. 323 34	0. 287 75	0. 813 29	3. 962 42	27. 625 72	251. 444	2828. 67
4	0. 206 79	0. 191 02	0. 551 32	2. 721 43	19. 142 33	175. 339	1981. 89

Values of $c_3^{(i,j)}$

i	$c_3^{(1)}$	$c_3^{(i,1)}$	$c_3^{(i,2)}$	$c_3^{(i,3)}$	$c_3^{(i,4)}$	$c_3^{(i,5)}$	$c_3^{(i,6)}$
0	7. 2279	33. 5297	225. 087	1995. 49	22 027. 5	291 213	4487 M
1	6. 4171	27. 1033	175. 665	1529. 10	16 691. 0	218 998	3356 M
2	5. 3430	21. 4411	135. 651	1164. 42	12 595. 5	164 213	2504 M
3	4. 3068	16. 7217	103. 928	882. 30	9 472. 4	122 827	1866 M
4	3. 4030	12. 9089	79. 137	665. 82	7 103. 1	91 673	1387 M

Values of $c_5^{(i,j)}$

i	$c_5^{(i)}$	$c_5^{(i,1)}$	$c_5^{(i,2)}$	$c_5^{(i,3)}$	$c_5^{(i,4)}$	$c_5^{(i,5)}$
0	44.884	444.91	5223.7	71 814.	1136 700	20 400 M
1	43.645	396.85	4443.8	59 288.	919 280	16 251 M
2	40.606	345.62	3724.1	48 450.	737 960	12 874 M
3	36.520	295.22	3082.0	39 244.	588 530	10 146 M
4	31.990	248.19	2523.5	31 544.	466 650	7 953 M

Values of $c_7^{(i,j)}$

i	$c_7^{(i)}$	$c_7^{(i,1)}$	$c_7^{(i,2)}$	$c_7^{(i,3)}$
0	338.062	5 126.62	86 684.	1631 M
1	333.500	4 752.89	77 063.	1407 M
2	320.607	4 326.60	67 599.	1202 M
3	301.022	3 874.36	58 586.	1017 M
4	276.781	3 418.92	50 226.	853 M

Values of $c_9^{(i,j)}$

i	$c_9^{(i)}$	$c_9^{(i,1)}$	$c_9^{(i,2)}$
0	2735.06	55 798.6	1 234 M
1	2710.69	52 745.0	1 125 M
2	2639.75	49 160.0	1 015 M
3	2527.50	45 216.0	906 M
4	. . . . .	. . . . .	. . . . .

From these are derived the following values of $c_n^{(i)}$ and their derivatives:

i	$c_1^{(i)}$	$D\,c_1^{(i)}$	$D^2 c_1^{(i)}$	$D^3 c_1^{(i)}$	$D^4 c_1^{(i)}$	$D^5 c_1^{(i)}$	$D^6 c_1^{(i)}$
0	2.386 37	1.188 98	5 228 15	34.709	329.25	4118	64 077
1	0.942 41	1.643 75	5.5841	35.17	331.26	4133	64 226
2	0.527 58	1.497 17	5.9751	36.96	338.9	4182	64 691
3	0.323 3	1.257 8	6.0253	38.99	352.4	4276	65 536
4	0.206 8	1.018 2	5.7702	40.40	368.9	4416	66 851

i	$c_3^{(i)}$	$D\,c_3^{(i)}$	$D^2 c_3^{(i)}$	$D^3 c_3^{(i)}$	$D^4 c_3^{(i)}$
0	7.2279	40.76	366.43	4464	68 359
1	6.4171	39.94	363.95	4447	68 195
2	5.3430	37.47	355.3	4394	67 697
3	4.307	33.9	340.1	4299	66 839
4	3.403	29.9	319.1	4160	65 850

i	$c_5^{(i)}$	$D\,c_5^{(i)}$	$D^2 c_5^{(i)}$	$D^3 c_5^{(i)}$	$D^4 c_5^{(i)}$
0	44.884	534.68	8072.7	147 315	3149.5 M
1	43.645	527.78	8011.4	146 569	3138.0 M
2	40.606	508.04	7830.0	144 350	3104.0 M
3	36.520	478.	7538.	140 710	3047.8 M
4	31.990	440.	7150.	135 750	2803.0 M

NOTE.—The symbol M means 1000.

i	$c_7^{(i)}$	$D c_7^{(i)}$	$D^2 c_7^{(i)}$	$D^3 c_7^{(i)}$
0	338.1	6141.	130.74 M	3191.5 M
1	333.5	6087.	129.93 M	3176.7 M
2	320.6	5930.	127.53 M	3132.8 M
3	301.0	5680.	123.66 M	3061.2 M
4	276.8	5356.	118.49 M	2974.0 M

i	$c_9^{(i)}$	$D c_9^{(i)}$	$D^2 c_9^{(i)}$
0	2735	66.74 M	1836 M
1	2711	66.30 M	1826 M
2	2643	65.00 M	1798 M
3	2530	62.93 M	1753 M
4	2353	60.0 M	1691 M

VENUS AND MARS.

i	$c_1^{(i)}$	$D c_1^{(i)}$	$D^2 c_1^{(i)}$	$D^3 c_1^{(i)}$	$D^4 c_1^{(i)}$	$D^5 c_1^{(i)}$	$D^6 c_1^{(i)}$
0	2.129 67	0.299 91	0.794 08	2.5995	11.066	60.84	415.1
1	0.521 63	0.631 8	1.041 0	2.7621	11.270	61.71	419.6
2	0.187 73	0.419 7	1.049 7	3.183	12.582	65.69	435.4
3	0.074 67	0.242 6	0.834 9	3.160	13.80	72.6	469.0

i	$c_3^{(i)}$	$D c_3^{(i)}$	$D^2 c_3^{(i)}$	$D^3 c_3^{(i)}$	$D^4 c_3^{(i)}$
0	1.6727	3.803	14.032	73.0	481.3
1	1.0940	3.394	13.666	71.9	475.9
2	0.6295	2.540	11.950	67.5	458.8
3	0.3431	1.714	9.464	. . . . .	. . . . .

i	$c_5^{(i)}$	$D c_5^{(i)}$	$D^2 c_5^{(i)}$	$D^3 c_5^{(i)}$
0	1.90	9.51	60.9	. . . . .
1	1.61	8.82	58.5	. . . . .
2	1.17	7.25	51.8	. . . . .
3	0.77	5.44	42.6	. . . . .

VENUS AND JUPITER.

i	$c_1^{(i)}$	$D c_1^{(i)}$	$D^2 c_1^{(i)}$	$D^3 c_1^{(i)}$	$D^4 c_1^{(i)}$
0	2.009 77	0.019 758	0.040 391	0.084 355	0.183 443
1	0.140 048	0.142 113	0.148 410	0.167 818	0.228 689
2	0.014 62	0.029 47	0.059 91	0.123 78	0.263 84
3	0.001 69	0.005 11	0.015 48	. . . . .	. . . . .

VENUS AND SATURN.

i	$c_1^{(i)}$	$D c_1^{(i)}$	$D^2 c_1^{(i)}$	$D^3 c_1^{(i)}$	$D^4 c_1^{(i)}$
0	2.002 89	0.005 79	0.011 65	0.023 60	0.048 43
1	0.075 99	0.076 32	0.077 31	0.080 31	0.089 44
2	0.004 32	0.008 67	0.017 42	0.035 17	0.071 68

NOTE.—The symbol M means 1000.

VENUS AND URANUS.

i	$c_1^{(i)}$	$D\,c_1^{(i)}$	$D^2\,c_1^{(i)}$	$D^3\,c_1^{(i)}$	$D^4\,c_1^{(i)}$
0	2.000 70	0.001 42	0.002 85	0.005 72	0.011 50
1	0.037 73	0.037 77	0.037 89	0.038 25	0.039 33
2	0.001 07	0.002 14	0.004 28	0.008 58	0.017 26

VENUS AND NEPTUNE.

i	$c_1^{(i)}$	$D\,c_1^{(i)}$	$D^2\,c_1^{(i)}$
0	2.000 29	0.000 58	0.001 16
1	0.024 09	0.024 10	0.024 13
2	0.000 44	0.000 87	0.001 74

EARTH AND MARS.

Values of $c_1^{(i,j)}$

i	$c_1^{(i)}$	$c_1^{(i,1)}$	$c_1^{(i,2)}$	$c_1^{(i,3)}$	$c_1^{(i,4)}$	$c_1^{(i,5)}$	$c_1^{(i,6)}$	$c_1^{(i,7)}$	$c_1^{(i,8)}$
0	2.291 14	0.805 99	1.341 29	4.183 02	19.2642	117.583	894.521	8153.5	86 612
1	0.804 57	0.423 51	0.773 17	2.507 78	11.7982	72.977	560.226	. . . . .	. . . . .
2	0.405 59	0.239 71	0.458 93	1.526 40	7.2919	45.567	352.370	. . . . .	. . . . .
3	0.224 61	0.140 64	0.277 36	0.938 95	4.5379	28.590	222.417	. . . . .	. . . . .
4	0.129 98	0.084 30	0.169 71	0.582 14	2.8393	18.008	140.808	. . . . .	. . . . .

Values of $c_3^{(i,j)}$

i	$c_3^{(i)}$	$c_3^{(i,1)}$	$c_3^{(i,2)}$	$c_3^{(i,3)}$	$c_3^{(i,4)}$	$c_3^{(i,5)}$	$c_3^{(i,6)}$
0	4.499 92	14.5486	67.5954	413.987	3154.61	28 778.6	305 884.
1	3.759 32	10.8896	48.5462	290.966	2187.96	19 782.0	208 906.
2	2.890 75	7.9201	34.3687	202.760	1508.94	13 543.3	142 245.
3	2.136 93	5.6569	24.0826	140.364	1035.89	9 240.8	96 606.
4	1.543 11	3.9899	16.7445	96.659	708.45	6 287.2	65 464.

Values of $c_5^{(i,j)}$

i	$c_5^{(i)}$	$c_5^{(i,1)}$	$c_5^{(i,2)}$	$c_5^{(i,3)}$	$c_5^{(i,4)}$	$c_5^{(i,5)}$
0	16.369	115.850	954.49	9143.9	100 500	1249 800
1	15.592	98.562	764.42	7052.6	75 549	922 210
2	13.863	80.867	598.67	5357.8	56 177	674 890
5	11.738	64.528	460.49	4019.3	41 383	490 340
4	9.589	50.377	349.00	2983.1	30 240	354 010

Values of $c_7^{(i,j)}$

i	$c_7^{(i)}$	$c_7^{(i,1)}$	$c_7^{(i,2)}$	$c_7^{(i,3)}$
0	72.2967	789.950	9469.0	125 182.
1	70.6267	709.483	8043.4	102 251.
2	66.0617	618.443	6689.3	82 220.
3	59.5300	525.612	5461.9	65 196.
4	51.9767	437.060	4388.4	51 064.

Values of $c_9^{(i,j)}$

i	$c_9^{(i)}$	$c_9^{(i,1)}$	$c_9^{(i,2)}$
0	343.533	5070.00	79 976.0
1	338.400	4680.33	70 366.7
2	323.392	4220.64	60 792.9
3	300.593	3729.00	51 642.5
4	272.360	3225.40	43 208.0

From these are derived the following values of $c_n^{(i)}$ and their derivatives:

i	$c_1^{(i)}$	$D c_1^{(i)}$	$D^2 c_1^{(i)}$	$D^3 c_1^{(i)}$	$D^4 c_1^{(i)}$	$D^5 c_1^{(i)}$	$D^6 c_1^{(i)}$	$D^7 c_1^{(i)}$	$D^8 c_1^{(i)}$
0	2.291 14	0.805 995	2.953 280	15.4547	113.464	1095.02	131 34	188.4 M	3147 M
1	0.804 57	1.228 08	3.271 79	15.7765	114.511	1101.45	131 84	. . . . .	. . . . .
2	0.405 59	1.050 89	3.519 56	16.9902	118.968	1123.50	133 43	. . . . .	. . . . .
3	0.224 61	0.814 46	3.423 91	18.0550	126.233	1166.51	136 47	. . . . .	. . . . .

i	$c_3^{(i)}$	$D c_3^{(i)}$	$D^2 c_3^{(i)}$	$D^3 c_3^{(i)}$	$D^4 c_3^{(i)}$	$D^5 c_3^{(i)}$	$D^6 c_3^{(i)}$	$D^7 c_3^{(i)}$	$D^8 c_3^{(i)}$
0	4.4999	19.048	130.29	1216.0	142 85	202.1 M	3342 M	. . . . .	. . . . . .
1	3.7593	18.408	128.92	1208.5	142 29	201.5 M	3336 M	[63] M^2	[1.34] M^3
2	2.8908	16.592	123.75	1184.4	140 56	199.9 M	3315 M	. . . . .	. . . . . .
3	2.137	14.205	114.84	1140.5	137 53	. . . . .	. . . . .	. . . . .	. . . . . .

i	$c_5^{(i)}$	$D c_5^{(i)}$	$D^2 c_5^{(i)}$	$D^3 c_5^{(i)}$	$D^4 c_5^{(i)}$
0	16.369	148.59	1715.07	23 972.	392 990.
1	15.592	145.338	1693.24	23 769.	390 604.
2	13.863	136.319	1629.15	23 167.	383 502.
3	11.738	123.218	1528.28	22 190.	371 864.

i	$c_7^{(i)}$	$D c_7^{(i)}$	$D^2 c_7^{(i)}$	$D^3 c_7^{(i)}$
0	72.297	1006.840	16 439.3	307 877
1	70.627	991.990	16 268.3	305 473
2	66.062	948.752	15 762.1	298 363
3	59.530	882.792	14 963.5	286 930
4	51.977	800.897	13 928.2	271 732

i	$c_9^{(i)}$	$D c_9^{(i)}$	$D^2 c_9^{(i)}$
0	343.5	6444.1	136 173.
1	338.4	6372.3	134 991.
2	323.4	6161.0	131 524.
3	300.6	5833.2	126 036.
4	270.	5404.3	118 696.

Note.—The symbol M means 1000.

EARTH AND JUPITER.

i	$c_1^{(i)}$	$D\,c_1^{(i)}$	$D^2\,c_1^{(i)}$	$D^3\,c_1^{(i)}$	$D^4\,c_1^{(i)}$
0	2.018 86	0.038 540	0.080 401	0.174 62	0.407 80
1	0.194 93	0.200 51	0.217 79	0.272 39	0.450 63
2	0.028 14	0.057 19	0.118 04	0.251 34	0.567 15
3	0.004 51	0.013 69	0.041 83	. . . . .	. . . . .

i	$c_3^{(i)}$	$D\,c_3^{(i)}$	$D^2\,c_3^{(i)}$
0	0.418 31	0.4902	0.7230
1	0.118 94	0.2550	0.5824
2	0.028 44	0.0892	0.2874

i	$c_5^{(i)}$	$D\,c_5^{(i)}$	$D^2\,c_5^{(i)}$
0	0.093 05	0.2290	0.6495
1	0.041 84	0.1395	0.4941
2	0.013 76	0.0590	0.2612

EARTH AND SATURN.

i	$c_1^{(i)}$	$D\,c_1^{(i)}$	$D^2\,c_1^{(i)}$	$D^3\,c_1^{(i)}$	$D^4\,c_1^{(i)}$
0	2.005 53	0.011 13	0.022 54	0.046 21	0.097 01
1	0.105 27	0.106 15	0.108 80	0.116 88	0.141 76
2	0.008 28	0.016 64	0.033 59	0.068 43	0.141 94

i	$c_3^{(i)}$	$D\,c_3^{(i)}$	$D^2\,c_3^{(i)}$
0	0.214 94	0.225 67	0.258 61
1	0.033 66	0.068 72	0.143 14
2	0.004 40	0.013 38	0.041 02

EARTH AND URANUS.

i	$c_1^{(i)}$	$D\,c_1^{(i)}$	$D^2\,c_1^{(i)}$	$D^3\,c_1^{(i)}$	$D^4\,c_1^{(i)}$
0	2.001 36	0.002 73	0.005 47	0.011 00	0.022 26
1	0.052 18	0.052 29	0.052 61	0.053 57	0.056 46
2	0.002 04	0.004 09	0.008 19	0.016 45	0.033 21

EARTH AND NEPTUNE.

i	$c_1^{(i)}$	$D\,c_1^{(i)}$	$D^2\,c_1^{(i)}$	$D^3\,c_1^{(i)}$	$D^4\,c_1^{(i)}$
0	2.000 56	0.001 11	0.002 22	0.004 45	0.008 95
1	0.033 31	0.033 33	0.033 41	0.033 65	0.034 37
2	0.000 83	0.001 66	0.003 33	0.006 68	0.013 42

MARS AND JUPITER.

i	$c_1^{(i)}$	$D\,c_1^{(i)}$	$D^2\,c_1^{(i)}$	$D^3\,c_1^{(i)}$	$D^4\,c_1^{(i)}$	$D^5\,c_1^{(i)}$	$D^6\,c_1^{(i)}$
0	2.045 08	0.094 854	0.209 66	0.5069	1.4152	4.782	19.80
1	0.302 82	0.323 89	0.392 00	0.6229	1.4638	4.845	20.23
2	0.066 76	0.138 70	0.299 34	0.6946	1.826	5.761	22.47
3	0.016 32	0.050 30	0.157 90	0.5111	1.739	6.391	26.19

i	$c_3^{(i)}$	$D\,c_3^{(i)}$	$D^2\,c_3^{(i)}$	$D^3\,c_3^{(i)}$	$D^4\,c_3^{(i)}$
0	0.715 9	1.015	2.087	6.31	25.07
1	0.304 51	0.717	1.922	6.20	24.58
2	0.110 2	0.367	1.303	5.06	22.15

i	$c_5^{(i)}$	$D\,c_5^{(i)}$	$D^2\,c_5^{(i)}$
0	0.293	0.902	3.420
1	0.186	0.707	3.023
2	0.090	0.425	2.139

MARS AND SATURN.

i	$c_1^{(i)}$	$D\,c_1^{(i)}$	$D^2\,c_1^{(i)}$	$D^3\,c_1^{(i)}$	$D^4\,c_1^{(i)}$
0	2.012 95	0.026 27	0.054 08	0.114 51	0.255 48
1	0.161 29	0.164 44	0.174 12	0.204 21	0.299 91
2	0.019 34	0.039 11	0.079 93	0.166 88	0.362 94

MARS AND URANUS.

i	$c_1^{(i)}$	$D\,c_1^{(i)}$	$D^2\,c_1^{(i)}$	$D^3\,c_1^{(i)}$	$D^4\,c_1^{(i)}$
0	2.003 17	0.006 35	0.012 79	0.025 94	0.053 34
1	0.079 62	0.080 00	0.081 14	0.084 59	0.095 09
2	0.004 74	0.009 51	0.019 12	0.038 64	0.078 90

MARS AND NEPTUNE.

i	$c_1^{(i)}$	$D\,c_1^{(i)}$	$D^2\,c_1^{(i)}$	$D^3\,c_1^{(i)}$	$D^4\,c_1^{(i)}$
0	2.001 29	0.002 58	0.005 18	0.010 43	0.021 13
1	0.050 78	0.050 87	0.051 17	0.052 07	0.054 78
2	0.001 93	0.003 87	0.007 75	0.015 56	0.031 38

§ 9.

NUMERICAL VALUES OF THE P-COEFFICIENTS FOR EACH PAIR OF PLANETS.

MERCURY AND VENUS.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for—		
	1600	1850	2100
$P^{0,0}_{0,0} = 2.172\,16 - 1.624\,45\,\sigma^2 + 7.414\,\sigma^4 - 41\,\sigma^6 + 269\,\sigma^8$	. . . . $-$ 0.122 390	. . . . $-$ 0.121 572	. . . . $-$ 0.120 753
$P^{2,0}_{0,0} = 0.406\,112 - 8.230\,\sigma^2 + 103.3\,\sigma^4 - 1198\,\sigma^6 + 13\,M\,\sigma^8$	$+$ 0.394 336 $-$ 0.6059	$+$ 0.394 494 $-$ 0.6021	$+$ 0.394 652 $-$ 0.5982
$P^{2,2}_{0,0} = 2.058 - 107.5\,\sigma^2 + 2.5\,M\,\sigma^4 - \ldots$	$+$ 1.907 $-$ 7.68	$+$ 1.909 $-$ 7.64	$+$ 1.911 $-$ 7.59
$P^{2,4}_{0,0} = 10.6 - 1001\,\sigma^2 + \ldots$	$+$ 9.1 $-$75.	$+$ 9.2 $-$75.	$+$ 9.2 $-$75.
$P^{4,0}_{0,0} = 0.207\,30 - 15.520\,\sigma^2 + 486\,\sigma^4 - 10.5\,M\,\sigma^6$	$+$ 0.185 69 $-$ 1.081	$+$ 0.185 97 $-$ 1.075	$+$ 0.186 24 $-$ 1.070
$P^{4,2}_{0,0} = 3.88 - 494\,\sigma^2 + \ldots$	$+$ 3.24 $-$36.	$+$ 3.24 $-$36.	$+$ 3.24 $-$36.
$P^{0,4}_{0,0} = 1.025 - 42.3\,\sigma^2 + 1.0\,M\,\sigma^4 - \ldots$	$+$ 0.97 $-$ 3.0	$+$ 0.97 $-$ 3.0	$+$ 0.97 $-$ 3.0
$P^{6,0}_{0,0} = 0.151 - 26\,\sigma^2 + \ldots$	$+$ 0.119 $-$ 1.9	$+$ 0.119 $-$ 1.9	$+$ 0.119 $-$ 1.9
$P^{8,0}_{0,0} = 0.136 - \ldots$	$+$ 0.09	$+$ 0.09	$+$ 0.09 . . . .
$P^{1,1}_{-1,1} = -0.260\,96 + 7.0938\,\sigma^2 - 97.1\,\sigma^4 + 1.15\,M\,\sigma^6 - \ldots$	$-$ 0.250 82 $+$ 0.520 52	$-$ 0.250 96 $+$ 0.517 21	$-$ 0.251 10 $+$ 0.513 91
$P^{3,1}_{-1,1} = -0.6301 + 39.72\,\sigma^2 - 1.15\,M\,\sigma^4 + \ldots$	$-$ 0.5747 $+$ 2.781	$-$ 0.5753 $+$ 2.765 ± 3	$-$ 0.5761 $+$ 2.749
$P^{5,1}_{-1,1} = -0.755 + 110.2\,\sigma^2 - \ldots$	$-$ 0.608 $+$ 7.5	$-$ 0.611 $+$ 7.5	$-$ 0.613 $+$ 7.5
$P^{7,1}_{-1,1} = -0.7 + [200]\,\sigma^2 - \ldots$	$-$ 0.4 $+$14	$-$ 0.4 ± 1 $+$14. ± 2	$-$ 0.4 $+$14
$P^{1,3}_{-1,1} = -1.28 + 64.5\,\sigma^2 - 1.64\,M\,\sigma^4 + 24\,M\,\sigma^6 - \ldots$	$-$ 1.19 $+$ 4.58	$-$ 1.19 $+$ 4.55	$-$ 1.19 $+$ 4.52
$P^{2,2}_{-2,2} = 0.36 - 22\,\sigma^2 + \ldots$	$+$ 0.33 $-$ 1.6	$+$ 0.33 $-$ 1.6	$+$ 0.33 $-$ 1.6
$P^{4,2}_{-2,2} = 1.20 - 152\,\sigma^2 + \ldots$	$+$ 1.00 $-$11.3	$+$ 1.00 $-$11.2 ± 3	$+$ 1.00 $-$11.1
$P'^{\,2,0}_{-2,0} = 4.0982\,\sigma^2 - 52.2\,\sigma^4 + 0.608\,M\,\sigma^6 - \ldots$	$+$ 0.005 863 $+$ 0.301 54	$+$ 0.005 784 0.299 62	$+$ 0.005 706 $+$ 0.297 69
$P'^{\,4,0}_{-2,0} = 7.69\,\sigma^2 - 291\,\sigma^4 + 6.7\,M\,\sigma^6 - \ldots$	$+$ 0.010 61 $+$ 0.5256	$+$ 0.010 47 $+$ 0.5228	$+$ 0.010 32 $+$ 0.5200
$P'^{\,6,0}_{-2,0} = 12.5\,\sigma^2 - [1.0\,M]\,\sigma^4 + \ldots$	$+$ 0.0161 $+$ 0.77	$+$ 0.0159 $+$ 0.77	$+$ 0.0157 $+$ 0.77

NOTE.—Numbers in brackets are derived by induction.

MERCURY AND VENUS—Continued.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for—		
	1600	1850	2100
$P'^{2,2}_{-2,0} = 42.3\,\sigma^2 - 1.31\,M\,\sigma^4 + 27\,M\,\sigma^6 - \ldots$	+ 0.0590 + 2.95	+ 0.0582 2.94	+ 0.0574 + 2.92
$P'^{4,2}_{-2,0} = 223\,\sigma^2 - \ldots$	+ 0.30 +15	+ 0.30 ± 2 15. ± 2	+ 0.30 +15.
$P'^{1,\ 1}_{-1,-1} = -3.6774\,\sigma^2 + 67.0\,\sigma^4 - 0.90\,M\,\sigma^6$	− 0.005 219 − 0.266 28	− 0.005 149 − 0.264 65	− 0.005 080 − 0.263 01
$P'^{3,\ 1}_{-1,-1} = -20.18\,\sigma^2 + 780\,\sigma^4 - 18.4\,M\,\sigma^6$	− 0.027 82 − 1.376	− 0.027 45 − 1.368	− 0.027 08 − 1.361
$P'^{5,\ 1}_{-1,-1} = -55.5\,\sigma^2 + [4\,M]\,\sigma^4 - \ldots$	− 0.077 − 4.2	− 0.076 ± 2 − 4.2 ± 3	− 0.075 − 4.2
$P'^{1,\ 3}_{-1,-1} = -32.9\,\sigma^2 + 1.11\,M\,\sigma^4 - \ldots$	− 0.046 − 2.30	− 0.045 − 2.28	− 0.044 − 2.26
$P'^{0,\ 2}_{0,-2} = 0.829\,\sigma^2 - 19.8\,\sigma^4 + 0.30\,M\,\sigma^6 - \ldots$	+ .001 17 + 0.0590	+ 0.001 15 + 0.0586	+ 0.001 14 + 0.0583
$P'^{2,\ 2}_{0,-2} = 15.6\,\sigma^2 - 643\,\sigma^4 + 15.7\,M\,\sigma^6$	+ 0.0214 + 1.06	+ 0.0211 + 1.05	+ 0.0208 + 1.04
$P'^{4,\ 2}_{0,-2} = 80\,\sigma^2 - [6\,M]\,\sigma^4$	+ 0.105 + 5	+ 0.104 ± 4 + 5. ± 1	+ 0.102 + 5
$P'^{3,1}_{-3,1} = -15.6\,\sigma^2 + 471\,\sigma^4 - 9.5\,M\,\sigma^6$	− 0.0217 − 1.091	− 0.0215 − 1.084	− 0.0211 − 1.076
$P'^{5,1}_{-3,1} = -50\,\sigma^2 + [2\,M]\,\sigma^4$	− 0.068 − 3.4	− 0.068 − 3.4	− 0.068 − 3.4
$P''^{4,0}_{-4,0} = 50\,\sigma^4 - \ldots$	+ 0.000 10 + 0.010	+ 0.000 10 + 0.010	+ 0.000 10 + 0.010

MERCURY AND EARTH.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for—		
	1600	1850	2100
$P^{0,0}_{0,0} = 2.082 - 0.610\,09\,\sigma^2 + 1.426\,\sigma^4$	. . . . − 0.073 167	. . . . − 0.073 214	. . . . − 0.073 262
$P^{2,0}_{0,0} = 0.152\,522 - 1.7301\,\sigma^2 + 9.26\,\sigma^4$	+ 0.146 208 − 0.202 74	+ 0.146 199 − 0.202 87	+ 0.146 191 − 0.203 00
$P^{4,0}_{0,0} = 0.0256 - 1.05\,\sigma^2$	+ 0.0217 − 0.128	+ 0.0217 − 0.128	+ 0.0217 − 0.128
$P^{2,2}_{0,0} = 0.432 - 10.1\,\sigma^2$	+ 0.395 − 1.23	+ 0.395 − 1.23	+ 0.395 − 1.23
$P^{0,4}_{0,0} = 0.3$	+ 0.3	+ 0.3	+ 0.3 . . . .
$P^{6,0}_{0,0} = 0.005 - 1.0\,\sigma^2$	+ 0.001 − 0.12	+ 0.001 − 0.12	+ 0.001 − 0.12

MERCURY AND EARTH—Continued.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for—		
	1600	1850	2100
$P^{4,2}_{0,0} = 0.26$	+ 0.26	+ 0.26	+ 0.26
$P^{1,1}_{-1,1} = -0.072\,350 + 1.2466\,\sigma^2 - 8.1\,\sigma^4$	− 0.067 820 + 0.144 78	− 0.067 814 + 0.144 87	− 0.067 808 + 0.144 96
$P^{3,1}_{-1,1} = -0.099\,33 + 3.10\,\sigma^2$	− 0.087 78 + 0.378	− 0.087 77 + 0.379	− 0.087 75 + 0.379
$P^{1,3}_{-1,1} = -0.248 + 6.3\,\sigma^2$	− 0.226 + 0.77	− 0.225 + 0.77	− 0.226 + 0.77
$P^{5,1}_{-1,1} = -0.038$	− 0.038	− 0.038	− 0.038
$P^{3,3}_{-1,1} = -0.55$	− 0.55	− 0.55	− 0.55
$P^{2,2}_{-2,2} = 0.051 - 1.7\,\sigma^2$	+ 0.045 − 0.207	+ 0.045 − 0.208	+ 0.045 − 0.208
$P^{4,2}_{-2,2} = 0.07$	+ 0.07	+ 0.07	+ 0.07
$P'^{2,0}_{-2,0} = 1.0678\,\sigma^2 - 5.43\,\sigma^4$	+ 3.9015 *m* + 0.1254	+ 3.9067 *m* + 0.1255	+ 3.9116 *m* + 0.1255
$P'^{4,0}_{-2,0} = 0.59\,\sigma^2$	+ 2.20 *m* + 0.072	+ 2.20 *m* + 0.072	+ 2.20 *m* + 0.072
$P'^{2,2}_{-2,0} = 4.9\,\sigma^2$	+18.3 *m* + 0.60	+18.3 *m* + 0.60	+18.3 *m* + 0.60
$P'^{1,\ 1}_{-1,-1} = -0.6595\,\sigma^2 + 5.8\,\sigma^4$	− 2.376 *m* − 0.0752	− 2.379 *m* − 0.0753	− 2.382 *m* − 0.0753
$P'^{3,\ 1}_{-1,-1} = -1.60\,\sigma^2$	− 5.96 *m* − 0.195	− 5.97 *m* − 0.195	− 5.98 *m* − 0.196
$P'^{1,\ 3}_{-1,-1} = -3.3\,\sigma^2$	− 0.0123 − 0.40	− 0.0123 − 0.40	− 0.0123 − 0.40
$P'^{0,\ 2}_{0,-2} = 0.10\,\sigma^2 - 1.3\,\sigma^4$	+ 0.37 *m* + 0.011	+ 0.37 *m* + 0.011	+ 0.37 *m* + 0.011
$P'^{2,\ 2}_{0,-2} = 1.0\,\sigma^2$	+ 3.7 *m* + 0.12	+ 3.7 *m* + 0.12	+ 3.7 *m* + 0.12
$P'^{3,1}_{-3,1} = -1.70\,\sigma^2$	− 6.3 *m* − 0.207	− 6.3 *m* − 0.208	− 6.3 *m* − 0.208
$P'^{1,\ 3}_{1,-3} = -0.2\,\sigma^2$	− 0.7 *m* − 0.02	− 0.7 *m* − 0.02	− 0.7 *m* − 0.02

NOTE.—The symbol *m* means .001.

MERCURY AND MARS.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for— 1600	1850	2100
$P^{0,0}_{0,0} = 2.033 - 0.219\,47\,\sigma^2 + 0.33\,\sigma^4$	. . . . − 0.019 593	. . . . − 0.019 611	. . . . − 0.019 629
$P^{2,0}_{0,0} = 0.054\,8676 - 0.4346\,\sigma^2 + 1.1\,\sigma^4$	+ 0.053 9955 − 0.038 64	+ 0.053 9939 − 0.038 67	+ 0.053 9922 − 0.038 71
$P^{4,0}_{0,0} = 0.003\,02 - 0.08\,\sigma^2$	+ 0.002 86 − 0.007	+ 0.002 86 − 0.007	+ 0.002 86 − 0.007
$P^{2,2}_{0,0} = 0.1086 - 1.34\,\sigma^2$	+ 0.1059 − 0.120	+ 0.1059 − 0.121	+ 0.1059 − 0.121
$P^{0,4}_{0,0} = 0.0787 - 0.8\,\sigma^2$	+ 0.0771 − 0.072	+ 0.0771 − 0.072	+ 0.0771 − 0.072
$P^{1,1}_{-1,1} = -0.017\,25 + 0.2294\,\sigma^2 - 0.9\,\sigma^4$	− 0.016 79 + 0.020 27	− 0.016 79 + 0.020 27	− 0.016 79 + 0.020 27
$P^{3,1}_{-1,1} = -0.0168 + 0.31\,\sigma^2$	− 0.0162 + 0.028	− 0.0162 + 0.028	− 0.0162 + 0.028
$P^{1,3}_{-1,1} = -0.0492 + 0.78\,\sigma^2$	− 0.0476 + 0.070	− 0.0476 + 0.070	− 0.0476 + 0.070
$P^{5,1}_{-1,1} = -0.0020$	− 0.0020	− 0.0020	− 0.0020 . . . .
$P^{3,3}_{-1,1} = -0.06$	− 0.06	− 0.06	− 0.06 . . . .
$P^{1,5}_{-1,1} = -0.10$	− 0.10	− 0.10	− 0.10 . . . .
$P^{2,2}_{-2,2} = 0.0065 - 0.14\,\sigma^2$	+ 0.0062 − 0.013	+ 0.0062 − 0.013	+ 0.0062 − 0.013
$P^{4,2}_{-2,2} = 0.005$	+ 0.005	+ 0.005	+ 0.005 . . . .
$P^{3,3}_{-3,3} = -0.002$	− 0.002	− 0.002	− 0.002 . . . .
$P'^{2,0}_{-2,0} = 0.3149\,\sigma^2 - 0.75\,\sigma^4$	+ 0.6321 *m* + 0.028 01	+ 0.6333 *m* + 0.028 04	+ 0.6344 *m* + 0.028 06
$P'^{4,0}_{-2,0} = 0.052\,\sigma^2$	+ 0.105 *m* + 4.7 *m*	+ 0.105 *m* + 4.7 *m*	+ 0.105 *m* + 4.7 *m*
$P'^{2,2}_{-2,0} = 0.80\,\sigma^2$	+ 1.61 *m* + 0.072	+ 1.62 *m* + 0.072	+ 1.62 *m* + 0.072
$P'^{1,1}_{-1,-1} = -0.1234\,\sigma^2 + 0.7\,\sigma^4$	− 0.2460 *m* −10.83 *m*	− 0.2465 *m* −10.84 *m*	− 0.2470 *m* −10.85 *m*
$P'^{3,1}_{-1,-1} = -0.164\,\sigma^2$	− 0.331 *m* −14.7 *m*	− 0.331 *m* −14.7 *m*	− 0.332 *m* −14.8 *m*
$P'^{1,3}_{-1,-1} = -0.42\,\sigma^2$	− 0.85 *m* −38 *m*	− 0.85 *m* −38 *m*	− 0.85 *m* −38 *m*

NOTE.—The symbol *m* means .001.

MERCURY AND MARS—Continued.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for—		
	1600	1850	2100
$P'^{0,\ 2}_{0,-2} = 0.012\ \sigma^2$	$+24\ m^2$ $+\ 1.1\ m$	$+24\ m^2$ $+\ 1.1\ m$	$+24\ m^2$ $+\ 1.1\ m$
$P'^{2,\ 2}_{0,-2} = 0.08\ \sigma^2$	$+0.16\ m$ $+\ 7\ m$	$+0.16\ m$ $+\ 7\ m$	$+0.16\ m$ $+\ 7\ m$
$P'^{0,\ 4}_{0,-2} = 0.05\ \sigma^2$	$+\ 0.1\ m$ $+\ 4\ m$	$+\ 0.1\ m$ $+\ 4\ m$	$+\ 0.1\ m$ $+\ 4\ m$
$P'^{\ 3,1}_{-3,1} = -0.23\ \sigma^2$	$-\ 0.46\ m$ $-21\ m$	$-\ 0.46\ m$ $-21\ m$	$-\ 0.47\ m$ $-21\ m$
$P'^{1,\ 3}_{1,-3} = -0.01\ \sigma^2$	$-\ 0.2\ m$ $-\ 1\ m$	$-\ 0.2\ m$ $-\ 1\ m$	$-\ 0.2\ m$ $-\ 1\ m$

MERCURY AND JUPITER.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for—		
	1600	1850	2100
$P^{0,0}_{0,0} = 2.0028 - 16.780\ m\ \sigma^2 + 17.4\ m\ \sigma^4 - 1.3\ m\ \sigma^6$	. . . . $-\ 1.828\ 27\ m$	. . . . $-\ 1.829\ 04\ m$	. . . . $-\ 1.829\ 88\ m$
$P^{2,0}_{0,0} = 4.195\ 18\ m - 25.789\ m\ \sigma^2 + 29\ m\ \sigma^4$	$+\ 4.117\ 94\ m$ $-\ 2.8083\ m$	$+\ 4.117\ 88\ m$ $-\ 2.8095\ m$	$+\ 4.117\ 81\ m$ $-\ 2.8108\ m$
$P^{4,0}_{0,0} = 16\ m^2 - 0.3\ m\ \sigma^2 + 2\ m\ \sigma^4$	$+\ 15\ m^2$ $-\ 0.03\ m$	$+\ 15\ m^2$ $-\ 0.03\ m$	$+\ 15\ m^2$ $-\ 0.03\ m$
$P^{2,2}_{0,0} = 6.446\ m - 41.8\ m\ \sigma^2 + 0.05\ m\ \sigma^4$	$+\ 6.321\ m$ $-\ 4.55\ m$	$+\ 6.321\ m$ $-\ 4.55\ m$	$+\ 6.321\ m$ $-\ 4.55\ m$
$P^{0,4}_{0,0} = 5.304\ m - 33.4\ m\ \sigma^2 + 0.04\ \sigma^4$	. . . . $-\ 3.64\ m$	. . . . $-\ 3.64\ m$	. . . . $-\ 3.64\ m$
$P^{\ 1,1}_{-1,1} = -389.89\ m^2 + 4.357\ m\ \sigma^2 - 11\ m\ \sigma^4$	$-376.90\ m^2$ $+471\ m^2$	$-376.89\ m^2$ $+471\ m^2$	$-376.88\ m^2$ $+471\ m^2$
$P^{\ 3,1}_{-1,1} = -299\ m^2 + 3.5\ m\ \sigma^2 - 0.01\ \sigma^4$	$-289\ m^2$ $+\ 0.38\ m$	$-289\ m^2$ $+\ 0.38\ m$	$-289\ m^2$ $+\ 0.38\ m$
$P^{\ 1,3}_{-1,1} = -986\ m^2 + 11.2\ m\ \sigma^2 - 0.02\ \sigma^4$	$-952\ m^2$ $+\ 1.2\ m$	$-952\ m^2$ $+\ 1.2\ m$	$-952\ m^2$ $+\ 1.2\ m$
$P^{\ 3,3}_{-1,1} = -0.76\ m + 0.01\ \sigma^2 - 0.03\ \sigma^4$	$-\ 0.73\ m$ $+\ 1\ m$	$-\ 0.73\ m$ $+\ 1\ m$	$-\ 0.73\ m$ $+\ 1\ m$
$P^{\ 2,2}_{-2,2} = 39\ m^2 - 0.6\ m\ \sigma^2$	$+\ 37\ m^2$ $-\ 0.05\ m$	$+\ 37\ m^2$ $-\ 0.05\ m$	$+\ 37\ m^2$ $-\ 0.05\ m$
$P^{\ 3,3}_{-3,3} = 3.3\ m^2$	$-\ 3.3\ m^2$	$-\ 3.3\ m^2$	$-\ 3.3\ m^2$ . . . .
$P'^{\ 2,0}_{-2,0} = 21.2176\ m\ \sigma^2 - 23.40\ m\ \sigma^4$	$+\ 63.550\ m^2$ $+\ 2.311\ m$	$+\ 63.603\ m^2$ $+\ 2.312\ m$	$+\ 63.662\ m^2$ $+\ 2.313\ m$
$P'^{\ 4,0}_{-2,0} = 0.24\ m\ \sigma^2 - 1\ m\ \sigma^4$	$+\ 0.7\ m^2$ $+\ 26\ m^2$	$+\ 0.7\ m^2$ $+\ 26\ m^2$	$+\ 0.7\ m^2$ $+\ 26\ m^2$

MERCURY AND JUPITER—Continued.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for—		
	1600	1850	2100
$P'^{2,2}_{-2,0} = 33.47\, m\, \sigma^2 - 45\, m\, \sigma^4$	$+100.2\, m^2$ $+\ 3.6\, m$	$+100.2\, m^2$ $+\ 3.6\, m$	$+100.3\, m^2$ $+\ 3.6\, m$
$P'^{1,\ 1}_{-1,-1} = -2.3736\, m\, \sigma^2 + 8.25\, m\, \sigma^4$	$-\ 7.06\, m^2$ $-255\, m^2$	$-\ 7.07\, m^2$ $-255\, m^2$	$-\ 7.08\, m^2$ $-255\, m^2$
$P'^{3,\ 1}_{-1,-1} = -1.88\, m\, \sigma^2 + 7\, m\, \sigma^4$	$-\ 5.6\, m^2$ $-203\, m^2$	$-\ 5.6\, m^2$ $-203\, m^2$	$-\ 5.6\, m^2$ $-203\, m^2$
$P'^{1,\ 3}_{-1,-1} = -6.09\, m\, \sigma^2 + 22\, m\, \sigma^4$	$-\ 18\, m^2$ $-\ 0.66\, m$	$-\ 18\, m^2$ $-\ 0.66\, m$	$-\ 18\, m^2$ $-\ 0.66\, m$
$P'^{0,\ 2}_{0,-2} = 66.3\, m^2\, \sigma^2 - 0.39\, m\, \sigma^4$	$+\ 0.20\, m^2$ $+\ 7\, m^2$	$+\ 0.20\, m^2$ $+\ 7\, m^2$	$+\ 0.20\, m^2$ $+\ 7\, m^2$
$P'^{2,\ 2}_{0,-2} = 341\, m^2\, \sigma^2 - 2\, m\, \sigma^4$	$+\ 1\, m^2$ $+\ 0.03\, m$	$+\ 1\, m^2$ $+\ 0.03\, m$	$+\ 1\, m^2$ $+\ 0.03\, m$
$P'^{0,\ 4}_{0,-2} = 0.24\, m\, \sigma^2 - 1\, m\, \sigma^4$	$+\ 1\, m^2$ $+\ 0.02\, m$	$+\ 1\, m^2$ $+\ 0.02\, m$	$+\ 1\, m^2$ $+\ 0.02\, m$
$P'^{3,1}_{-3,1} = -3.54\, m\, \sigma^2 + 8\, m\, \sigma^4$	$-\ 11\, m^2$ $-\ 0.3\, m$	$-\ 11\, m^2$ $-\ 0.3\, m$	$-\ 11\, m^2$ $-\ 0.3\, m$

MERCURY AND SATURN.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ fo:—		
	1600	1850	2100
$P^{0,0}_{0,0} = 2.000\,82 - 4.9558\, m\, \sigma^2 + 5.1\, m\, \sigma^4$	. . . . $-551.79\, m^2$	. . . . $-549.35\, m^2$	. . . . $-546.93\, m^2$
$P^{2,0}_{0,0} = 1.238\,95\, m - 7.487\, m\, \sigma^2 + 7\, m\, \sigma^4$	$+\ 1.215\,51\, m$ $-834.1\, m^2$	$+\ 1.215\,72\, m$ $-830.4\, m^2$	$+\ 1.215\,93\, m$ $-826.9\, m^2$
$P^{4,0}_{0,0} = 1.4\, m^2 - 0.03\, m\, \sigma^2$	$+\ 1.3\, m^2$ $-\ 3\, m^2$	$+\ 1.3\, m^2$ $-\ 3\, m^2$	$+\ 1.3\, m^2$ $-\ 3\, m^2$
$P^{2,2}_{0,0} = 1.872\, m - 11\, m\, \sigma^2 + 0.01\, \sigma^4$	$+\ 1.838\, m$ $-\ 1.2\, m$	$+\ 1.838\, m$ $-\ 1.2\, m$	$+\ 1.838\, m$ $-\ 1.2\, m$
$P^{0,4}_{0,0} = 1.5\, m - 9\, m\, \sigma^2$	. . . . $-\ 1.0\, m$	. . . . $-\ 1.0\, m$	. . . . $-\ 1.0\, m$
$P^{4,2}_{0,0} = 7\, m^2 - 0.2\, m\, \sigma^2$	$+\ 6\, m^2$ $-\ 0.02\, m$	$+\ 6\, m^2$ $-\ 0.02\, m$	$+\ 6\, m^2$ $-\ 0.02\, m$
$P^{2,4}_{0,0} = 2.3\, m - 0.01\, \sigma^2$	$+\ 2.3\, m$ $-\ 1\, m$	$+\ 2.3\, m$ $-\ 1\, m$	$+\ 2.3\, m$ $-\ 1\, m$
$P^{1,1}_{-1,1} = -62.83\, m^2 + 694.4\, m^2\, \sigma^2 - 1.61\, m\, \sigma^4$	$-\ 60.66\, m^2$ $+\ 76.70\, m^2$	$-\ 60.68\, m^2$ $+\ 76.36\, m^2$	$-\ 60.70\, m^2$ $+\ 76.05\, m^2$
$P^{3,1}_{-1,1} = -47.5\, m^2 + 0.53\, m\, \sigma^2 - 1\, m\, \sigma^4$	$45.9\, m^2$ $+\ 58.8\, m^2$	$-\ 45.9\, m^2$ $+\ 58.6\, m^2$	$-\ 45.9\, m^2$ $+\ 58.3\, m^2$
$P^{1,3}_{-1,1} = -158\, m^2 + 1.7\, m\, \sigma^2 - 4\, m\, \sigma^4$	$-153\, m^2$ $+\ 0.19\, m$	$-153\, m^2$ $+\ 0.19\, m$	$-153\, m^2$ $+\ 0.19\, m$

MERCURY AND SATURN—Continued.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for—		
	1600	1850	2100
$P_{-1,1}^{3,3} = -0.11\,m + 1\,m\,\sigma^2$	− 0.11 m + 0.1 m	− 0.11 m + 0.1 m	− 0.11 m + 0.1 m
$P'^{\,2,0}_{-2,0} = 6.216\,m\,\sigma^2 - 6.41\,m\,\sigma^4$	+ 19.46 m^2 +692.2 m^2	+ 19.28 m^2 +689.1 m^2	+ 19.11 m^2 +686.2 m^2
$P'^{\,4,0}_{-2,0} = 20\,m^2\,\sigma^2 - 0.11\,m\,\sigma^4$	+ 0.06 m^2 + 2.1 m^2	+ 0.06 m^2 + 2.1 m^2	+ 0.06 m^2 + 2.1 m^2
$P'^{\,2,2}_{-2,0} = 9.466\,m\,\sigma^2 - 10.42\,m\,\sigma^4$	+ 29.62 m^2 + 1.054 m	+ 29.36 m^2 + 1.049 m	+ 29.19 m^2 + 1.044 m
$P'^{\,1,\ 1}_{-1,-1} = -378.6\,m^2\,\sigma^2 + 1.28\,m\,\sigma^4$	− 1.18 m^2 − 41.52 m^2	− 1.17 m^2 − 41.34 m^2	− 1.16 m^2 − 41.18 m^2
$P'^{\,3\ \ 1}_{-1-1} = -288\ 7\,m^2\,\sigma^2 + 1.00\,m\,\sigma^4$	− 0.90 m^2 − 31.65 m^2	− 0.89 m^2 − 31.51 m^2	− 0.88 m^2 − 31.39 m^2

MERCURY AND URANUS.

General values of P.	Special values of P and $\frac{dP'}{d\sigma}$ for—		
	1600	1850	2100
$P^{0,0}_{0,0} = 2.0002 - 1.223\,m\,\sigma^2 + 1.2\,m\,\sigma^4$	. . . . −134.32 m^2	. . . . −134.03 m^2	. . . . −133.77 m^2
$P^{2,0}_{0,0} = 305.61\,m^2 - 1.83\,m\,\sigma^2 + 1.8\,m\,\sigma^4$	+300.06 m^2 −201.1 m^2	+300.08 m^2 −200.7 m^2	+300.10 m^2 −200.3 m^2
$P^{2,2}_{0,0} = 459\,m^2 - 2.77\,m\,\sigma^2 + 3\,m\,\sigma^4$	+451 m^2 − 30 m^2	+451 m^2 − 30 m^2	+451 m^2 − 30 m^2
$P^{1,1}_{-1,1} = -7.21\,m^2 + 85\,m^2\,\sigma^2$	− 6.95 m^2 + 9.4 m^2	− 6.95 m^2 + 9.4 m^2	− 6.95 m^2 + 9.4 m^2
$P^{3,1}_{-1,1} = -5.79\,m^2 + 64\,m^2\,\sigma^2$	− 5.60 m^2 + 7.1 m^2	− 5.60 m^2 + 7.1 m^2	− 5.60 m^2 + 7.1 m^2
$P'^{\,2,0}_{-2,0} = 1.529\,m\,\sigma^2 - 1.5\,m\,\sigma^4$	+ 4.65 m^2 +168.0 m^2	+ 4.63 m^2 +167.6 m^2	+ 4.61 m^2 +167.3 m^2
$P'^{\,2,2}_{-2,0} = 2.3\,m\,\sigma^2 - 2.4\,m\,\sigma^4$	+ 6.9 m^2 + 0.25 m	+ 6.9 m^2 + 0.25 m	+ 6.9 m^2 + 0.25 m
$P'^{\,1,\ 1}_{-1,-1} = -0.05\,m\,\sigma^2 + 0.2\,m\,\sigma^4$	− 0.15 m^2 − 5.5 m^2	− 0.15 m^2 − 5.5 m^2	+ 0.15 m^2 − 5.5 m^2

NOTE.—The symbol m means .001.

MERCURY AND NEPTUNE.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for—		
	1600	1850	2100
$P^{0,0}_{0,0} = 2.000 - 498\, m^2 \sigma^2 + 0.5\, m\, \sigma^4$	. . . . $-60.60\, m^3$	. . . . $-60.56\, m^2$	. . . . $-60.52\, m^2$
$P^{2,0}_{0,0} = 124.61\, m^2 - 748\, m^2 \sigma^2 + 0.7\, m\, \sigma^4$	$+121.81\, m^2$ $-91.09\, m^2$	$+121.81\, m^2$ $-91.03\, m^2$	$+121.81\, m^2$ $-90.97\, m^2$
$P^{2,2}_{0,0} = 187.04\, m^2 - 1.12\, m\, \sigma^2 + 1.0\, m\, \sigma^4$	$+182.82\, m^2$ $-136.2\, m^2$	$+182.83\, m^2$ $-136.1\, m^2$	$+182.84\, m^2$ $-136.0\, m^2$
$P^{1,1}_{-1,1} = -2.01\, m^2 + 22\, m^2 \sigma^2$	$-\ 1.93\, m^2$ $+\ 2.7\, m^2$	$-\ 1.93\, m^2$ $+\ 2.7\, m^2$	$-\ 1.93\, m^2$ $+\ 2.7\, m^2$
$P^{3,1}_{-1,1} = -1.51\, m^2 + 0.02\, m\, \sigma^2$	$-\ 1.43\, m^2$ $+\ 2\, m^2$	$-\ 1.43\, m^2$ $+\ 2\, m^2$	$-\ 1.43\, m^2$ $+\ 2\, m^2$
$P'^{\,2,0}_{-2,0} = 623.2\, m^2 \sigma^2 - 0.62\, m\, \sigma^4$	$+\ 2.34\, m^2$ $+\ 75.9\, m^2$	$+\ 2.33\, m^2$ $+\ 75.8\, m^2$	$+\ 2.33\, m^2$ $+\ 75.8\, m^2$
$P'^{\,2,2}_{-2,0} = 936.4\, m^2 \sigma^2 - 0.94\, m\, \sigma^4$	$+\ 3.51\, m^2$ $+114\, m^2$	$+\ 3.51\, m^2$ $+114\, m^2$	$+\ 3.50\, m^2$ $+114\, m^2$
$P'^{\,1,\ 1}_{-1,-1} = -12.0\, m^2 \sigma^2 + 0.04\, m\, \sigma^4$	$-\ 45\, m^3$ $-\ 1.4\, m^2$	$-\ 45\, m^3$ $-\ 1.4\, m^2$	$-\ 45\, m^3$ $-\ 1.4\, m^2$

VENUS AND EARTH.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for—		
	1600	1850	2100
$P^{0,0}_{0,0} = 2{:}386 - 6.4171\, \sigma^2 + 97.78\, \sigma^4 - 2064\, \sigma^6 + 51.3\, M\, \sigma^8$	. . . . $-\ 0.369\,89$	. . . . $-\ 0.370\,08$	. . . . $-\ 0.370\,37$
$P^{2,0}_{0,0} = 1.604\,28 - 100.97\, \sigma^2 + 4791\, \sigma^4 - 212\, M\, \sigma^6 + 9.0\, M^2 \sigma^8$	$+\ 1.519\,42$ $-\ 5.5062$	$+\ 1.519\,32$ $-\ 5.5090$	$+\ 1.519\,19$ $-\ 5.5123$
$P^{4,0}_{0,0} = 4.0 - 922\, \sigma^2$	$+\ 3.2$ $-\ 54.56$	$+\ 3.2$ $-\ 54.59$	$+\ 3.2$ $-\ 54.64$
$P^{2,2}_{0,0} = 25.2 - 4841\, \sigma^2$	$+\ 21.0$ -286.5	$+\ 21.0$ -286.6	$+\ 21.0$ -286.8
$P^{0,4}_{0,0} = 9.41 - 1549\, \sigma^2$	$+\ 8.06$ $-\ 91.7$	$+\ 8.06$ $-\ 91.7$	$+\ 8.06$ $-\ 91.8$
$P^{0,6}_{0,0} = 71 - [28\, M]\, \sigma^2$	$+\ 47$ $-\ 1.66\, M$	$+\ 46$ $-\ 1.66\, M$	$+\ 46$ $-\ 1.66\, M$
$P^{2,4}_{0,0} = 387 - [140\, M]\, \sigma^2$	$+264$ $-\ 8.29\, M$	$+264$ $-\ 8.29\, M$	$+264$ $-\ 8.30\, M$
$P^{1,1}_{-1,1} = -1.335\,76 + 96.86\, \sigma^2 - 4706\, \sigma^4 + 209\, M\, \sigma^6$	$-\ 1.254\,43$ $+\ 5.273$	$-\ 1.254\,33$ $+\ 5.275$	$-\ 1.254\,21$ $+\ 5.279$

VENUS AND EARTH—Continued.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for—		
	1600	1850	2100
$P_{-1,1}^{3,1} = -9.9 + 2094\,\sigma^2 - 218\,M\,\sigma^4 + [18\,M^2]\,\sigma^6$	− 8.2 +103.8	− 8.2 +103.8	− 8.2 +103.9
$P_{-1,1}^{1,3} = -14.9 + 2708\,\sigma^2 - 262\,M\,\sigma^4 + [21\,M^2]\,\sigma^6$	− 12.7 +136.0	− 12.7 +136.0	− 12.7 +136.1
$P_{-1,1}^{1,5} = -164$	−164	−164	−164 . . . .
$P_{-2,2}^{2,2} = 5.7$	+ 5.7	+ 5.7	+ 5.7 . . . .
$P_{-2,2}^{2,4} = 124$	+124	+124	+124 . . . .
$P'^{\,2,0}_{-2,0} = 37.7\,\sigma^2 - 2.03\,M\,\sigma^4 + 94\,M\,\sigma^6$	+ 0.0315 + 2.033	+ 0.0316 + 2.034	+ 0.0316 + 2.036
$P'^{\,2,2}_{-2,0} = 1549\,\sigma^2 - 192\,M\,\sigma^4$	+ 1.209 + 71.8	+ 1.211 + 71.8	+ 1.212 + 71.8
$P'^{\,1,\ 1}_{-1,-1} = -49.09\,\sigma^2 + 3.17\,M\,\sigma^4 - 158\,M\,\sigma^6$	− 40.66 *m* − 2.598	− 40.71 *m* − 2.599	− 40.78 *m* − 2.600
$P'^{\,1,\ 3}_{-1,-1} = -1360\,\sigma^2 + 176\,M\,\sigma^4$	− 1.056 − 62.26	− 1.057 − 62.26	− 1.059 − 62.28
$P'^{\,0,\ 2}_{0,-2} = 16.06\,\sigma^2 - 1.21\,M\,\sigma^4 + 65\,M\,\sigma^6$	+ 13.18 *m* + 0.834	+ 13.19 *m* + 0.834	+ 13 21 *m* + 0.835
$P'^{\,0,\ 4}_{0,-2} = 387\,\sigma^2 - 52\,M\,\sigma^4$	+ 0.299 + 17.5	+ 0.299 + 17.5	+ 0.299 + 17.5
$P'^{\,2,\ 2}_{0,-2} = 922\,\sigma^{`} - 133\,M\,\sigma^4$	+ 0.706 + 40.8	+ 0.707 + 40.8	+ 0.708 + 40.8
$P'^{\,1,\ 3}_{1,-3} = -262\,\sigma^2 + 39\,M\,\sigma^4$	− 0.200 − 11.4	− 0.200 − 11.5	− 0.200 − 11.5
$P'^{\,3,1}_{-3,1} = -557\,\sigma^2 + 68\,M\,\sigma^4$	− 0.443 − 26.5	− 0.444 − 26.5	− 0.445 − 26.5

NOTE—The symbol M means 1000.
The symbol *m* means .001.
Numbers in brackets have been derived wholly or in part by induction from the law of progression of the various series.

VENUS AND MARS.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for—		
	1600	1850	2100
$P^{0,0}_{0,0} = 2.129 - 1.0940\,\sigma^2 + 3.7\,\sigma^4$	. . . . − 0.036 649	. . . . − 0.036 854	. . . . − 0.037 073
$P^{2,0}_{0,0} = 0.273\,50 - 4.26\,\sigma^2$	+ 0.272 30 − 0.143	+ 0.272 29 − 0.144	+ 0.272 27 − 0.145
$P^{2,2}_{0,0} = 1.066 - 40\,\sigma^2$	+ 1.055 − 1.34	+ 1.055 − 1.35	+ 1.054 − 1.36
$P^{0,4}_{0,0} = 0.581 - 17\,\sigma^2$	+ 0.576 − 0.57	+ 0.576 − 0.57	+ 0.576 − 0.58
$P^{2,4}_{0,0} = 4.2$	+ 4.2	+ 4.2	+ 4.2 . . . .
$P^{0,6}_{0,0} = 1.4$	+ 1.4	+ 1.4	+ 1.4 . . . .
$P^{\ 1,1}_{-1,1} = -0.157\,38 + 3.46\,\sigma^2$	− 0.156 41 + 0.116	− 0.156 39 + 0.117	− 0.156 38 + 0.117
$P^{\ 3,1}_{-1,1} = -0.3$	− 0.3	− 0.3	− 0.3 . . . .
$P^{\ 1,3}_{-1,1} = -0.651 + 24\,\sigma^2$	− 0.644 + 0.8	− 0.644 + 0.8	− 0.644 + 0.8
$P^{\ 1,5}_{-1,1} = -2.2$	− 2.2	− 2.2	− 2.2 . . . .
$P^{\ 2,2}_{-2,2} = 0.16$	+ 0.16	+ 0.16	+ 0.16 . . . .
$P'^{\ 2,0}_{-2,0} = 2.3\,\sigma^2$	+ 0.6 m + 0.08	+ 0.7 m + 0.08	+ 0.7 m + 0.08
$P'^{\ 2,2}_{-2,0} = 17\,\sigma^2$	+ 5 m + 0.6	+ 5 m + 0.6	+ 5 m + 0.6
$P'^{\ 1,\ \ 1}_{-1,-1} = -1.81\,\sigma^2$	− 0.5 m −61 m	− 0.5 m −61 m	− 0.5 m −61 m
$P'^{\ 1,\ \ 3}_{-1,-1} = -12\,\sigma^2$	− 3.4 m − 0.4	− 3.4 m − 0.4	− 3.5 m − 0.4
$P'^{0,\ \ 2}_{0,-2} = 0.36\,\sigma^2$	+ 0.101 m + 0.012	+ 0.103 m + 0.012	+ 0.104 m + 0.012
$P'^{0,\ \ 4}_{0,-2} = 2.6\,\sigma^2$	+ 0.7 m + 0.09	+ 0.7 m + 0.09	+ 0.7 m + 0.09

VENUS AND JUPITER.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for		
	1600	1850	2100
$P_{0,0}^{0,0} = 2.009\,77 - 60.148\,m\,\sigma^2 + 69\,m\,\sigma^4 - 0.01\,\sigma^6$	. . . . $-\ 2.3451\,m$	. . . . $-\ 2.3630\,m$	. . . . $-\ 2.3816\,m$
$P_{0,0}^{2,0} = 15.0371\,m - 98.14\,m\,\sigma^2 + 140\,m\,\sigma^4 - 0.07\,\sigma^6$	$+\ 14.9997\,m$ $-\ 3.8255\,m$	$+\ 14.9992\,m$ $-\ 3.8547\,m$	$+\ 14.9986\,m$ $-\ 3.8851\,m$
$P_{0\,0}^{4,0} = 216.5\,m^2 - 4.9\,m\,\sigma^2 + 0.03\,\sigma^4$	$+214.6\,m^2$ $-191\,m^2$	$+214.6\,m^2$ $-192\,m^2$	$+214.6\,m^2$ $-193\,m^2$
$P_{0,0}^{2,2} = 24.533\,m - 0.190\,\sigma^2 + 0.5\,\sigma^4$	$+\ 24.461\,m$ $-\ 7.39\,m$	$+\ 24.460\,m$ $-\ 7.45\,m$	$+\ 24.459\,m$ $-\ 7.50\,m$
$P_{0,0}^{0,4} = 19.56\,m - 140\,m\,\sigma^2 + 0.2\,\sigma^4$	. . . . $-\ 5.45\,m$	. . . . $-\ 5.49\,m$	. . . . $-\ 5.53\,m$
$P_{0,0}^{0,0} = 4\,m^2$	$+\ 4\,m^2$	$+\ 4\,m^2$	$+\ 4\,m^2$ . . . .
$P_{0,0}^{4,2} = 1.2\,m - 0.03\,\sigma^2$	$+\ 1.2\,m$ $-\ 1.2\,m$	$+\ 1.2\,m$ $-\ 1.2\,m$	$+\ 1.2\,m$ $-\ 1.2\,m$
$P_{0,0}^{2,4} = 34.8\,m - 0\ 2\,\sigma^2$	$+\ 34.7\,m$ $-\ 8\,m$	$+\ 34.7\,m$ $-\ 8\,m$	$+\ 34.7\,m$ $-\ 8\,m$
$P_{0\,0}^{0,6} = 24.1\,m - 0.2\,\sigma^2$	. . . . $-\ 8\,m$	. . . . $-\ 8\,m$	. . . . $-\ 8\,m$
$P_{-1,1}^{1,1} = -2.606\,85\,m + 30.31\,m\,\sigma^2 - 82\,m\,\sigma^4$	$-2.595\,32\,m$ $+\ 1.1804\,m$	$-2.595\,14\,m$ $+\ 1.1893\,m$	$-2.594\,96\,m$ $+\ 1.1987\,m$
$P_{-1,1}^{3,1} = -2.115\,m + 27.7\,m\,\sigma^2 - 98\,m\,\sigma^4$	$-\ 2.104\,m$ $+\ 1.079\,m$	$-\ 2.104\,m$ $+\ 1.087\,m$	$-\ 2.104\,m$ $+\ 1.095\,m$
$P_{-1,1}^{1,3} = -6.767\,m + 83.3\,m\,\sigma^2 - 0.27\,\sigma^4$	$-\ 6.734\,m$ $+\ 3.243\,m$	$-\ 6.734\,m$ $+\ 3.268\,m$	$-\ 6.734\,m$ $+\ 3.294\,m$
$P_{-1,1}^{5,1} = -0.06\,m + 3\,m\,\sigma^2$	$-\ 0.06\,m$ $+\ 0.12\,m$	$-\ 0.06\,m$ $+\ 0.12\,m$	$-\ 0.06\,m$ $+\ 0.12\,m$
$P_{-1,1}^{3,3} = -5.9\,m + 0.09\,\sigma^2$	$-\ 5.9\,m$ $+\ 3.5\,m$	$-\ 5.9\,m$ $+\ 3.5\,m$	$-\ 5.9\,m$ $+\ 3.5\,m$
$P_{-1,1}^{1,5} = -0.01 + 0.1\,\sigma^2$	$-\ 0.01$ $+\ 4\,m$	$-\ 0.01$ $+\ 4\,m$	$-\ 0.01$ $-\ 4\,m$
$P_{-2,2}^{2,2} = 0.49\,m - 8.6\,m\,\sigma^2$	$+\ 0.49\,m$ $-336\,m$	$+\ 0.49\,m$ $-338\,m^2$	$+\ 0.49\,m$ $-341\,m^2$
$P_{-2,2}^{4,2} = 0.30\,m - 7\,m\,\sigma^2$	$+\ 0.30\,m$ $-\ 0.28\,m$	$+\ 0.30\,m$ $-\ 0.28\,m$	$+\ 0.30\,m$ $-\ 0.28\,m$
$P_{-2,2}^{2,4} = 1.8\,m - 0.04\,\sigma^2$	$+\ 1.8\,m$ $-\ 1.6\,m$	$+\ 1.8\,m$ $-\ 1.6\,m$	$+\ 1.8\,m$ $-\ 1.6\,m$
$P_{-3,3}^{3,3} = -0.08\,m$	$-\ 0.08\,m$	$-\ 0.08\,m$	$-\ 0.08\,m$ . . . .
$P'^{\,2,0}_{-2,0} = 78.276\,m\,\sigma^2 - 107.4\,m\,\sigma^4$	$+\ 29.78\,m^2$ $+3.051\,45\,m$	$+\ 30.25\,m^2$ $+3.074\,67\,m$	$+\ 30.72\,m^2$ $+3.098\,97\,m$
$P'^{\,4,0}_{-2,0} = 3.24\,m\,\sigma^2 - 22\,m\,\sigma^4$	$+\ 1\,m^2$ $+125\,m^2$	$+\ 1\,m^2$ $+126\,m^2$	$+\ 1\,m^2$ $+127\,m^2$

VENUS AND JUPITER—Continued.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for—		
	1600	1850	2100
$P'^{2,2}_{-2,0} = 139.3\, m\, \sigma^2 - 305\, m\, \sigma^4$	$+53\, m^2$ $+5.428\, m$	$+54\, m^2$ $+5.468\, m$	$+55\, m^2$ $+5.512\, m$
$P'^{1,1}_{-1,-1} = -16.4611\, m\, \sigma^2 + 63.57\, m\, \sigma^4$	$-6.26\, m^2$ $-640.48\, m^2$	$-6.36\, m^2$ $-645.33\, m^2$	$-6.46\, m^2$ $-650.42\, m^2$
$P'^{3,1}_{-1,-1} = -14.9\, m\, \sigma^2 + 74\, m\, \sigma^4$	$-6\, m^2$ $-580\, m^2$	$-6\, m^2$ $-584\, m^2$	$-6\, m^2$ $-588\, m^2$
$P'^{1,3}_{-1,-1} = -45.1\, m\, \sigma^2 + 0.20\, \sigma^4$	$-17\, m^2$ $-1.754\, m$	$-17\, m^2$ $-1.767\, m$	$-17\, m^2$ $-1.781\, m$
$P'^{0,2}_{0,-2} = 866\, m^2\, \sigma^2 - 5.52\, m\, \sigma^4$	$+329\, m^3$ $+33.65\, m^2$	$+334\, m^3$ $+33.90\, m^2$	$+339\, m^3$ $+34.16\, m^2$
$P'^{2,2}_{0,-2} = 4.79\, m\, \sigma^2 - 34\, m\, \sigma^4$	$+2\, m^2$ $+186\, m^2$	$+2\, m^2$ $+187\, m^2$	$+2\, m^2$ $+189\, m^2$
$P'^{0,4}_{0,-2} = 3.24\, m\, \sigma^2 - 22.2\, m\, \sigma^4$	$+1\, m^2$ $+126\, m^2$	$+1\, m^2$ $+127\, m^2$	$+1\, m^2$ $+128\, m^2$
$P'^{3,1}_{-3,1} = -25.9\, m\, \sigma^2 + 78\, m\, \sigma^4$	$-10\, m^2$ $-1.01\, m$	$-10\, m^2$ $-1.01\, m$	$+10\, m^2$ $-1.01\, m$
$P'^{1,3}_{1,-3} = -0.35\, m\, \sigma^2 + 3.3\, m\, \sigma^4$	$0\, m^2$ $-14\, m^2$	$0\, m^2$ $-14\, m^2$	$0\, m^2$ $-14\, m^2$

VENUS AND SATURN.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for—		
	1600	1850	2100
$P^{0,0}_{0,0} = 2.002\,89 - 17.438\, m\, \sigma^2 - 18\, m\, \sigma^4 - 1.4\, m\, \sigma^6$	. . . . $-624.49\, m^2$	. . . . $-624.47\, m^2$	. . . . $-624.59\, m^2$
$P^{2,0}_{0,0} = 4.359\,53\, m - 26.822\, m\, \sigma^2 + 30.3\, m\, \sigma^4 - 7.3\, m\, \sigma^6$	$+4.350\,92\, m$ $-960.50\, m^2$	$+4.350\,92\, m$ $-960.47\, m^2$	$-4.350\,92\, m$ $-960.65\, m^2$
$P^{4,0}_{0,0} = 17.9\, m^2 - 0.37\, m\, \sigma^2 + 2\, m\, \sigma^4$	$+17.8\, m^2$ $-13\, m^2$	$+17.8\, m^2$ $-13\, m^2$	$+17.8\, m^2$ $-13\, m^2$
$P^{2,2}_{0,0} = 6.7056\, m - 43.6\, m\, \sigma^2 + 0.05\, \sigma^4$	$+6.6916\, m$ $-1.560\, m$	$+6.6916\, m$ $-1.560\, m$	$+6.6916\, m$ $-1.560\, m$
$P^{0,4}_{0,0} = 5.514\, m - 34.9\, m\, \sigma^2 + 0.04\, \sigma^4$	. . . . $-1.249\, m$	. . . . $-1.249\, m$	. . . . $-1.249\, m$
$P^{2,4}_{0,0} = 8.7\, m - 0.06\, \sigma^2$	$+8.7\, m$ $-2.2\, m$	$+8.7\, m$ $-2.2\, m$	$+8.7\, m$ $-2.2\, m$
$P^{1,1}_{-1,1} = -412.932\, m^2 + 4.618\, m\, \sigma^2 - 11\, m\, \sigma^4$	$-411.450\, m^2$ $+165.24\, m^2$	$-411.449\, m^2$ $+165.24\, m^2$	$-411.449\, m^2$ $+165.27\, m^2$
$P^{3,1}_{-1,1} = -317.1\, m^2 + 3.7\, m\, \sigma^2$	$-315.9\, m^2$ $+132\, m^2$	$-315.9\, m^2$ $+132\, m^2$	$-315.9\, m^2$ $+132\, m^2$

N TE.—The symbol m means .001.

VENUS AND SATURN—Continued.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for— 1600	1850	2100
$P_{-1,1}^{1,3} = -1.044\, m + 11.8\, m\, \sigma^2$	$-\ 1.040\, m$ $+423\, m^2$	$-\ 1.040\, m$ $+423\, m^2$	$-\ 1.040\, m$ $+423\, m^2$
$P_{-1,1}^{1,5} = -1.9\, m + 0.02\, \sigma^2$	$-\ 1.9\, m$ $+\ 0.7\, m$	$-\ 1.9\, m$ $+\ 0.7\, m$	$-\ 1.9\, m$ $+\ 0.7\, m$
$P_{-2,2}^{2,2} = 42\, m^2 - 0.68\, m\, \sigma^2$	$+\ 42\, m^2$ $-\ 24.4\, m^2$	$+\ 42\, m^2$ $+\ 24.4\, m^2$	$+\ 42\, m^2$ $+\ 24.4\, m^2$
$P'^{2,0}_{-2,0} = 22.0585\, m\, \sigma^2 - 24.43\, m\, \sigma^4$	$+\ 7.08\, m^2$ $+789.93\, m^2$	$+\ 7.08\, m^2$ $+789.91\, m^2$	$+\ 7.08\, m^2$ $+790.05\, m^2$
$P'^{2,2}_{-2,0} = 35.0\, m\, \sigma^2 - 47.\, m\, \sigma^4$	$+\ 11\, m^2$ $+\ 1.252\, m$	$+\ 11\, m^2$ $+\ 1.252\, m$	$+\ 11\, m^2$ $+\ 1.252\, m$
$P'^{1,\ 1}_{1,-1} = -2.515\,35\, m\, \sigma^2 + 8.7\, m\, \sigma^4$	$-\ 0.81\, m^2$ $-\ 89.94\, m^2$	$-\ 0.81\, m^2$ $-\ 89.94\, m^2$	$+\ 0.81\, m^2$ $-\ 89.95\, m^2$
$P'^{1,\ 3}_{-1,-1} = -6.5\, m\, \sigma^2 + 0.02\, \sigma^4$	$-\ 2\, m^2$ $-233\, m^2$	$-\ 2\, m^2$ $-233\, m^2$	$-\ 2\, m^2$ $-233\, m^2$
$P'^{0,\ 2}_{0-2} = 71\, m^2\, \sigma^2 - 0.4\, m\, \sigma^4$	$+\ 0.02\, m^2$ $+\ 2.55\, m^2$	$+\ 0.02\, m^2$ $+\ 2.55\, m^2$	$+\ 0.02\, m^2$ $+\ 2.55\, m^2$
$P'^{0,\ 4}_{0,-2} = 0.26\, m\, \sigma^2 - 1.6\, m\, \sigma^4$	$+\ 0.1\, m^2$ $+\ 9\, m^2$	$+\ 0.1\, m^2$ $+\ 9\, m^2$	$+\ 0.1\, m^2$ $+\ 9\, m^2$

VENUS AND URANUS.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for— 1600	1850	2100
$P_{0,0}^{0,0} = \ldots\ldots -4.2766\, m\, \sigma^2 + 4.4\, m\, \sigma^4$	. . . . $-195.66\, m^2$	. . . . $-195.43\, m^2$	. . . . $-195.24\, m^2$
$P_{0,0}^{2,0} = 1.069\,15\, m - 6.4549\, m\, \sigma^2 + 6.66\, m\, \sigma^4$	$+1.065\,78\, m$ $-295.31\, m^2$	$+1.065\,79\, m$ $-294.96\, m^2$	$+1.065\,79\, m$ $-294.68\, m^2$
$P_{0,0}^{2,2} = 1.6137\, m - 9.87\, m\, \sigma^2$	$+\ 1.6085\, m$ $-452\, m^2$	$+\ 1.6085\, m$ $-451\, m^2$	$+\ 1.6085\, m$ $-451\, m^2$
$P_{0,0}^{0,4} = 1.34\, m - 8.0\, m\, \sigma^2$	. . . . $-139.7\, m^2$	. . . . $-139.5\, m^2$	. . . . $-139.4\, m^2$
$P_{-1,1}^{1,1} = -50.3845\, m^2 + 556.46\, m^2\, \sigma^2 - 1.28\, m\, \sigma^4$	$-\ 50.093\, m^2$ $+\ 25.49\, m^2$	$-\ 50.094\, m^2$ $+\ 25.46\, m^2$	$-\ 50.094\, m^2$ $+\ 25.43\, m^2$
$P_{-1,1}^{3,1} = -38.0\, m^2 + 423\, m^2\, \sigma^2$	$-\ 37.8\, m^2$ $+\ 19.3\, m^2$	$-\ 37.8\, m^2$ $+\ 19.3\, m^2$	$-\ 37.8\, m^2$ $+\ 19.3\, m^2$
$P_{-1,1}^{1,3} = -127\, m^2 + 1.40\, m\, \sigma^2$	$-126\, m^2$ $+\ 64.1\, m^2$	$-126\, m^2$ $+\ 64.1\, m^2$	$-126\, m^2$ $+\ 64.1\, m^2$
$P_{-2,2}^{2,2} = 2.5\, m^2$	$+\ 2.5\, m^2$	$+\ 2.5\, m^2$	$+\ 2.5\, m^2$ . . . .

VENUS AND URANUS—Continued.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for—		
	1600	1850	2100
$P'^{2,0}_{-2,0} = +5.36\, m\, \sigma^2 - 5.50\, m\, \sigma^4$	$+\ 2.81\, m^2$ $+245.2\, m^2$	$+\ 2.80\, m^2$ $+244.9\, m^2$	$+\ 2.80\, m^2$ $+244.7\, m^2$
$P'^{2,2}_{-2,0} = +8.15\, m\, \sigma^2$	$+\ 4.3\, m^2$ $+373\, m^2$	$+\ 4.3\, m^2$ $+373\, m^2$	$+\ 4.3\, m^2$ $+372\, m^2$
$P'^{1,\ 1}_{-1,-1} = -303.423\, m^2\, \sigma^2 + 1.02\, m\, \sigma^4$	$-\ 0.16\, m^2$ $-\ 13.85\, m^2$	$-\ 0.16\, m^2$ $-\ 13.83\, m^2$	$-\ 0.16\, m^2$ $-\ 13.82\, m^2$
$P'^{3,\ 1}_{-1,-1} = -231\, m^2\, \sigma^2 + 0.80\, m\, \sigma^4$	$-\ 0.1\, m^2$ $-\ 11\, m^2$	$-\ 0.1\, m^2$ $-\ 11\, m^2$	$-\ 0.1\, m^2$ $-\ 11\, m^2$
$P'^{1,\ 3}_{-1,-1} = -763.75\, m^2\, \sigma^2 + 2.60\, m\, \sigma^4$	$-\ 0.4\, m^2$ $-\ 34\, m^2$	$-\ 0.4\, m^2$ $-\ 34\, m^2$	$-\ 0.4\, m^2$ $-\ 34\, m^2$
$P'^{0,\ 2}_{0,-2} = +4.27\, m^2\, \sigma^2$	$+\ 0.00\, m^2$ $+\ 0.21\, m^2$	$+\ 0.00\, m^2$ $+\ 0.21\, m^2$	$+\ 0.00\, m^2$ $+\ 0.21\, m^2$

VENUS AND NEPTUNE.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for—		
	1600	1850	2100
$P^{0,0}_{0,0} = 2.000\,29 - 1.7417\, m\, \sigma^2 + 1.7\, m\, \sigma^4$	. . . . $-\ 83.73\, m^2$	. . . . $-\ 84.32\, m^2$	. . . . $-\ 84.92\, m^2$
$P^{2,0}_{0,0} = -435.41\, m^2 - 2.619\, m\, \sigma^2$	$+433.89\, m^2$ $-126.04\, m^2$	$+433.89\, m^2$ $-126.94\, m^2$	$+433.89\, m^2$ $-127.84\, m^2$
$P^{2,2}_{-0,0} = +654.8\, m^2 - 3.96\, m\, \sigma^2$	$+\ 0.65\, m$ $-191\, m^2$	$+\ 0.65\, m$ $-192\, m^2$	$+\ 0.65\, m$ $-193\, m^2$
$P^{1,1}_{-1,1} = -13.10\, m^2 + 144\, m^2\, \sigma^2$	$-\ 13.02\, m^2$ $+\ 6.92\, m^2$	$-\ 13.02\, m^2$ $+\ 6.97\, m^2$	$-\ 13.02\, m^2$ $+\ 7.02\, m^2$
$P'^{2,0}_{-2,0} = +2.1797\, m\, \sigma^2$	$+\ 1.26\, m^2$ $+104.90\, m^2$	$+\ 1.28\, m^2$ $+105.65\, m^2$	$+\ 1.30\, m^2$ $+106.40\, m^2$
$P'^{1,\ 1}_{-1,-1} = -78.75\, m\, \sigma^2 + 0.26\, m\, \sigma^4$	$-\ 0.05\, m^2$ $-\ 3.78\, m^2$	$-\ 0.05\, m^2$ $-\ 3.81\, m^2$	$-\ 0.05\, m^2$ $-\ 3.83\, m^2$
$P'^{0,\ 2}_{0,-2} = +711\, m^3\, \sigma^2$	$-\ 0.00\, m^2$ $+\ 0.03\, m^2$	$-\ 0.00\, m^2$ $+\ 0.03\, m^2$	$-\ 0.00\, m^2$ $+\ 0.03\, m^2$

NOTE.—The symbol m means .001.

EARTH AND MARS.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for—		
	1600	1850	2100
$P^{0,0}_{0,0} = 2.291\,14 - 3.759\,\sigma^2 + 34.9\,\sigma^4 - 434\,\sigma^6$	. . . . $-$ 0.1210	. . . . $-$ 0.1209	. . . . $-$ 0.1208
$P^{2,0}_{0,0} = 0.939\,820 - 36\,83\,\sigma^2 + 1030\,\sigma^4 - 27\,M\,\sigma^6$	$+$ 0.930 266 $-$ 1.173	$+$ 0.930 286 $-$ 1.172	$+$ 0.930 298 $-$ 1.172
$P^{4,0}_{0,0} = 1.27 - 183\,\sigma^2 + 12\,M\,\sigma^4$	$+$ 1.22 $-$ 5.7	$+$ 1.22 $-$ 5.7	$+$ 1.22 $-$ 5.7
$P^{2,2}_{0,0} = 9.208 - 1048\,\sigma^2 + 62\,M\,\sigma^4$	$+$ 8.94 $-$ 32.8	$+$ 8.94 $-$ 32.8	$+$ 8.94 $-$ 32.8
$P^{0,4}_{0,0} = 3.8049 - 360\,\sigma^2 + 19\,M\,\sigma^4$	$+$ 3 712 $-$ 11.3	$+$ 3.712 $-$ 11.3	$+$ 3.712 $-$ 11.3
$P^{4,2}_{0,0} = 46 - 12.1\,M\,\sigma^2$	$+$ 43 $-$391	$+$ 43 $-$391	$+$ 43 $-$391
$P^{2,4}_{0,0} = 90 - 19.6\,M\,\sigma^2$	$+$ 85 $-$634	$+$ 85 $-$633	$+$ 85 $-$633
$P^{0,6}_{0,0} = 18.92 - 3.42\,M\,\sigma^2$	$+$ 18.0 $-$111	$+$ 18.0 $-$110	$+$ 18.0 $-$110
$P^{2,6}_{0,0} = 855 - [298]\,M\,\sigma^2$	$+$785 $-$ 9.64 M	$+$785 $-$ 9.62 M	$+$785 $-$ 9.62 M
$P^{0,8}_{0,0} = 107 - [32]\,M\,\sigma^2$	$+$100 $-$ 1.03 M	$+$100 $-$ 1.03 M	$+$100 $-$ 1.03 M
$P^{1,1}_{-1,1} = -0.722\,68 + 34.36\,\sigma^2 - 1000\,\sigma^4 + 26.2\,M\,\sigma^6$	$-$ 0.713 77 $+$ 1.094	$-$ 0.713 79 $+$ 1.093	$-$ 0.713 80 $+$ 1.092
$P^{3,1}_{-1,1} = -3.35 + 432\,\sigma^2$	$-$ 3.24 $+$ 14.0	$-$ 3.24 $+$ 14.0	$-$ 3.24 $+$ 14.0
$P^{1,3}_{-1,1} = -5.603 + 602\,\sigma^2 - 34\,M\,\sigma^4$	$-$ 5.448 $+$ 18.9	$-$ 5.449 $+$ 18.9	$-$ 5.449 $+$ 18.9
$P^{3,3}_{-1,1} = -64 + 15.3\,M\,\sigma^2$	$-$ 60 $+$495	$-$ 60 $+$494	$-$ 60 $+$494
$P^{1,5}_{-1,1} = -40.4 + 8.11\,M\,\sigma^2$	$-$ 38.3 $+$262.2	$-$ 38.3 $+$261.9	$-$ 38.3 $+$261.8
$P^{1,7}_{-1,1} = -300 + 80\,M\,\sigma^2$	$-$279 $+$ 2.59 M	$-$281 $+$ 2.58 M	$-$279 $+$ 2.58 M
$P^{2,2}_{-2,2} = 1.95 - 240\,\sigma^2$	$+$ 1.89 $-$ 7.8	$+$ 1.89 $-$ 7.8	$+$ 1.89 $-$ 7.7
$P^{4,2}_{-2,2} = 15 - 3.9\,M\,\sigma^2$	$+$ 14 $-$126	$+$ 14 $-$126	$+$ 14 $-$126
$P^{2,4}_{-2,2} = 28 - 6.3\,M\,\sigma^2$	$+$ 26 $-$203.7	$+$ 26 $-$203.5	$+$ 26 $-$203.4
$P^{2,6}_{-2,2} = 306 - 90\,M\,\sigma^2$	$+$282 $-$ 2.91 M	$+$285 $-$ 2.91 M	$+$283 $-$ 2.91 M
$P^{3,3}_{-3,3} = -6 + 5\,M\,\sigma^2$	$-$ 5 $+$162	$-$ 5 $+$161	$-$ 5 $+$161

NOTE.—The symbol M means 1000.

EARTH AND MARS—Continued.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for—		
	1600	1850	2100
${P'}^{2,0}_{-2,0} = 15.2\,\sigma^2 - 464\,\sigma^4$	+ 0.003 95 + 0.48	+ 0.003 94 + 0.48	+ 0.003 93 0.48
${P'}^{2,2}_{-2,0} = 360\,\sigma^2 - 26\,M\,\sigma^4$	+ 0.0922 +11.2	+ 0.0921 +11.2	+ 0.0920 +11.2
${P'}^{1,1}_{-1,-1} = -17.54\,\sigma^2 + 678\,\sigma^4 - 20\,M\,\sigma^6$	− 0.004 54 − 0.556	− 0.004 53 − 0.555	− 0.004 52 − 0.555
${P'}^{3,1}_{-1,-1} = -218\,\sigma^2 + 18\,M\,\sigma^4$	− 0.0557 − 6.7	− 0.0556 − 6.7	− 0.0555 − 6.7
${P'}^{1,3}_{-1,-1} = -304\,\sigma^2 + 23\,M\,\sigma^4$	− 0.0779 − 9.4	− 0.0777 − 9.4	− 0.0776 − 9.4
${P'}^{1,5}_{-1,-1} = -4.07\,M\,\sigma^2 + [538]\,M\,\sigma^4$	− 1.0 −125	− 1.0 −125	− 1.0 −125
${P'}^{0,2}_{0,-2} = 5.08\,\sigma^2 - 239\,\sigma^4 + 8\,M\,\sigma^6$	+ 0.001 312 + 0.160	+ 0.001 309 + 0.160	+ 0.001 308 + 0.160
${P'}^{2,2}_{0,-2} = 183\,\sigma^2 - 16\,M\,\sigma^4$	+ 0.0468 + 5.6	+ 0.0467 + 5.6	+ 0.0466 + 5.6
${P'}^{0,4}_{0,-2} = 81\,\sigma^2 - 7\,M\,\sigma^4$	+ 0.0206 + 2.5	+ 0.0206 + 2.5	+ 0.0206 + 2.5
${P'}^{3,1}_{-3,1} = -133\,\sigma^2 + 9\,M\,\sigma^4$	− 0.0342 − 4.2	− 0.0342 − 4.2	− 0.0341 − 4.1
${P'}^{1,3}_{1,-3} = -50\,\sigma^2 + 4\,M\,\sigma^4$	− 0.013 − 1.55	− 0.013 − 1.55	− 0.013 − 1.55
${P''}^{1,4}_{-1,-3} - 1\ 2\,M\,\sigma^4$	− 0.000 08 − 0.020	− 0.000 08 − 0.020	− 0.000 08 − 0.020
${P''}^{0,4}_{0,-4} = 0.16\,M\,\sigma^4$	+ 0.000 011 + 0.0027	+ 0.000 011 + 0.0027	+ 0.000 011 + 0.0027

EARTH AND JUPITER.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for—		
	1600	1850	2100
$P^{0,0}_{0,0} = 2.018\,86 - 118.940\,m\,\sigma^2 + 0.14\,\sigma^4 - 0.07\,\sigma^6$	. . . . −2.750 66 m	. . . . −2.722 06 m	. . . . −2.693 49 m
$P^{2,0}_{0,0} = 29.7346\,m - 200.3\,m\,\sigma^2 + 388\,m\,\sigma^4 - 0.50\,\sigma^6$	+ 29.7066 m − 4.840 m	+ 29.7072 m − 4.789 m	+ 29.7078 m − 4.739 m
$P^{4,0}_{0,0} = 862\,m^2 - 20\,m\,\sigma^2 + 0.13\,\sigma^4$	+859 m^2 − 0.46 m	+859 m^2 − 0.46 m	+859 m^2 − 0.46 m
$P^{2,2}_{0,0} = 52.33\,m - 494\,m\,\sigma^2 + 0.3\,\sigma^4$	+ 52.27 m − 11.4 m	+ 52.27 m − 11.3 m	+ 52.27 m − 11.2 m

EARTH AND JUPITER—Continued.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for—		
	1600	1850	2100
$P^{0,4}_{0,0} = 40.16\,m - 331\,m\,\sigma^2 + 1.0\,\sigma^4$	.... $-\ 7.64\,m$	.... $-\ 7.56\,m$	.... $-\ 7.48\,m$
$P^{2,4}_{0,0} = 82.7\,m - 1.1\,\sigma^2$	$+\ 82.6\,m$ $-\ 25\,m$	$+\ 82.6\,m$ $-\ 25\,m$	$+\ 82.6\,m$ $-\ 25\,m$
$P^{0,6}_{0,0} = 52.2\,m - 0.6\,\sigma^2$	.... $-\ 14\,m$	.... $-14\,m$	.... $-\ 14\,m$
$P^{1,1}_{-1,1} = -7.1105\,m + 87.03\,m\,\sigma^2 - 0.28\,\sigma^4$	$-\ 7.0989\,m$ $+\ 2.0115\,m$	$-\ 7.0991\,m$ $+\ 1.9906\,m$	$-\ 7.0993\,m$ $+\ 1.9697\,m$
$P^{3,1}_{-1,1} = -6.195\,m + 93.4\,m\,\sigma^2 - 0.41\,\sigma^4$	$-\ 6.183\,m$ $+\ 2.158\,m$	$-\ 6.183\,m$ $+\ 2.136\,m$	$-\ 6.183\,m$ $+\ 2.113\,m$
$P^{1,3}_{-1,1} = -19.1\,m + 0.26\,\sigma^2 - 1.0\,\sigma^4$	$-\ 19.1\,m$ $+\ 6.0\,m$	$-\ 19.1\,m$ $+\ 6.0\,m$	$-\ 19.1\,m$ $+\ 6.0\,m$
$P^{2,2}_{-2,2} = 1.93\,m - 37\,m\,\sigma^2$	$+\ 1.93\,m$ $-\ 0.85\,m$	$+\ 1.93\,m$ $-\ 0.84\,m$	$+\ 1.93\,m$ $-\ 0.83\,m$
$P'^{2,0}_{-2,0} = 160.69\,m\,\sigma^2 - 280\,m\,\sigma^4$	$+\ 21.49\,m^2$ $+\ 3.7156\,m$	$+\ 21.05\,m^2$ $+\ 3.6770\,m$	$+\ 20.61\,m^2$ $+\ 3.6384\,m$
$P'^{4,0}_{-2,0} = 13.7\,m\,\sigma^2 - 0.11\,\sigma^4$	$+\ 1.8\,m^2$ $+316\,m^2$	$+\ 1.9\,m^2$ $+313\,m^2$	$+\ 1.9\,m^2$ $+309\,m^2$
$P'^{2,2}_{-2,0} = 331\,m\,\sigma^2 - 1.09\,\sigma^4$	$+\ 44.3\,m^2$ $+\ 7.64\,m$	$+\ 43.4\,m^2$ $+\ 7.56\,m$	$+\ 42.5\,m^2$ $+\ 7.48\,m$
$P'^{1,\ 1}_{-1,-1} = -47.07\,m\,\sigma^2 + 206\,m\,\sigma^4$	$-\ 6.29\,m^2$ $-\ 1.087\,m$	$-\ 6.16\,m^2$ $-\ 1.076\,m$	$-\ 6.03\,m^2$ $-\ 1.064\,m$
$P'^{3,\ 1}_{-1,-1} = -49\,m\,\sigma^2 + 0.31\,\sigma^4$	$-\ 6.6\,m^2$ $-\ 1.1\,m$	$-\ 6.5\,m^2$ $-\ 1.1\,m$	$-\ 6.3\,m^2$ $-\ 1.1\,m$
$P'^{1,\ 3}_{-1,-1} = -139\,m\,\sigma^2 + 0.7\,\sigma^4$	$-\ 18.6\,m^2$ $-\ 3.21\,m$	$-\ 18.2\,m^2$ $-\ 3.18\,m$	$-\ 17.8\,m^2$ $-\ 3.15\,m$
$P'^{0,\ 2}_{0,-2} = 3.45\,m\,\sigma^2 - 24\,m\,\sigma^4$	$+\ 0.46\,m^2$ $+\ 79.6\,m^2$	$+\ 0.45\,m^2$ $+\ 78.8\,m^2$	$+\ 0.44\,m^2$ $+\ 78.0\,m^2$
$P'^{2,\ 2}_{0,-2} = 20.8\,m\,\sigma^2 - 0.17\,\sigma^4$	$+\ 2.8\,m^2$ $+480\,m^2$	$+\ 2.7\,m^2$ $+475\,m^2$	$+\ 2.7\,m^2$ $+470\,m^2$
$P'^{0,\ 4}_{0,-2} = 13.7\,m\,\sigma^2 - 0.11\,\sigma^4$	$+\ 1.8\,m^2$ $+316\,m^2$	$+\ 1.8\,m^2$ $+313\,m^2$	$+\ 1.8\,m^2$ $+309\,m^2$
$P'^{3,1}_{-3,1} = -79\,m\,\sigma^2 + 0.31\,\sigma^4$	$-\ 11\,m^2$ $-\ 1.83\,m$	$-\ 11\,m^2$ $-\ 1.81\,m$	$-\ 11\,m^2$ $-\ 1.79\,m$
$P'^{1,\ 3}_{1,-3} = -2.0\,m\,\sigma^2 + 21\,m\,\sigma^4$	$-\ 0.27\,m^2$ $-\ 46\,m^2$	$-\ 0.27\,m^2$ $-\ 46\,m^2$	$-\ 0.27\,m^2$ $-\ 46\,m^2$

EARTH AND SATURN.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for— 1600	1850	2100
$P^{0,0}_{0,0} = 2.005\,53 - 33.661\, m\, \sigma^2 + 36\, m\, \sigma^4 - 5.2\, m\, \sigma^6$	. . . . $-1.469\,13\, m$	. . . . $-1.463\,91\, m$	. . . . $-1.458\,35\, m$
$P^{2,0}_{0,0} = 8.415\,28\, m - 52.970\, m\, \sigma^2 + 65.8\, m\, \sigma^4 - 29\, m\, \sigma^6$	$+8.390\,01\, m$ $-\ 2.311\, m$	$+8.390\,19\, m$ $-\ 2.302\, m$	$+8.390\,38\, m$ $-\ 2.294\, m$
$P^{4,0}_{0,0} = 67.2\, m^2 - 1.4\, m\, \sigma^2$	$+\ 66.5\, m^2$ $-\ 0.06\, m$	$+\ 66.5\, m^2$ $-\ 0.06\, m$	$+\ 66.5\, m^2$ $-\ 0.06\, m$
$P^{2,2}_{0,0} = 13.243\, m - 92.4\, m\, \sigma^2 + 0.15\, \sigma^4$	$+\ 13.199\, m$ $-\ 4.03\, m$	$+\ 13.199\, m$ $-\ 4.02\, m$	$+\ 13.199\, m$ $-\ 4.00\, m$
$P^{0,4}_{0,0} = 10.76\, m - 71.2\, m\, \sigma^2 + 0.10\, \sigma^4$	. . . . $-\ 3.11\, m$	. . . . $-\ 3.10\, m$	. . . . $-\ 3.09\, m$
$P^{2,4}_{0,0} = 17.7\, m - 0.13\, \sigma^2$	$+\ 17.6\, m$ $-\ 6\, m$	$+\ 17.6\, m$ $-\ 6\, m$	$+\ 17.6\, m$ $-\ 6\, m$
$P^{0,6}_{0,0} = 12.9\, m - 0.08\, \sigma^2$	. . . . $-\ 3\, m$	. . . . $-\ 3\, m$	. . . . $-\ 3\, m$
$P^{1,1}_{-1,1} = -1.101\,23\, m + 12.51\, m\, \sigma^2$	$-1.095\,26\, m$ $+547\, m^2$	$-1.095\,30\, m$ $+544\, m^2$	$-1.095\,35\, m$ $+542\, m^2$
$P^{3,1}_{-1,1} = -863.6\, m^2 + 10.51\, m\, \sigma^2$	$-858.6\, m^2$ $+\ 0.46\, m$	$-858.6\, m^2$ $+\ 0.46\, m$	$-858.6\, m^2$ $+\ 0.46\, m$
$P^{1,3}_{-1,1} = -2.812\, m + 33.1\, m\, \sigma^2$	$-\ 2.796\, m$ $+\ 1.44\, m$	$-\ 2.796\, m$ $+\ 1.44\, m$	$-\ 2.796\, m$ $+\ 1.44\, m$
$P^{3,3}_{-1,1} = -2.3\, m + 0.04\, \sigma^2$	$-\ 2.3\, m$ $+\ 2\, m$	$-\ 2.3\, m$ $+\ 2\, m$	$-\ 2.3\, m$ $+\ 2\, m$
$P^{1,5}_{-1,1} = -5.0\, m + 0.07\, \sigma^2$	$-\ 5.0\, m$ $+\ 3\, m$	$-\ 5.0\, m$ $+\ 3\, m$	$-\ 5.0\, m$ $+\ 3\, m$
$P^{2,2}_{-2,2} = 0.16\, m - 3\, m\, \sigma^2$	$+\ 0.16\, m$ $-\ 0.1\, m$	$+\ 0.16\, m$ $-\ 0.1\, m$	$+\ 0.16\, m$ $-\ 0.1\, m$
$P'^{2,0}_{-2,0} = 43.0469\, m\, \sigma^2 - 51.98\, m\, \sigma^4$	$+\ 20.53\, m^2$ $+\ 1.878\, m$	$+\ 20.38\, m^2$ $+\ 1.872\, m$	$+\ 20.24\, m^2$ $+\ 1.864\, m$
$P'^{2,2}_{-2,0} = 71.26\, m\, \sigma^2 - 119.4\, m\, \sigma^4$	$+\ 34.0\, m^2$ $+\ 3.11\, m$	$+\ 33.8\, m^2$ $+\ 3.10\, m$	$+\ 33.5\, m^2$ $+\ 3.09\, m$
$P'^{1,1}_{-1,-1} = -6.8017\, m\, \sigma^2 + 24.73\, m\, \sigma^4$	$-\ 3.24\, m^2$ $-296\, m^2$	$-\ 3.22\, m^2$ $-295\, m^2$	$+\ 3.20\, m^2$ $-294\, m^2$
$P^{3,1}_{-1,-1} = -5.690\, m\, \sigma^2 + 24\, m\, \sigma^4$	$-\ 2.7\, m^2$ $-\ 0.25\, m$	$-\ 2.7\, m^2$ $-\ 0.25\, m$	$-\ 2.7\, m^2$ $-\ 0.25\, m$
$P'^{1,3}_{-1,-1} = -17.922\, m\, \sigma^2 + 70.9\, m\, \sigma^4$	$-\ 8.5\, m^2$ $-\ 0.78\, m$	$-\ 8.5\, m^2$ $-\ 0.78\, m$	$-\ 8.5\, m^2$ $-\ 0.78\, m$
$P'^{0,2}_{0,-2} = 268.80\, m^2\, \sigma^2 - 1.631\, m\, \sigma^4$	$+\ 0.13\, m^2$ $+\ 12\, m^2$	$+\ 0.13\, m^2$ $+\ 12\, m^2$	$+\ 0.13\, m^2$ $+\ 12\, m^2$
$P'^{2,2}_{0,-2} = 1.424\, m\, \sigma^2 - 9.3\, m\, \sigma^4$	$+\ 0.7\, m^2$ $+\ 0.06\, m$	$+\ 0.7\, m^2$ $+\ 0.06\, m$	$+\ 0.7\, m^2$ $+\ 0.06\, m$
$P'^{3,1}_{-3,1} = -10.35\, m\, \sigma^2 + 27\, m\, \sigma^4$	$-\ 4.9\, m^2$ $-\ 0.45\, m$	$-\ 4.9\, m^2$ $-\ 0.45\, m$	$-\ 4.9\, m^2$ $-\ 0.45\, m$

NOTE.—The symbol m means .001.

EARTH AND URANUS

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for—		
	1600	1850	2100
$P_{0,0}^{0,0} = 2.001\,36 - 8.1936\,m\,\sigma^2 + 8.3\,m\,\sigma^4$	. . . . $-109.65\,m^2$	. . . . $-110.45\,m^2$	. . . . $-111.33\,m^2$
$P_{0,0}^{2,0} = 2.048\,42\,m - 12.438\,m\,\sigma^2 + 13.1\,m\,\sigma^4$	$+2.047\,86\,m$ $-166.4\,m^2$	$+2.047\,86\,m$ $-167.7\,m^2$	$+2.047\,85\,m$ $-169.0\,m^2$
$P_{0,0}^{2,2} = 3.109\,m - 19.3\,m\,\sigma^2$	$+\ 3.108\,m$ $-258\,m^2$	$+\ 3.108\,m$ $-260\,m^2$	$+\ 3.108\,m$ $-262\,m^2$
$P_{0,0}^{0,4} = 2.574\,m - 16\,m\,\sigma^2$	. . . . $-\ 0.21\,m$	. . . . $-\ 0.21\,m$	. . . . $-\ 0.22\,m$
$P_{-1,1}^{1,1} = -133.429\,m^2 + 1.479\,m\,\sigma^2$	$-133.36\,m^2$ $+\ 20\,m^2$	$-133.36\,m^2$ $+\ 20\,m^2$	$-133.36\,m^2$ $+\ 20\,m^2$
$P_{-1,1}^{3,1} = -100.1\,m^2 + 1.1\,m\,\sigma^2$	$-100.1\,m^2$ $+\ 0.01\,m$	$-100.1\,m^2$ $+\ 0.01\,m$	$-100.1\,m^2$ $+\ 0.01\,m$
$P_{-1,1}^{1,3} = -335.3\,m^2$	$-335.3\,m^2$	$-335.3\,m^2$	$-335.3\,m^2$ . . . .
$P_{-2,2}^{2,2} = 9.2\,m^2$	$+\ 9.2\,m^2$	$+\ 9.2\,m^2$	$+\ 9.2\,m^2$ . . . .
$P'^{\,2,0}_{-2,0} = 10.300\,m\,\sigma^2$	$+461\,m^3$ $+137.8\,m^2$	$+468\,m^3$ $+138.8\,m^2$	$+475\,m^3$ $+140.0\,m^2$
$P'^{\,2,2}_{-2,0} = 15.85\,m\,\sigma^2$	$+\ 0.7\,m^2$ $+\ 0.21\,m$	$+\ 0.7\,m^2$ $+\ 0.21\,m$	$+\ 0.7\,m^2$ $+\ 0.21\,m$
$P'^{\,1,\ 1}_{-1,-1} = -806.311\,m^2\,\sigma^2 + 2.745\,m\,\sigma^4$	$-\ 36\,m^3$ $-\ 10.8\,m^2$	$-\ 36\,m^3$ $-\ 10.9\,m^2$	$-\ 36\,m^3$ $-\ 11.0\,m^2$
$P'^{\,1,\ 3}_{-1,-1} = -2.1\,\sigma^2$	$-\ 0.1\,m^2$ $-\ 0.03\,m$	$-\ 0.1\,m^2$ $-\ 0.03\,m$	$-\ 0.1\,m^2$ $-\ 0.03\,m$
$P'^{\,0,\ 2}_{0,-2} = 15\,m^2\,\sigma^2$	$+\ 0.00\,m^2$ $+\ 0.2\,m$	$+\ 0.00\,m^2$ $+\ 0.2\,m^2$	$+\ 0.00\,m^2$ $+\ 0.2\,m^2$

EARTH AND NEPTUNE.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for—		
	1600	1850	2100
$P_{0,0}^{0,0} = 2.000\,56 - 3.3320\,m\,\sigma^2$	. . . . $-105.11\,m^2$	. . . . $-103.73\,m^2$	. . . . $-102.34\,m^2$
$P_{0,0}^{2,0} = 833.024\,m^2 - 5.023\,m\,\sigma^2$	$+831.775\,m^2$ $-158\,m^2$	$+831.807\,m^2$ $-156\,m^2$	$+831.841\,m^2$ $-154\,m^2$
$P_{0,0}^{2,2} = 1.255\,m = 7.6\,m\,\sigma^2$	$+\ 1.254\,m$ $-\ 0.2\,m$	$+\ 1.254\,m$ $-\ 0.2\,m$	$+\ 1.254\,m$ $-\ 0.2\,m$
$P_{0,0}^{0,4} = -1.044\,m$	. . . .	. . . .	. . . .
$P_{-1,1}^{1,1} = -34.662\,m^2 + 382.6\,m^2\,\sigma^2$	$-\ 34.57\,m^2$ $+\ 12\,m^2$	$-\ 34.57\,m^2$ $+\ 12\,m^2$	$-\ 34.57\,m^2$ $+\ 12\,m^2$

EARTH AND NEPTUNE—Continued.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for— 1600	1850	2100
$P^{3,1}_{-1,1} = -26.11\, m^2$	$-26.11\, m^2$	$-26.11\, m^2$	$-26.11\, m^2$ . . . .
$P^{1,3}_{-1,1} = -86.83\, m^2$	$-86.83\, m^2$	$-86.83\, m^2$	$-86.83\, m^2$ . . . .
$P^{2,2}_{-2,2} = 1.52\, m^2$	$+1.52\, m^2$	$+1.52\, m^2$	$+1.52\, m^2$ . . . .
$P'^{2,0}_{-2,0} = 4.1746\, m\, \sigma^2$	$+1.04\, m^2$ $+132\, m^2$	$+1.01\, m^2$ $+130\, m^2$	$+0.98\, m^2$ $+128\, m^2$
$P'^{1,1}_{-1,-1} = -208.58\, m^2\, \sigma^2$	$-0.05\, m^2$ $-6.6\, m^2$	$-0.05\, m^2$ $-6.5\, m^2$	$-0.05\, m^2$ $-6.4\, m^2$
$P'^{0,2}_{0,-2} = 2.605\, m^2\, \sigma^2$	$+0.00\, m^2$ $+0.08\, m^2$	$+0.00\, m^2$ $+0.08\, m^2$	$+0.00\, m^2$ $+0.08\, m^2$

MARS AND JUPITER.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for— 1600	1850	2100
$P^{0,0}_{0,0} = +2.04508 - 304.50\, m\, \sigma^2 + 0.51\, \sigma^4 - 0.58\, \sigma^6$	. . . . $-7.5624\, m$	. . . . $-7.6229\, m$	. . . . $-7.6867\, m$
$P^{2,0}_{0,0} = 76.1262\, m - 659.5\, m\, \sigma^2 + 2.10\, \sigma^4 - 5.5\, \sigma^6$	$+76.0245\, m$ $-16.371\, m$	$+76.0229\, m$ $-16.503\, m$	$+76.0211\, m$ $-16.641\, m$
$P^{4,0}_{0,0} = 5.95\, m - 0.18\, \sigma^2 + 1.6\, \sigma^4$	$+5.92\, m$ $-4.5\, m$	$+5.92\, m$ $-4.6\, m$	$+5.92\, m$ $-4.6\, m$
$P^{2,2}_{0,0} = 164.9\, m - 2.42\, \sigma^2 + 16\, \sigma^4$	$+164.5\, m$ $-60.0\, m$	$+164.5\, m$ $-60.5\, m$	$+164.5\, m$ $-61.0\, m$
$P^{0,4}_{0,0} = 114.5\, m - 1.36\, \sigma^2$	. . . . $-33.8\, m$	. . . . $-34.1\, m$	. . . . $-34.4\, m$
$P^{6,0}_{0,0} = 0.56\, m$	$+0.56\, m$	$+0.56\, m$	$+0.56\, m$ . . . .
$P^{4,2}_{0,0} = 46\, m - 2\, \sigma^2$	$+46\, m$ -0.05	$+46\, m$ -0.05	$+46\, m$ -0.05
$P^{2,4}_{0,0} = 339\, m - 8\, \sigma^2$	$+338\, m$ -0.20	$+338\, m$ -0.20	$+338\, m$ -0.20
$P^{0,6}_{0,0} = 171\, m - 3\, \sigma^2$	. . . . -0.07	. . . . -0.07	. . . . -0.07
$P^{1,1}_{-1,1} = -27.5608\, m + 389.7\, m\, \sigma^2 - 1.71\, \sigma^4$	$-27.5007\, m$ $+9.672\, m$	$-27.4997\, m$ $+9.748\, m$	$-27.4987\, m$ $+9.831\, m$
$P^{3,1}_{-1,1} = -29.23\, m + 0.61\, \sigma^2 - 5\, \sigma^4$	$-29.14\, m$ $+15.1\, m$	$-29.14\, m$ $+15.3\, m$	$-29.14\, m$ $+15.4\, m$

MARS AND JUPITER—Continued.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for—		
	1600	1850	2100
$P^{1,3}_{-1,1} = -81.9\,m + 1.46\,\sigma^2$	$-81.7\,m$ $+36.2\,m$	$-81.7\,m$ $+36.5\,m$	$-81.7\,m$ $+36.8\,m$
$P^{5,1}_{-1,1} = -4.9\,m$	$-4.9\,m$	$-4.9\,m$	$-4.9\,m$
$P^{3,3}_{-1,1} = -0.12$	-0.12	-0.12	-0.12
$P^{1,5}_{-1,1} = -0.17 + 4\,\sigma^2$	-0.17 $+0.10$	-0.17 $+0.10$	-0.17 $+0.10$
$P^{2,2}_{-2,2} = 12.7\,m - 0.30\,\sigma^2$	$+12.7\,m$ $-7.4\,m$	$+12.7\,m$ $-7.5\,m$	$+12.7\,m$ $-7.6\,m$
$P^{4,2}_{-2,2} = 12.2\,m - 0.5\,\sigma^2$	$+12.1\,m$ $-12\,m$	$+12.1\,m$ $-13\,m$	$+12.1\,m$ $-13\,m$
$P^{2,4}_{-2,2} = 55\,m - 1.5\,\sigma^2$	$+55\,m$ $-37\,m$	$+55\,m$ $-38\,m$	$+55\,m$ $-38\,m$
$P^{3,3}_{-3,3} = -5.4\,m$	$-5.4\,m$	$-5.4\,m$	$-5.4\,m$
$P'^{2,0}_{-2,0} = 458.1\,m\,\sigma^2 - 1.36\,\sigma^4$	$+70.68\,m^2$ $+11.35\,m$	$+71.83\,m^2$ $+11.47\,m$	$+73.03\,m^2$ $+11.56\,m$
$P'^{4,0}_{-2,0} = 111\,m\,\sigma^2 - 1.3\,\sigma^4$	$+17.1\,m^2$ $+2.74\,m$	$+17.4\,m^2$ $+2.77\,m$	$+17.7\,m^2$ $+2.79\,m$
$P'^{2,2}_{-2,0} = 1.36\,\sigma^2$	$+209\,m^2$ $+33.8\,m$	$+213\,m^2$ $+34.1\,m$	$+217\,m^2$ $+34.4\,m$
$P'^{1,1}_{-1,-1} = -208.7\,m\,\sigma^2 + 1.26\,\sigma^4$	$-32.21\,m^2$ $-5.176\,m$	$+32.73\,m^2$ $-5.217\,m$	$-33.29\,m^2$ $-5.261\,m$
$P'^{3,1}_{-1,-1} = -325\,m\,\sigma^2 + 3.5\,\sigma^4$	$-49.9\,m^2$ $-8.05\,m$	$-50.7\,m^2$ $-8.13\,m$	$-51.6\,m^2$ $-8.18\,m$
$P'^{1,3}_{-1,-1} = -775\,m\,\sigma^2 + 6.9\,\sigma^4$	$-120\,m^2$ $-19.21\,m$	$-121\,m^2$ $-19.36\,m$	$-124\,m^2$ $-19.52\,m$
$P'^{0,2}_{0,-2} = 23.8\,m\,\sigma^2 - 0.22\,\sigma^4$	$+3.67\,m^2$ $+590\,m^2$	$+3.73\,m^2$ $+595\,m^2$	$+3.80\,m^2$ $+600\,m^2$
$P'^{2,2}_{0,-2} = 181\,m\,\sigma^2$	$+28.0\,m^2$ $+4.50\,m$	$+28.4\,m^2$ $+4.54\,m$	$+28.9\,m^2$ $+4.57\,m$
$P'^{0,4}_{0,-2} = 111\,m\,\sigma^2$	$+17.1\,m^2$ $+2.76\,m$	$+17.4\,m^2$ $+2.78\,m$	$+17.7\,m^2$ $+2.80\,m$
$P'^{3,1}_{-3,1} = -424\,m\,\sigma^2$	$-65.5\,m^2$ $-10.5\,m$	$-66.4\,m^2$ $-10.6\,m$	$-67.6\,m^2$ $-10.7\,m$
$P'^{1,3}_{1,-3} = -27\,m\,\sigma^2 + 0.4\,\sigma^4$	$-4.2\,m^2$ $-0.66\,m$	$-4.3\,m^2$ $-0.67\,m$	$-4.3\,m^2$ $-0.68\,m$

NOTE.—The symbol m means .001.

MARS AND SATURN.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for—		
	1600	1850	2100
$P^{0,0}_{0,0} = 2.0129 - 80.348\, m\, \sigma^2 + 95\, m\, \sigma^4 - 32\, m\, \sigma^6$	. . . . $-3.310\,98\, m$	. . . . $-3.312\,24\, m$	. . . . $-3.313\,58\, m$
$P^{2,0}_{0,0} = 20.0871\, m - 134.64\, m\, \sigma^2 + 213\, m\, \sigma^4$	$+\ 20.0299\, m$ $-\ 5.546\, m$	$+\ 20.0298\, m$ $-\ 5.548\, m$	$+\ 20.0297\, m$ $-\ 5.551\, m$
$P^{4,0}_{0,0} = 389\, m^2 - 8.8\, m\, \sigma^2 + 0.05\, \sigma^4$	$+385\, m^2$ $-\ 0.35\, m$	$+385\, m^2$ $-\ 0.35\, m$	$+385\, m^2$ $-\ 0.35\, m$
$P^{2,2}_{0,0} = 33.66\, m - 280\, m\, \sigma^2 + 0.8\, \sigma^4$	$+\ 33.55\, m$ $-\ 11.53\, m$	$+\ 33.5\, m$ $-\ 11.53\, m$	$+\ 33.55\, m$ $-\ 11.54\, m$
$P^{0,4}_{0,0} = 26.47\, m - 198\, m\, \sigma^2$	. . . . $-\ 8.2\, m$	. . . . $-\ 8.2\, m$	. . . . $-\ 8.2\, m$
$P^{4,2}_{0,0} = 2.2\, m - 0.05\, \sigma^2$	$+\ 2.2\, m$ $-\ 2.1\, m$	$+\ 2.2\, m$ $-\ 2.1\, m$	$+\ 2.2\, m$ $-\ 2.1\, m$
$P^{2,4}_{0,0} = 49.6\, m - 0.6\, \sigma^2$	$+\ 49.3\, m$ $-\ 25\, m$	$+\ 49.3\, m$ $-\ 25\, m$	$+\ 49.3\, m$ $-\ 25\, m$
$P^{1,1}_{-1,1} = -3.997\,89\, m + 47.34\, m\, \sigma^2 - 134\, m\, \sigma^4$	$-3.977\,77\, m$ $+\ 1.949\, m$	$-3.977\,76\, m$ $+\ 1.950\, m$	$-3.977\,74\, m$ $+\ 1.950\, m$
$P^{3,1}_{-1,1} = -3.326\, m + 45.8\, m\, \sigma^2 - 179\, m\, \sigma^4$	$-\ 3.306\, m$ $+\ 1.88\, m$	$-\ 3.306\, m$ $+\ 1.88\, m$	$-\ 3.306\, m$ $+\ 1.88\, m$
$P^{1,3}_{-1,1} = -10.505\, m + 134\, m\, \sigma^2 - 0.47\, \sigma^4$	$-\ 10.448\, m$ $+\ 5.5\, m$	$-\ 10.448\, m$ $+\ 5.5\, m$	$-\ 10\ 448\, m$ $+\ 5.5\, m$
$P^{5,1}_{-1,1} = -0.14\, m + 4\, m\, \sigma^2$	$-\ 0.14\, m$ $+\ 0.2\, m$	$-\ 0.14\, m$ $+\ 0.2\, m$	$-\ 0.14\, m$ $+\ 0.2\, m$
$P^{3,3}_{-1,1} = -9.6\, m + 0.16\, \sigma^2$	$-\ 9.5\, m$ $+\ 6\, m$	$-\ 9.5\, m$ $+\ 6\, m$	$-\ 9.5\, m$ $+\ 6\, m$
$P^{1,5}_{-1,1} = -20\, m + 0.3\, \sigma^2$	$-\ 20\, m$ $+\ 0.01$	$-\ 20\, m$ $+\ 0.01$	$-\ 20\, m$ $+\ 0.01$
$P^{2,2}_{-2,2} = 883\, m^2 - 16\, m\, \sigma^2$	$+876\, m^2$ $-\ 0.6\, m$	$+876\, m^2$ $-\ 0.6\, m$	$+876\, m^2$ $-\ 0.6\, m$
$P^{4,2}_{-2,2} = 0.55\, m - 12\, m\, \sigma^2$	$+\ 0.55\, m$ $-\ 0.5\, m$	$+\ 0.55\, m$ $+\ 0.5\, m$	$+\ 0.55\, m$ $+\ 0.5\, m$
$P^{2,4}_{-2,2} = 3.3\, m - 0.06\, \sigma^2$	$+\ 3.3\, m$ $-\ 2\, m$	$+\ 3.3\, m$ $-\ 2\, m$	$+\ 3.3\, m$ $-\ 2\, m$
$P^{3,3}_{-3,3} = -0.18\, m$	$-\ 0.18\, m$	$-\ 0.18\, m$	$-\ 0.18\, m$ . . . .
$P'^{2,0}_{-2,0} = 105.942\, m\, \sigma^2 - 158.90\, m\, \sigma^4$	$+\ 45.04\, m^2$ $+\ 4.364\, m$	$+\ 45.07\, m^2$ $+\ 4.366\, m$	$+\ 45.10\, m^2$ $+\ 4.368\, m$
$P'^{4,0}_{-2,0} = 5.947\, m\, \sigma^2 - 43.4\, m\, \sigma^4$	$+\ 2.5\, m^2$ $+\ 0.24\, m$	$+\ 2.5\, m^2$ $+\ 0.24\, m$	$+\ 2.5\, m^2$ $+\ 0.24\, m$
$P'^{2,2}_{-2,0} = 198.62\, m\, \sigma^2 - 513\, m\, \sigma^4$	$+\ 84.4\, m^2$ $+\ 8.2\, m$	$+\ 84.5\, m^2$ $+\ 8.2\, m$	$+\ 84.5\, m^2$ $+\ 8.2\, m$
$P'^{1,\ 1}_{-1,-1} = -25.666\, m\, \sigma^2 + 103.7\, m\, \sigma^4$	$-\ 10.9\, m^2$ $-\ 1.0551\, m$	$-\ 10.9\, m^2$ $-\ 1.0555\, m$	$-\ 10.9\, m^2$ $-\ 1.0559\, m$
$P'^{3,\ 1}_{-1,-1} = -24.60\, m\, \sigma^2 + 136\, m\, \sigma^4$	$-\ 10\, m^2$ $-\ 1.01\, m$	$-\ 10\, m^2$ $-\ 1.01\, m$	$-\ 10\, m^2$ $-\ 1.01\, m$

MARS AND SATURN—Continued.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for—		
	1600	1850	2100
$P'^{1,\,3}_{-1,-1} = -72.3\, m\, \sigma^2 + 348\, m\, \sigma^4$	$-\ 0.03\, m$ $-\ 2.97\, m$	$-\ 0.03\, m$ $-\ 2.97\, m$	$-\ 0.03\, m$ $-\ 2.97\, m$
$P'^{0,\,2}_{0,-2} = 1.5562\, m\, \sigma^2 - 10.3\, m\, \sigma^4$	$+\ 0.66\, m^2$ $+\ 64\, m^2$	$+\ 0.66\, m^2$ $+\ 64\, m^2$	$-\ 0.66\, m^2$ $+\ 64\, m^2$
$P'^{2,\,2}_{0,-2} = 8.89\, m\, \sigma^2$	$+\ 3.8\, m^2$ $+\ 0.37\, m$	$+\ 3.8\, m^2$ $+\ 0.37\, m$	$+\ 3.8\, m^2$ $+\ 0.37\, m$
$P'^{0,\,4}_{0,-2} = 5.9\, m\, \sigma^2$	$+\ 3\, m^2$ $+\ 0.24\, m$	$+\ 3\, m^2$ $+\ 0.24\, m$	$+\ 3\, m^2$ $+\ 0.24\, m$
$P'^{3,\,1}_{-3,\,1} = -41.4\, m\, \sigma^2$	$-\ 17.6\, m^2$ $-\ 1.70\, m$	$-\ 17.6\, m^2$ $-\ 1.70\, m$	$-\ 17.6\, m^2$ $-\ 1.70\, m$

MARS AND URANUS.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for—		
	1600	1850	2100
$P^{0,0}_{0,0} = 2.003\,17 - 19.152\, m\, \sigma^2 + 20\, m\, \sigma^4$	. . . . $-403.73\, m^2$	. . . . $-399.21\, m^2$	. . . . $-394.91\, m^2$
$P^{2,0}_{0,0} = 4.787\,95\, m - 29.529\, m\, \sigma^2$	$+4.784\,67\, m$ $-623\, m^2$	$+4.784\,74\, m$ $-616\, m^2$	$+4.784\,81\, m$ $-609\, m^2$
$P^{4,0}_{0,0} = 21.6\, m^2 - 0.45\, m\, \sigma^2$	$+\ 21.5\, m^2$ $-\ 0.01\, m$	$+\ 21.5\, m^2$ $-\ 0.01\, m$	$+\ 21.5\, m^2$ $-\ 0.01\, m$
$P^{2,2}_{0,0} = 7.383\, m - 48.4\, m\, \sigma^2$	$+\ 7.329\, m$ $-\ 1.0\, m$	$+\ 7.330\, m$ $-\ 1.0\, m$	$+\ 7.332\, m$ $-\ 1.0\, m$
$P^{0,4}_{0,0} = 6.062\, m - 38.5\, m\, \sigma^2$	. . . . $-\ 0.8\, m$	. . . . $-\ 0.8\, m$	. . . . $-\ 0.8\, m$
$P^{4,2}_{0,0} = 0.11\, m$	$+\ 0.11\, m$	$+\ 0.11\, m$	$+\ 0.11\, m$ . . . .
$P^{2,4}_{0,0} = 9.61\, m$	$+\ 9.61\, m$	$+\ 9.61\, m$	$+\ 9.61\, m$ . . . .
$P^{1,1}_{-1,1} = -474.990\, m^2 + 5.321\, m\, \sigma^2$	$-474.40\, m^2$ $+112.19\, m^2$	$-474.41\, m^2$ $+110.94\, m^2$	$-474.42\, m^2$ $+109.75\, m^2$
$P^{3,1}_{-1,1} = -365.5\, m^2 + 4.26\, m\, \sigma^2$	$-365.0\, m^2$ $+\ 90\, m^2$	$-365.0\, m^2$ $+\ 89\, m^2$	$-365.0\, m^2$ $+\ 88\, m^2$
$P^{1,3}_{-1,1} = -1.201\, m + 13.7\, m\, \sigma^2$	$-\ 1.200\, m$ $+\ 0.29\, m$	$-\ 1.200\, m$ $+\ 0.29\, m$	$-\ 1.200\, m$ $+\ 0.29\, m$
$P^{3,3}_{-1,1} = -0.94\, m$	$-\ 0.94\, m$	$-\ 0.94\, m$	$-\ 0.94\, m$ . . . .
$P^{1,5}_{-1,1} = -2.1\, m$	$-\ 2.1\, m$	$-\ 2.1\, m$	$-\ 2.1\, m$ . . . .

NOTE.—The symbol m means .001.

MARS AND URANUS—Continued.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for—		
	1600	1850	2100
$P_{-2,2}^{2,2} = 50.1\,m^2$	$+50.1\,m^2$	$+50.1\,m^2$	$+50.1\,m^2$ ....
$P_{-2,2}^{4,2} = 27\,m^2$	$+27\,m^2$	$+27\,m^2$	$+27\,m^2$ ....
$P_{-2,2}^{2,4} = 0.18\,m$	$+0.18\,m$	$+0.18\,m$	$+0.18\,m$ ....
$P'^{2,0}_{-2,0} = 24.26\,m\,\sigma^2 - 26\,m\,\sigma^4$	$+2.70\,m^2$ $+511.4\,m^2$	$+2.64\,m^2$ $+505.7\,m^2$	$+2.58\,m^2$ $+500.3\,m^2$
$P'^{4,0}_{-2,0} = 309\,m^2\,\sigma^2 - 1.9\,m\,\sigma^4$	$+0.03\,m^2$ $+6\,m^2$	$+0.03\,m^2$ $+6\,m^2$	$+0.03\,m^2$ $+6\,m^2$
$P'^{2,2}_{-2,0} = 38.53\,m\,\sigma^2 - 0.05\,\sigma^4$	$+4.3\,m^2$ $+812\,m^2$	$+4.2\,m^2$ $+803\,m^2$	$+4.1\,m^2$ $+794\,m^2$
$P'^{1,1}_{-1,-1} = -2.897\,63\,m\,\sigma^2 + 10.144\,m\,\sigma^4$	$-0.32\,m^2$ $-61.0\,m^2$	$-0.31\,m^2$ $-60.3\,m^2$	$-0.31\,m^2$ $-59.7\,m^2$
$P'^{3,1}_{-1,-1} = -2.314\,m\,\sigma^2 + 9.1\,m\,\sigma^4$	$-0.3\,m^2$ $-0.05\,m$	$-0.3\,m^2$ $-0.05\,m$	$-0.3\,m^2$ $-0.05\,m$
$P'^{1,3}_{-1,-1} = -7.47\,m\,\sigma^2 + 0.03\,\sigma^4$	$-1\,m^2$ $-156\,m^2$	$-1\,m_2$ $-155\,m^2$	$-1\,m^2$ $-153\,m^2$
$P'^{0,2}_{0,-2} = 84.25\,m^2\,\sigma^2 - 0.51\,m\,\sigma^4$	$-0.01\,m^2$ $+1.8\,m^2$	$-0.01\,m^2$ $+1.7\,m^2$	$-0.01\,m^2$ $+1.7\,m^2$
$P'^{2,2}_{0,-2} = 0.45\,m\,\sigma^2$	$+0.05\,m^2$ $+9\,m^2$	$+0.05\,m^2$ $+9\,m^2$	$+0.05\,m^2$ $+9\,m^2$
$P'^{3,1}_{-3,1} = -4.34\,m\,\sigma^2 + 10\,m\,\sigma^4$	$-0.5\,m^2$ $-91\,m^2$	$-0.5\,m^2$ $-90\,m^2$	$-0.5\,m^2$ $-89\,m^2$

MARS AND NEPTUNE.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for—		
	1600	1850	2100
$P^{0,0}_{0,0} = 2.001\,29 - 7.757\,m\,\sigma^2 + 8.0\,m\,\sigma^4$	.... $-321.15\,m^2$	.... $-321.65\,m^2$	.... $-322.21\,m^2$
$P^{2,0}_{0,0} = 1.939\,30\,m - 11.77\,m\,\sigma^2$	$+1.934\,25\,m$ $-487.8\,m^2$	$+1.934\,23\,m$ $-488.6\,m^2$	$+1.934\,21\,m$ $-489.3\,m^2$
$P^{4,0}_{0,0} = 3.53\,m^2$	$+3.53\,m^2$	$+3.53\,m^2$	$+3.53\,m^2$ ....
$P^{2,2}_{0,0} = 2.941\,m$	$+2.941\,m$	$+2.941\,m$	$+2.941\,m$ ....
$P^{0,4}_{0,0} = 2.436\,m$	....	....	....
$P^{1,1}_{-1,1} = -122.930\,m^2 + 1.362\,m\,\sigma^2$	$-122.35\,m^2$ $+56.44\,m^2$	$-122.35\,m^2$ $+56.53\,m^2$	$-122.34\,m^2$ $+56.63\,m^2$

NOTE.—The symbol m means .001.

MARS AND NEPTUNE—Continued.

General values of P.	Special values of P and $\frac{dP}{d\sigma}$ for—		
	1600	1850	2100
$P_{-1,1}^{2,1} = -93.16\, m^2$	$-93.16\, m^2$	$-93.16\, m^2$	$-93.16\, m^2$ . . . .
$P_{-1,1}^{1,3} = -308.8\, m^2$	$-308.8\, m^2$	$-308.8\, m^2$	$-308.8\, m^2$ . . . .
$P_{-2,2}^{2,2} = 8.23\, m^2$	$+ 8.23\, m^2$	$+ 8.23\, m^2$	$+ 8.23\, m^2$ . . . .
$P'^{\,2,0}_{-2,0} = 9.748\, m\, \sigma^2$	$+ 4.18\, m^2$ $+403.95\, m^2$	$+ 4.20\, m^2$ $+404.59\, m^2$	$+ 4.21\, m^2$ $+405.28\, m^2$
$P'^{\,2,2}_{-2,0} = 14.97\, m\, \sigma^2$	$+ 6\, m^2$ $+620.4\, m^2$	$+ 6\, m^2$ $+621.5\, m^2$	$+ 6\, m^2$ $+622.4\, m^2$
$P'^{\,1,\ 1}_{-1,-1} = -742.585\, m^2\, \sigma^2$	$- 0.32\, m^2$ $- 30.77\, m^2$	$- 0.32\, m^2$ $- 30.82\, m^2$	$- 0.32\, m^2$ $- 30.87\, m^2$
$P'^{\,3,\ 1}_{-1,-1} = -571.7\, m^2\, \sigma^2$	$- 0.2\, m^2$ $- 23.7\, m^2$	$- 0.2\, m^2$ $- 23.7\, m^2$	$- 0.2\, m^2$ $- 23.7\, m^2$
$P'^{\,1,\ 3}_{-1,-1} = -1.8795\, m\, \sigma^2$	$- 0.8\, m^2$ $- 77.9\, m^2$	$- 0.8\, m^2$ $- 78.0\, m^2$	$- 0.8\, m^2$ $- 78.2\, m^2$
$P'^{\,0,\ 2}_{0,-2} = 14.13\, m^2\, \sigma^2$	$+ 0.01\, m^2$ $+ 0.59\, m^2$	$+ 0.01\, m^2$ $+ 0.59\, m^2$	$+ 0.01\, m^2$ $+ 0.59\, m^2$
$P'^{\,2,\ 2}_{0,-2} = 71.69\, m^2\, \sigma^2$	$+ 0.03\, m^2$ $+ 3.0\, m^2$	$+ 0.03\, m^2$ $+ 3.1\, m^2$	$+ 0.03\, m^2$ $+ 3.2\, m^2$
$P'^{\,3,1}_{-3,1} = -1.0941\, m\, \sigma^2$	$- 0.5\, m^2$ $- 45.3\, m^2$	$- 0.5\, m^2$ $- 45.4\, m^2$	$- 0.5\, m^2$ $- 45.5\, m^2$

§ 10.

CONCLUDED RESULTS FOR THE THREE FUNDAMENTAL EPOCHS.

The results of the preceding investigation are shown in the following tables. Such points connected with the results as will facilitate their interpretation and use may be recapitulated.

The symbol $D_t e$ represents the secular variation of the eccentricity, and $e\, D_t\, \pi_1$ represents the product of the secular motion of the perihelion into the eccentricity. The unit of time in all cases is a JULIAN century, and the eccentricity, which appears as a factor, is that for each separate epoch. Both are computed from formulæ (16), page 332. Hence the motion of the perihelion is measured along the plane of the orbit itself, without reference to any other plane of reference.

The quantities $D_t\, i$ and $\sin i\, D_t\, \theta$ may be taken to represent the instantaneous motion of the plane of the orbit around two lines situated in it, one being the ascending node; the other a line emanating from the Sun at right angles to the node, in

a longitude 90° greater than that of the node. As the numbers are given the node is, in each case, that upon the ecliptic of 1850. These two motions may be compounded into the single motion defined in § 7, namely, a rotation around a certain line in the plane of the orbit. The result of this composition is given in the tables of values of $\varkappa$ and L, which are computed independently from the theory of § 7. Since the axis on which the plane turns is different for the action of the different planets, no use could be made of the separate values of $\varkappa$ and L for the separate planets. The action of each planet is therefore resolved into the two components, $\varkappa \sin L$ and $\varkappa \cos L$. The final value of $\varkappa$ derived from the sum is given at the bottom of the left-hand table, and the position of the axis of rotation is defined by the quantity L, given at the bottom of the right-hand table. Here the point Q is so situated in the plane of the orbit as to be as far from the node as the equinox of 1850 is from the node, and L is the value of ν for this special case. In each case the node on the fixed ecliptic of 1850 is adopted.

It follows that the quantities given should be connected by the relations:

$$\begin{aligned} D_t\, i &= \varkappa \cos (L-\theta) \\ \sin i\, D_t\, \theta &= \varkappa \sin (L-\theta) \end{aligned} \qquad (\alpha)$$

θ being the longitude of the node on the ecliptic of 1850, already given on p. 338.

In the case of the Earth the motion for the extreme epochs is referred to the assumed nodes given on p. 338 and not to the node as computed from the motion, which may be slightly different. But for 1850 the motion is referred to the computed axis of rotation, which is in longitude 173° 35′. 13″.

To find the secular variations of i and θ for the extreme dates as referred to the fixed ecliptic of those dates, we may still use the formula (α) only correcting $L-\theta$ for the motion of the ecliptic during the interval from 1850. This correction may be computed by mechanical integration from the equation

$$\triangle'(L-\theta) = \int \varkappa'' \operatorname{cosec} i \sin (L''-\theta)\, dt$$

which is derived from the equations of p. 336.

The results all rest on the values of the masses given on p. 337. It does not seem necessary to develop the formulæ further until definite corrections to these masses are derived.

The degree of precision aimed at in the numerical work has been that of 0″.01 per century in the sum of the theoretical results. This required that the individual action of the several planets should be correct to very nearly 0″.001, and, owing to the large number of small terms which came in, most of the separate quantities were computed to yet one more place of decimals. In printing, however, the fourth decimal of seconds has in most cases been dropped.

MERCURY.

Action of—	$D_t e$ 1600	$D_t e$ 1850	$D_t e$ 2100	$e D_t \pi_1$ 1600	$e D_t \pi_1$ 1850	$e D_t \pi_1$ 2100
	″	″	″	″	″	″
Venus	+ 2.813	+ 2.761	+ 2.708	+ 56.814	+ 56.826	+ 56.842
Earth	+ 1.126	+ 1.147	+ 1.168	+ 18.815	+ 18.799	+ 18.783
Mars	— 0.060	— 0.061	— 0.061	+ 0.512	+ 0.511	+ 0.510
Jupiter	+ 0.317	+ 0.320	+ 0.321	+ 31.685	+ 31.664	+ 31.643
Saturn	+ 0.054	+ 0.053	+ 0.053	+ 1.503	+ 1.503	+ 1.502
Uranus	+ 0.001	+ 0.001	+ 0.001	+ 0.029	+ 0.029	+ 0.029
Neptune	0	0	0	+ 0.009	+ 0.009	+ 0.009
Sum	+ 4.251	+ 4.221	+ 4.190	+109.367	+109.341	+109.318

Action of—	$D_t i$ 1600	$D_t i$ 1850	$D_t i$ 2100	$\sin i\, D_t \theta$ 1600	$\sin i\, D_t \theta$ 1850	$\sin i\, D_t \theta$ 2100
	″	″	″	″	″	″
Venus	—14.870	—14.739	—14.607	— 23.578	— 23.557	— 23.537
Earth	— 1.324	— 1.404	— 1.481	— 12.184	— 12.233	— 12.281
Mars	— 0.031	— 0.030	— 0.030	— 0.234	— 0.235	— 0.236
Jupiter	— 4.866	— 4.905	— 4.945	— 17.986	— 18.037	— 18.090
Saturn	— 0.421	— 0.421	— 0.421	— 0.852	— 0.850	— 0.848
Uranus	— 0.002	— 0.002	— 0.002	— 0.016	— 0.016	— 0.016
Neptune	— 0.002	— 0.002	— 0.002	— 0.005	— 0.005	— 0.005
Sum	—21.516	—21.503	—21.488	— 54.855	— 54.933	— 55.013

Action of—	$\kappa \sin L$ 1600	$\kappa \sin L$ 1850	$\kappa \sin L$ 2100	$\kappa \cos L$ 1600	$\kappa \cos L$ 1850	$\kappa \cos L$ 2100
	″	″	″	″	″	″
Venus	—26.971	—26.900	—26.829	+ 7.0390	+ 6.9660	+ 6.8960
Earth	— 9.297	— 9.432	— 9.565	+ 7.9866	+ 7.9153	+ 7.8455
Mars	— 0.1822	— 0.1833	— 0.1843	+ 0.1498	+ 0.1496	+ 0.1495
Jupiter	—15.849	—15.965	—16.083	+ 9.7984	+ 9.7216	+ 9.6447
Saturn	— 0.8899	— 0.8905	— 0.8910	+ 0.3342	+ 0.3277	+ 0.3213
Uranus	— 0.0130	— 0.0131	— 0.0131	+ 0.0104	+ 0.0103	+ 0.0102
Neptune	— 0.0052	— 0.0052	— 0.0052	+ 0.0026	+ 0.0026	+ 0.0025
Sum	—53.207	—53.389	—53.571	+ 25.3210	+ 25.0931	+ 24.8697
	″	″	″	° ′	° ′	° ′
κ and L	κ: 58.925	58.993	59.062	L: 295 27.00	295 10.48	294 54.15

VENUS.

Action of—	$D_t e$ 1600	$D_t e$ 1850	$D_t e$ 2100	$e D_t \pi_1$ 1600	$e D_t \pi_1$ 1850	$e D_t \pi_1$ 2100
	″	″	″	″	″	″
Mercury	− 1.306	− 1.301	− 1.296	− 0.804	− 0.814	− 0.825
Earth	− 5.031	− 4.896	− 4.759	− 3.768	− 3.852	− 3.932
Mars	− 0.188	− 0.196	− 0.205	+ 0.514	+ 0.510	+ 0.507
Jupiter	− 3.074	− 3.117	− 3.158	+ 4.554	+ 4.491	+ 4.430
Saturn	− 0.070	− 0.067	− 0.065	+ 0.057	+ 0.054	+ 0.052
Uranus	+ 0.001	+ 0.001	+ 0.001	+ 0.002	+ 0.002	+ 0.002
Neptune	0	0	0	+ 0.001	+ 0.001	+ 0.001
Sum	− 9.668	− 9.576	− 9.482	+ 0.556	+ 0.392	+ 0.235

Action of—	$D_t i$ 1600	$D_t i$ 1850	$D_t i$ 2100	$\sin i \, D_t \theta$ 1600	$\sin i \, D_t \theta$ 1850	$\sin i \, D_t \theta$ 2100
	″	″	″	″	″	″
Mercury	+ 0.958	+ 0.949	+ 0.941	+ 0.523	+ 0.531	+ 0.539
Earth	+ 0.419	+ 0.004	− 0.408	− 43.146	− 43.169	− 43.199
Mars	+ 0.133	+ 0.132	+ 0.131	− 0.278	− 0.281	− 0.284
Jupiter	− 3.705	− 3.865	− 4.025	− 16.029	− 16.122	− 16.218
Saturn	− 0.521	− 0.523	− 0.525	− 0.490	− 0.488	− 0.486
Uranus	0	0	0	− 0.017	− 0.017	− 0.017
Neptune	− 0.003	− 0.003	− 0.003	− 0.005	− 0.005	− 0.005
Sum	− 2.719	− 3.306	− 3.889	− 59.442	− 59.551	− 59.670

Action of—	$\kappa \sin L$ 1600	$\kappa \sin L$ 1850	$\kappa \sin L$ 2100	$\kappa \cos L$ 1600	$\kappa \cos L$ 1850	$\kappa \cos L$ 2100
	″	″	″	″	″	″
Mercury	+ 1.055	+ 1.053	+ 1.050	− 0.2762	− 0.2733	− 0.2705
Earth	−10.0097	−10.9277	−11.8424	+ 41.971	+ 41.763	+ 41.546
Mars	+ 0.0624	+ 0.0568	+ 0.0513	+ 0.3016	+ 0.3046	+ 0.3080
Jupiter	− 7.4653	− 7.8224	− 8.1797	+ 14.660	+ 14.617	+ 14.571
Saturn	− 0.6239	− 0.6295	− 0.6349	+ 0.3500	+ 0.3395	+ 0.3294
Uranus	− 0.0039	− 0.0042	− 0.0044	+ 0.0167	+ 0.0166	+ 0.0165
Neptune	− 0.0039	− 0.0040	− 0.0040	+ 0.0038	+ 0.0038	+ 0.0037
Sum	−16.989	−18.278	−19.564	+ 57.027	+ 56.771	+ 56.504
	″	″	″	° ′	° ′	° ′
κ and L	κ: 59.503	59.642	59.796	L: 343 24.63	342 9.20	340 54.12

EARTH.

Action of—	$D_t e$ 1600	$D_t e$ 1850	$D_t e$ 2100	$e D_t \pi_1$ 1600	$e D_t \pi_1$ 1850	$e D_t \pi_1$ 2100
	″	″	″	″	″	″
Mercury	− 0.114	− 0.116	− 0.118	− 0.185	− 0.184	− 0.184
Venus	+ 1.390	+ 1.342	+ 1.295	+ 5.797	+ 5.766	+ 5.737
Mars	− 1.561	− 1.572	− 1.584	+ 1.645	+ 1.636	+ 1.627
Jupiter	− 8.111	− 8.182	− 8.255	+ 11.721	+ 11.677	+ 11.645
Saturn	− 0.047	− 0.043	− 0.040	+ 0.314	+ 0.314	+ 0.314
Uranus	+ 0.002	+ 0.002	+ 0.002	+ 0.010	+ 0.010	+ 0.009
Neptune	0	0	0	+ 0.003	+ 0.003	+ 0.003
Sum	− 8.441	− 8.569	− 8.700	+ 19.305	+ 19.222	+ 19.151

Action of—	$D_t i$ 1600	$D_t i$ 1850	$D_t i$ 2100	$\sin i\, D_t \theta$ 1600	$\sin i\, D_t \theta$ 1850	$\sin i\, D_t \theta$ 2100
	″	″	″	″	″	″
Mercury	− 0.236	+ 0.236	+ 0.238	+ 0.224	− 0.226	− 0.227
Venus	−29.015	+28.983	+28.957	+ 3.865	− 4.201	− 4.515
Mars	− 0.794	+ 0.785	+ 0.777	+ 0.539	− 0.550	− 0.560
Jupiter	−15.802	+15.666	+15.526	− 4.430	+ 4.288	+ 4.159
Saturn	− 1.246	+ 1.250	+ 1.253	− 0.704	+ 0.686	+ 0.669
Uranus	− 0.008	+ 0.008	+ 0.008	+ 0.001	− 0.001	− 0.002
Neptune	− 0.004	+ 0.004	+ 0.004	− 0.004	+ 0.004	+ 0.004
Sum	−47.105	+46.932	+46.763	− 0.509	0.000	− 0.472

Action of—	$\varkappa \sin L$ 1600	$\varkappa \sin L$ 1850	$\varkappa \sin L$ 2100	$\varkappa \cos L$ 1600	$\varkappa \cos L$ 1850	$\varkappa \cos L$ 2100
	″	″	″	″	″	″
Mercury	+ 0.247	+ 0.251	+ 0.254	− 0.212	− 0.210	− 0.208
Venus	+ 6.790	+ 7.412	+ 8.032	− 28.473	− 28.332	− 28.185
Mars	+ 0.617	+ 0.634	+ 0.651	− 0.735	− 0.719	− 0.703
Jupiter	− 2.804	− 2.511	− 2.224	− 16.170	− 16.047	− 15.919
Saturn	− 0.574	− 0.542	− 0.510	− 1.310	− 1.318	− 1.325
Uranus	+ 0.002	+ 0.002	+ 0.003	− 0.008	− 0.008	− 0.008
Neptune	− 0.004	− 0.004	− 0.004	− 0.004	− 0.004	− 0.004
Sum	+ 4.275	+ 5.242	+ 6.202	− 46.912	− 46.639	− 46.352
	″	″	″	° ′	° ′	° ′
κ'' and L″	κ'' 47.106	46.933	46.765	L″ 174 47.60	173 35.22	172 22.73

MARS.

Action of—	$D_t e$ 1600	$D_t e$ 1850	$D_t e$ 2100	$e D_t \pi_1$ 1600	$e D_t \pi_1$ 1850	$e D_t \pi_1$ 2100
	″	″	″	″	″	″
Mercury	+ 0.033	+ 0.033	+ 0.034	+ 0.058	+ 0.057	+ 0.057
Venus	+ 0.077	+ 0.079	+ 0.081	+ 4.586	+ 4.593	+ 4.599
Earth	+ 2.159	+ 2.148	+ 2.138	+ 21.342	+ 21.374	+ 21.408
Jupiter	+15.870	+15.818	+15.752	+116.348	+116.372	+116.375
Saturn	+ 0.638	+ 0.629	+ 0.619	+ 6.214	+ 6.226	+ 6.238
Uranus	− 0.002	− 0.001	− 0.001	+ 0.112	+ 0.112	+ 0.113
Neptune	0	0	0	+ 0.032	+ 0.032	+ 0.032
Sum	+18.775	+18.706	+18.623	+148.692	+148.766	+148.822

Action of—	$D_t i$ 1600	$D_t i$ 1850	$D_t i$ 2100	$\sin i\, D_t \theta$ 1600	$\sin i\, D_t \theta$ 1850	$\sin i\, D_t \theta$ 2100
	″	″	″	″	″	″
Mercury	+ 0.008	+ 0.007	+ 0.007	+ 0.048	+ 0.048	+ 0.048
Venus	− 1.276	− 1.278	− 1.281	+ 0.978	+ 0.994	+ 1.009
Earth	+ 0.115	+ 0.032	− 0.052	− 7.384	− 7.379	− 7.372
Jupiter	−25.347	−25.655	−25.967	− 26.812	− 26.850	− 26.904
Saturn	− 2.459	− 2.468	− 2.476	− 0.877	− 0.849	− 0.822
Uranus	− 0.006	− 0.006	− 0.006	− 0.024	− 0.024	− 0.024
Neptune	− 0.011	− 0.011	− 0.011	− 0.010	− 0.010	− 0.010
Sum	−28.976	−29.379	−29.786	− 34.081	− 34.070	− 34.075

Action of—	$\kappa \sin L$ 1600	$\kappa \sin L$ 1850	$\kappa \sin L$ 2100	$\kappa \cos L$ 1600	$\kappa \cos L$ 1850	$\kappa \cos L$ 2100
	″	″	″	″	″	″
Mercury	+ 0.0370	+ 0.0373	+ 0.0375	− 0.0308	− 0.0308	− 0.0307
Venus	− 0.3253	− 0.2963	− 0.2674	− 1.5741	− 1.5912	− 1.6082
Earth	− 4.746	− 4.875	− 5.004	+ 5.660	+ 5.540	+ 5.414
Jupiter	−36.712	−37.011	−37.315	+ 3.685	+ 3.044	+ 2.394
Saturn	− 2.4331	− 2.4089	− 2.3839	− 0.9463	− 1.0036	− 1.0599
Uranus	− 0.0206	− 0.0206	− 0.0207	+ 0.0144	+ 0.0138	+ 0.0132
Neptune	− 0.0144	− 0.0144	− 0.0144	+ 0.0006	+ 0.0004	+ 0.0002
Sum	−44.214	−44.589	−44.968	+ 6.809	+ 5.973	+ 5.123
	″	″	″	° ′	° ′	° ′
κ and L	κ: 44.735	44.990	45.258	L: 278 45.6	277 38.8	276 29.8

ON THE MASS OF JUPITER

AND

THE ORBIT OF POLYHYMNIA.

BY

SIMON NEWCOMB.

CONTENTS.

THE MASS OF JUPITER AND THE ORBIT OF POLYHYMNIA.

I.

INTRODUCTORY NOTE.

As far back as 1872 the author pointed out the great suitability of the planet Polyhymnia (33) for determining the mass of Jupiter, showing that the perturbations produced by the latter planet in the geocentric longitude of the other would probably amount to two or three degrees. It was therefore to be expected that a mass of Jupiter could thus be derived which would be more accurate than that attainable by other methods. A subsequent examination of the relative motions of the two bodies showed that remarkably favorable conjunctions occurred in 1859 and 1883, Polyhymnia being in each case within a few degrees of its aphelion. It was therefore to be expected that observations made from the discovery of the planet in 1854 up to 1888 would furnish all necessary data for the purpose in question.

In 1885 an arrangement was made with Mr. G. C. Comstock, now professor of astronomy in the University of Wisconsin, to compute the special perturbations from the time of discovery until 1888, and subsequently to do most of the numerical work required in the discussion of the observations and the formations of the coefficients of condition up to 1884.

Although this work was executed with the skill and precision of a master in the subject, it was subsequently found advisable to form more normal places than had first been used, and to construct the equations in a somewhat different way, as well as to carry the work on through 1888. In this reconstruction, involving the formation of geocentric places with changed data, the remaking of the comparison between theory and observation, and the construction of new equations of condition, the author was aided by Dr. Nobert Herz, who worked with him during a considerable part of 1893. Although Dr. Herz did not remain to complete the work, he left it in such a shape that the author had no difficulty in carrying it to a conclusion.

Quite likely no method of determining the mass of Jupiter better than that afforded by the perturbations of the planet now in question will soon be discovered. With improved places of the fixed stars, and a more careful and rigorous examination of the observations hitherto made, it will doubtless be found possible to improve the work here presented. The author is conscious that many defects may be found in its execution, and he publishes it with the hope that even should it not stand unchanged as a permanent discussion of the subject, it will facilitate the work of the future investigator.

A very serious drawback is the paucity of observations at several critical oppositions when they would have been of great value for the purpose in question.

II.

ON THE DETERMINATION OF THE MASS OF JUPITER BY OBSERVATIONS OF THE MINOR PLANETS.

The fact has long been recognized, that the perturbations of the small planets by Jupiter afford a valuable method of determining the mass of the latter planet. Such a determination has, therefore, formed a part of many discussions on the orbits of those bodies. It is obvious that there is a great difference in the accuracy with which the determination can be made by different planets. A large eccentricity of the orbit is advantageous, for three reasons: The greater the eccentricity, the nearer the planet may approach to Jupiter when at aphelion, and the nearer it may approach to the Earth when in perihelion. Moreover, the largest perturbations are those of the mean longitude, and these are magnified in the true longitude when the planet is near its perihelion—that is, in case of an opposition nearest to the Earth. A small inclination to the orbit of Jupiter is also a favorable circumstance.

The period over which observations extend is also an important consideration. Only by observations extending through a number of revolutions can the perturbations be completely separated from corrections to the elements; and a period of many years must sometimes elapse before the favorable conditions for the determination occur. Moreover, the number of the known planets has now become so great, that the observations upon any one are scattered and unsystematic. It follows that we must look mainly to the older ones for the attainment of the object in question.

Among the older small planets whose orbits are well situated must be mentioned Themis. A determination of the mass of Jupiter from the motions of this planet was made by KRUEGER, with results which appear very satisfactory.

But the planet whose orbit best fulfills the required conditions is Polyhymnia. Its orbit is the most eccentric of any discovered up to 1873, at least; so eccentric, in fact, that its greatest distance from the Sun is double its least distance. The inclination of its orbit to that of Jupiter is only about two degrees. On the whole, it seems to fulfill the required conditions much better than any other small planet yet known.

It is now of interest to inquire at what times favorable circumstances will occur. What is wanted is a conjunction with Jupiter near aphelion. To show the law of recurrence of conjunctions of this class, I have prepared the following table, extending from the discovery of the planet in 1854 to the year 1934. The first column gives the year and hundredth at which the planet passed its aphelion; the second, the longitude of the aphelion. Both of these columns have been obtained on the supposition that an approximate mean of the changes of the elements between 1855 and 1888 gave a value of the secular variation which continued uniformly through subsequent years. The error of this hypothesis may be two or three degrees, but is not important for our present purpose. The third column contains the mean longitude of Jupiter at the time when Polyhymnia reached its aphelion. The next column gives the difference of these longitudes. The mean longitude of Jupiter has been taken, instead of the true longitude, so as to show more clearly the law of progression. Near the aphelion of Polyhymnia the equation of the center of Jupiter is not far from $+5°$; by adding

this quantity to the small values of $\varDelta L$ we shall approximate a little nearer to the true value.

Table to show the aphelion conjunctions of Polyhymnia with Jupiter, and its passages through the perihelion.

Passage through aphelion.	Long. of aphelion.	Long. of Jupiter.	Diff.	Passage through perihelion.	$\varDelta t$ for Earth.
	°	°	°		y
1856.85	161	8	−153	1859.28	−.32
1861.71	161	155	− 6	1864.14	−.46
1866.57	161	303	+142	1869.00	+.40
1871.42	162	90	− 72	1873.85	+.25
1876.28	162	238	+ 76	1878.71	+.11
1881.14	163	25	−138	1883.57	−.03
1886.00	163	173	+ 10	1888.43	−.17
1890.86	164	320	+156	1893.28	−.32
1895.71	164	107	− 57	1898.14	−.46
1900.57	165	255	+ 90	1903.00	+.40
1905.43	165	42	−123	1907.86	+.26
1910.29	166	190	+ 24	1912.71	+.11
1915.15	166	337	+171	1917.57	−.03
1920.00	167	125	− 42	1922.43	−.17
1924.86	167	273	+106	1927.29	−.31
1929.72	167	60	−107	1932.14	−.46
1934.58	168	208	+ 40		

It will be noticed that the time of five revolutions of Polyhymnia corresponds closely to that of two revolutions of Jupiter. Consequently, at the end of twenty-four years the two bodies will return nearly to their original positions, and a favorable aphelion conjunction at any time will be followed by a close approach to another at the end of that period. We thus see that an extremely favorable aphelion conjunction occurred in 1861, and another, a little less favorable, at the beginning of 1886. From this point onward the deviation from commensurability is such that no other conjunction equally favorable with these two will occur until the year 1968. We must, therefore, regard the observations made between the time of the discovery of the planet and 1888 as of the most precious character.

But the approach of the planet to the Earth must also be considered. The column "perihelion" shows the time of each passage of the planet through its perihelion first following that through aphelion. The passage will be favorable for observation according as the Earth is at this time nearly in the same longitude as the planet. This depends upon the fraction of the year, since the earth passes the required point near 0.60 of each year.

The last column shows the fraction of the year which has elapsed between the passage of the Earth and that of the planet through the perihelion of the latter. It will be seen from this that the favorable passages occur in cycles of seven revolutions of the planet, or thirty-four years. The most favorable yet observed was that of 1883, which occurred just before the favorable conjunction with Jupiter in 1886. The conjunction of 1968 will be particularly unfavorable for the determination of the perturbation by observations from the Earth. Altogether, therefore, we may say that it is not until near the close of the twentieth century that so favorable an opportunity can again be found for attacking the problem now in hand.

It is noteworthy that a similar state of things exists in the case of Themis. The conjunction with Jupiter in 1854 was near aphelion; so was that of 1866. Not until near the middle of the twentieth century will the two planets come so near each other as on these two occasions. We must therefore regard it as fortunate that KRUEGER could command the data he did for his work.

III.

STATEMENT AND DISCUSSION OF ADOPTED METHODS.

The great extent of the present work, the necessity for a high degree of theoretical accuracy in its execution, the difficulty of fulfilling this condition when even the adopted methods had frequently to be determined by tentative processes, and the time through which the work has extended, have all resulted in a lack of continuity in its prosecution. An attempt to publish it in a perfectly homogeneous form would result in its not being published as it was actually done, and would, I conceive, detract from its value. I have therefore deemed it best to preface the results with such a history of the work as will best enable it to be utilized for any further researches in the same direction. The necessary subdivisions of the work are as follows:

I. The derivation of osculating elements, especially the mean motion at some mean epoch, so accurate that, when the perturbations are added, an ephemeris very close to the truth could be prepared for the entire thirty-four years of observation.

II. The computation of the perturbations through the whole period with all necessary precision.

III. The computation of ephemerides or geocentric places from the theory thus obtained, at or near the times of opposition.

IV. The discussion or correction of observations, including the derivation of star-places.

V. The formation and solution of the equations of condition, to which the comparison of theory and observation would give rise.

VI. The general discussion of results.

It is hardly necessary to say that no existing elements were accurate enough for the purpose. Indeed, the magnitude of the perturbations leads to a lack of rigor in the use of elliptic elements, which can be made harmless only by having the adopted elements very near the truth. The following considerations will show the nature of the difficulty.

Let a, b, c, etc., represent the elements adopted for any one epoch. If, as we may, we regard these elements as the fundamental data of the problem, to be corrected by observations in the usual way, then the place of the planet throughout the whole period of observation must be regarded as a function of these elements. Following the usual method, which is accurate enough in all ordinary cases, we regard the coordinates of a planet at any subsequent time as formed of two parts, an elliptic value given by the elements, a, b, c, etc., and a perturbation due to the action of the planets. These perturbations are commonly treated as known quantities which do not change with the elements, and the differential coefficients for correcting the elements are formed on this supposition. But, as a matter of fact, the perturbations are functions of the elements, and the rigorous values of the differential coefficients of the coordinates with respect to the elements a, b, c, etc., can be obtained only by finding the differentials of the perturbations of each coordinate with respect to each of the six elements. We should thus have eighteen combinations in all, the computation of which would be quite impracticable.

In the actual work, perturbations of the elements were used instead of those of the coordinates. When this system is adopted, osculating elements, a_0, b_0, c_0, ... which I shall call zero elements, are adopted for a certain epoch. From them the varying elements a, b, c ... are computed for all subsequent epochs. Thus the latter elements become functions of the zero elements. If small corrections, δa_0, δb_0, δc_0, ... be applied to the latter, the corresponding corrections of the actual elements at any date are given by the equation

$$\delta a = \frac{da}{da_0}\delta a_0 + \frac{da}{db_0}\delta b_0 + \frac{da}{dc_0}\delta c_0 + \quad . \quad . \quad .$$

On the usual system we have implicitly to suppose

$$\frac{da}{da_0} = 1; \quad \frac{da}{db_0} = \frac{da}{dc_0} = \quad . \quad . \quad . \quad = 0 \qquad (a)$$

As a matter of fact, however, this supposition is not rigorous, because each element is a function of all the zero elements, except at the date of osculation. The result is that the small corrections δa_0, δb_0, ... to the zero elements will not be rigorously those to be applied to the elements at any other epoch.

The computation of the rigorous derivatives of any coordinate v with respect to the zero elements would be obtained from equations of the form

$$\frac{dv}{da_0} = \frac{dv}{da}\,\frac{da}{da_0} + \frac{dv}{db}\,\frac{db}{da_0} + \quad . \quad . \quad . \quad .$$

The rigorous computation of the required derivatives would be in any case a very laborious process. Two methods might be adopted. One would involve the expression of the general values of the osculating elements in terms of the zero elements. The differentiation could then be immediately performed. Many terms would, however, have to be computed to obtain a fairly approximate result. Another method would be to compute not only the perturbations of the six elements themselves, but

to find, by finite integration, the sum of all of the derivatives of their unit variations as to each of the separate elements which entered into them. This would require the approximate summation of 36 different mechanical quadratures.

A careful examination of the numbers which enter into the theory leads me to believe that during the thirty-four years over which the present investigation extends, the error in the coefficients arising from the adoption of the hypotheses (a) would not amount to more than two or three per cent of their true values in consequence of the perturbations. Instead therefore of attempting to compute rigorous differential coefficients, I have sought to compute the perturbations with elements so nearly accurate that this amount of error would be unimportant in the final result. How far I have succeeded the future must show.

The determination of the first values of the elements being only provisional, no details of the work are necessary. A rough correction to SCHUBERT's elements was first obtained by comparing the ephemerides of the *Astronomisches Jahrbuch* with observations. The epoch chosen for the comparison was 1873, July 17.0, Berlin mean time. With osculating elements thus approximately derived for this epoch, approximate perturbations by Jupiter and Saturn were computed for the years 1868 to 1878, and a new set of elements derived, which were considered sufficiently accurate to be used as provisional elements. These were known in the work as elements III.

The next question which arose was the best method of computing the definitive perturbations, especially those by Jupiter and Saturn. The computation of general perturbations by these planets was of course not to be thought of, and special perturbations had therefore to be adopted. Among the astronomers whom I consulted on the subject there was a general consensus of opinion that the special perturbations of the elements were those which it was, on the whole, most convenient to use.

A troublesome question now arises as to whether the perturbations by other planets than these three, especially by the Earth and Uranus, might not be sensible in the course of the thirty-four years' observation. It was difficult to decide this question otherwise than by actual trial. In the case of the Earth, a new difficulty arises from the fact that the usual forty-day interval between the epochs of the computation of the disturbing force is too long, and when we shorten it, not only is the labor greatly increased, but the liability to an accumulation of small accidental errors must increase. This difficulty we must regard as almost fatal to the accurate construction of a theory of any planet by special perturbations which shall hold good through a great number of revolutions, unless, indeed, the individual computations are made with a high degree of precision. It was therefore deemed best to compute the general perturbations of the planet by the Earth to quantities of the first order as to the masses. In order to make the work correspond in principle as closely as possible to the other portions, perturbations of the elements were chosen. One reason for choosing them was that their computation was somewhat simpler than those of the coordinates themselves in the case of an eccentricity so great as that of the asteroid in question. It was, however, well understood that the perturbations of the elements thus obtained would be much larger than the actual perturbations of the coordinates. The introduction of the Earth perturbations, which was not made until after the others had been computed,

necessitated changes in some of the elliptic elements, the introduction of which was productive of some additional complication in the work, though not of any error. With these changes the elements are called IV.

With the perturbations by the four planets just mentioned, the project of computing the perturbations by Uranus being for the moment abandoned, Mr. COMSTOCK computed such geocentric places of the planet as were necessary for the formation of normal places, and the comparison of the elements with observation. In order to obtain normal places, however, it was not deemed necessary to compute a complete ephemeris from the provisional elements. In most cases the published comparison of observations with existing ephemerides was deemed sufficient. Proceeding in the usual way, the mean correction during an opposition was considered as applicable to the ephemeris at the mean of the times of observation, and the normal place derived accordingly. In the case of the first opposition the observations extended through so long a period that two normal places were formed. The coefficients of the equations of condition were also computed, and the whole work thus made complete up to the year 1885. As it was essential to its completeness that the observations up to 1888 should be included, it was necessary to lay it aside until after that year had passed.

In 1893, on taking up the work for examination, review, and completion, much reconstruction was found necessary. The special perturbations were found to have been computed with the greatest precision, so that no alteration whatever was necessary in them. In most other points, however, revision was necessary. At most of the oppositions the position of the planet and the relation of its geocentric coordinates to its elements changed so largely that a single normal place did not appear sufficient. It was therefore necessary to completely reconstruct the comparison with observation so as to form new normal places.

The question whether the general perturbations by the Earth were computed with sufficient rigor, and whether those by Uranus were insensible, also arose. To test this question, I adopted a plan by which a great saving of labor could be made in computing the special perturbations, when the variations of the elements of the planet were not rigorously taken account of. The principle of this method is as follows: Instead of computing the disturbing forces for epochs equidistant by an integral number of days, the time of one anomalistic revolution of the asteroid was divided into a certain number of equal parts. For the action of Uranus the number of parts was 36, and for the action of the Earth 72 were chosen. This amounted to choosing as epochs those at which the mean anomaly of the planet was, respectively, 0°, 5°, 10°, etc. With these values of the mean anomaly, the values of all the quantities which depended upon the position of the disturbing planet were computed once for all. To each of these values would correspond 7 positions of the disturbing planet, because the asteroid had made about 7 revolutions during the period of observation. The computation of the disturbing forces for these seven epochs being made, the variations of the elements were computed, with a common value for each of the quantities depending only on the asteroid. The differential variations thus obtained were then computed and summed in the usual way.

IV.

SYSTEMS OF PROVISIONAL ELEMENTS.

As already stated, rough corrections to SHUBERT's elements were first derived by comparing the ephemerides of the *Astronomische Jahrbuch* with observations. These elements were as follows:

Elements II of Polyhymnia.

Epoch, 1873. July, 17.0. Berlin Mean Time.

		°	′	″	
Mean Long.	$L=$	318	53	30.0	Mean Eq. 1870.0
Long. Per.	$\pi=$	342	20	25.8	
Long. Node.	$\Omega=$	9	6	19.2	
Inclination.	$i=$	1	56	25.5	
Angle of ecc.	$\varphi=$	19	52	43.9	
Mean motion.	$\mu=$	732″.8160			
Log. mean dist.	$\log a=$	0.456674			

With these preliminary elements the approximate perturbations by Jupiter and Saturn were computed from 1868 to 1878, and ephemerides thus computed compared with observations and corrected in the usual way. Thus elements III were derived.

Elements III of Polyhymnia.

	°	′	″
$L=$	318	53	26.9
$\pi=$	342	19	56.4
$\Omega=$	9	2	44.9
$i=$	1	56	20.0
$\varphi=$	19	52	54.7
$\mu=$	732″.79471		

With these elements definitive perturbations by Jupiter, Saturn, and Mars were computed. The addition of the general perturbations by the earth made necessary a further slight correction to all the elements except the node and inclination. The result of this correction is seen in Elements IV, the epoch being the same as before.

Elements IV of Polyhymnia.

	°	′	″
$L=$	318	53	28.03
$\pi=$	342	20	2.04
$\Omega=$	9	2	44.90
$i=$	1	56	20.00
$\varphi=$	19	52	49.45
$\mu=$	732″.81640		

These elements were used by Mr. COMSTOCK in the computation of ephemerides, and their comparison with observation, which were intended to give the definitive values of the absolute terms of the equations of condition. But in 1893 it was found

that slight corrections to the elements might be made which would result in the absolute terms being markedly smaller. The final provisional elements adopted for correction by comparison with observations are the following:

Elements V of Polyhymnia.

$$
\begin{aligned}
L &= 318^\circ\ 53'\ 30''.0 \\
M &= 336\ \ 33\ \ 22.5 \\
\pi &= 342\ \ 20\ \ \ \ 7.5 \\
\varphi &= 19\ \ 52\ \ 54.2 \\
\Omega &= 9\ \ \ \ 2\ \ 44.9 \\
i &= 1\ \ 56\ \ 20.0 \\
\mu &= 732''.81750
\end{aligned}
$$

V.

RESULTS FOR THE ELEMENTS OF POLYHYMNIA.

In order not to interrupt the continuity of the discussion by inserting the large mass of material on which it is based, this material has been given in tabular form at the end of the present paper.

Referring to the explanation of the tables for a detailed statement of the quantities in each table, it will suffice here to say, in a general way, that the first four tables contain the numerical data for determining the perturbations at any time between the date of discovery, in 1854, and the date of the last observations used, near the end of 1888.

In Table V the perturbations derived from the first four tables are applied to elements V, to obtain the osculating elements at the required dates of comparison between theory and observation. This work is given in Table V.

From these various systems of elements geocentric places are computed, and shown in Table VI.

Table VII shows the mean corrections given by observations to the geocentric places of Table VI.

In Table VIII these geocentric places form the absolute terms of certain equations of condition between the corrections to the elements.

In Table IX these equations are modified, so as to obtain equations of condition in the form best adapted for numerical solution.

The equations of condition thus formed give rise to the following system of normal equations:

$$
\begin{array}{rrrrrrrl}
551.7\,x & +224.5\,y & +177.0\,z & -440.5\,u & -\ 62.8\,v & +\ \ 2.6\,w & +\ 45.0\,t & =+337''.8 \\
224.5 & +347.3 & +219.1 & -197.0 & -\ 15.1 & -\ \ 6.3 & +175.3 & =-127.8 \\
177.0 & +219.1 & +630.8 & -119.6 & -\ 26.9 & +\ \ 4.6 & +117.8 & =-503.9 \\
-440.5 & -197.0 & -119.6 & +426.4 & +\ 39.5 & -\ \ 3.2 & -\ 20.3 & =-398.6 \\
-\ 62.8 & -\ 15.1 & -\ 26.9 & +\ 39.5 & +112.4 & -\ 27.3 & -\ 12.9 & =+245.5 \\
2.6 & -\ \ 6.3 & +\ \ 4.6 & -\ \ 3.2 & -\ 27.3 & +114.9 & -\ \ 2.7 & =-179.6 \\
45.0 & +175.3 & +117.8 & -\ 20.3 & -\ 12.9 & -\ \ 2.7 & +317.3 & =-771.1
\end{array}
$$

The solution of these equations gives the following values and weights of the unknowns, and of the corresponding corrections to Elements V:

$x=+0.5895$;	$w=$ 86.9	$\delta L_0=+\ 0''.5895\pm 0''.34$
$y=+0.9061$;	" 148.1	$1000\,\delta\mu=+\ 0.7182\pm 0.66$
$z=-0.7671$;	" 474.3	$\delta\varphi=-\ 0.7671\pm 0.15$
$u=-0.4439$;	" 67.5	$\delta\pi=-\ 0.8878\pm 0.80$
$v=+2.0351$;	" 96.6	$\delta\Omega=+60.15\ \pm 1.00$
$w=-1.0840$;	" 107.6	$\delta i=-\ 1.0840\pm 0.31$
$t=-2.6848$;	" 206.6	$\delta\frac{1}{m}=-\ 0.5370\pm 0.06$

These corrections being applied to Elements V give the definitive elements of the epoch 1873, July 17.0, Berlin, referred to the ecliptic and equinox of 1870.0. If strictly osculating elements are required, a modification is to be made, arising from the fact that the perturbations by the Earth were not taken so as to vanish at the above fundamental epoch. The corrections for reduction to osculating elements consist simply of the perturbations produced by the Earth at the epoch, which are given in Table V. With this modification, the definitive Elements VI are formed as follows:

	Elements V.	⊕ Pert.	Corr. per Obs.	Elements VI.
	° ′ ″	″	″ ″	° ′ ″
L	318 53 30.0	+1.27	+ 0.59 ± 0.34	318 53 31.86
π	342 20 7.5	−6.60	− 0.89 ± 0.80	342 20 0.01
φ	19 52 54.2	+2.33	− 0.77 ± 0.15	19 52 55.76
☊	9 2 44.9	0.00	+60.2 ± 10.0	9 3 45.1
i	1 56 20.0	0.00	− 1.08 ± 0.31	1 56 18.92
μ	732″.81750	−0.02496	+ 72 ± 6	732″.79326

Epoch, 1873, July 17.0, Berlin; ecliptic and equinox of 1870.0.

For the convenience of future investigators, we shall next reduce these definitive Elements VI to the epoch 1888, November 8.0, Berlin mean time. This reduction is effected by applying to Elements VI the perturbations from the fundamental epoch to the epoch of reduction, together with STRUVE's precession and the effect of the motion of the ecliptic, from 1870.0 until 1888.0. The perturbations employed in the reduction should be those resulting from the definitive mass of Jupiter. The necessary correction is given below as δ♃.

Reduction of Elements VI to 1888, Nov. 8.0, Berlin noon.

	L	π	φ	☊	i	μ
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
El. VI	17 22 4.56	342 20 0.01	19 52 55.76	9 3 45.1	1 56 18.92	732.79326
♃, ♄, ♂	− 2 29 57.67	+49 21.95	−24 18.65	− 5 26.5	− 1 22.27	− 3.48587
♅	− 1.60	+ 3.25	+ 0.03			+ .00006
⊕	+ 2 14.03	+15.92	− 0.64			+ .00202
Δ♃	− 4.57	+ 1.50	− 0.74	− 0.2	− 0.04	− .00177
Prec. to 1888.0	15 4.70	15 4.70		13 58.3	+ 8.18	
	15 9 19.45	343 24 47.33	19 28 35.76	9 12 16.7	1 55 4.79	729.30770

The ecliptic and equinox are those of 1888.0.

VI.

INVESTIGATION OF THE MASS OF JUPITER.

The first question we have to consider is that of the comparative accuracy of the two methods of determining this mass—by the motions of the satellites and by the perturbations of other bodies. Respecting the general method of determining the mass of a planet from measures of the mean distance of a satellite, it is to be remarked that, other conditions being equal, the greater the mass the less the degree of accuracy with which the determination can be made. Although in the case of Jupiter other conditions are not equal, yet a little examination will show that no determination founded on measures of the satellite can be entirely satisfactory. The main source of error in these measures has not received due consideration. Every observation for the angular distance of two neighboring bodies requires for its expression in arc that the value in arc of some unit on the measuring scale shall be known. In observations with the filar micrometer the required factor is that of one revolution of the micrometer. In the case of heliometer observations, the required factor is the angular value of the unit of the scale by which the distance of the two parts of the objective are read. Experience shows that neither of these determinations can be made with entire precision. A remarkable instance of this is seen in the discrepancy between the mass of Saturn found by the measures made by Professor Hall and myself with the Washington equatorial, and the other good determinations made in various ways, including those by H. Struve with the Pulkowa equatorial. It would seem that the discrepancy must be attributed to the determination of the value of the micrometer screw in the Washington instrument; yet this determination was made by methods which would seem best calculated to give an accurate result. I have especially in view in this connection the method employed by Professor Holden, of measuring the differences of declination between a row of stars taken chainwise, the extremes of the series being two well-known stars. In micrometer measures it is quite likely that the necessary obliquity of vision, in case the pointing on the two objects is made simultaneously, may be a source of error. The fact with which we have to deal is that a very important systematic error seems to have crept in, which would never have been known but by the erroneous results to which it led.

Measures with the heliometer are undoubtedly free from some sources of error that might affect those made with the filar micrometer—those arising from obliquity of the rays passing through the eye piece, for example; yet the heliometer, as commonly used, is subject to a possible source of systematic error which has not received the consideration which it deserves. A definite angular value can not be assigned to the scale of the heliometer, except by assuming a definite point as the stellar focus. We might go further and say that it presupposes that the rays through each half of the objective come to a definite focus on the retina of the eye of the observer, the lenses of the eye being so adjusted by accommodation that this condition shall be exactly fulfilled. Now rays of different colors necessarily come to different foci through defects of achromatism, both in the glasses of the objective and in the eye of the observer. The result of this is that the scale of the heliometer has a different

value for each ray of the spectrum that comes to a different focus. Moreover the mean value for all the rays is not any determinable quantity, but depends on the setting of the ocular, or the accommodation of the observer's eye, at the moment of observation.

Altogether I should think that we can not prudently rely upon a final result of a heliometer measure being absolutely correct within the six-thousandth part. Now a proportionate error of this sort is increased threefold in the mass of a planet. I conceive, therefore, that we can not rely upon any heliometer measure giving a result for the mass of Jupiter of which the denominator will not be subject to an error of half a unit, or more.

Yet another method of measuring the distance between the satellites, or between one satellite and the planet, is available, namely, the method of transits. This was employed by AIRY in his determination of the mass of Jupiter at Cambridge. The weakness of this method is that it may be affected by personal equations in estimating the respective times of transit of the limb of the planet and of the satellite over the thread. It is true that this error, considered as a rigorous and constant quantity, should be nearly eliminated between observations of the satellite made at opposite elongations. But I hardly think that we could practically count on the elimination being complete.

The method of making measures between the satellites *inter se* through a number of revolutions of each, and thus independently determining the orbit of each, which has been applied with so much success by HERMANN STRUVE at Pulkowa, is free from some of these objections. Yet if the heliometer is employed for this purpose it must still be affected by whatever uncertainty may inhere in the value of the scale.

The importance of the possible error thus arising may be seen by reflecting that a change of half a unit in the denominator expressing the mass of Jupiter corresponds to the addition to the planet of a mass more than half as large again as the entire mass of Mars.

BESSEL's mass of Jupiter, determined with the Königsberg heliometer, has been very carefully examined and corrected by SCHUR. He finds a correction to the value of the unit of BESSEL's scale, the result of which is to add nearly a unit to the denominator expressing the mass of Jupiter. As the action of Jupiter on other bodies seems to show very clearly that BESSEL's denominator is already too large, this result shows the uncertainty of the method.

Just after SCHUR's investigation, appeared that of Dr. KEMPF.* The original observations here discussed are those made by Dr. VOGEL during the years 1868–1870, but he has also rediscussed a number of previous determinations, including those of BESSEL and AIRY. The following four principal values of the reciprocal of the mass, found by measures of the satellites, is taken from this paper (page 125):

BESSEL (SCHUR), from heliometer measures	1048.629 ± 0.134
SCHUR	1047.232 ± 0.246
AIRY	1047.641 ± 0.488
VOGEL	1047.767 ± 0.310

*Publicationen des Astrophysikalischen Observatoriums zu Potsdam, dritter Band. Potsdam, 1883.

In view of the uncertainty which inheres in the heliometer scale, I consider that the indiscriminate mean of these results is more likely to be correct than a weighted mean. This indiscriminate mean is about 1047.82, agreeing very closely with BESSEL'S original determination, which has been a classic in astronomy. At the same time, for reasons already given, I think we must assign to this result a mean error of several tenths of a unit.

The mass of Jupiter has been determined from the action which it exerts both upon planets and upon comets. The use of some comets for the purpose of determining the mass in question has the great advantage that the perturbations are sometimes much greater than in the case of any one planet. But this advantage is compensated by the possible source of uncertainty in the constitution of a comet. Were the latter a well-defined, spherical, solid body, no uncertainty would arise from its constitution. As a matter of fact, however, it is not certain that even the nucleus of a telescopic comet, when there is one, can be considered as a well-defined spherical body. It is not even certain that most of the telescopic comets are more than clouds of what Professor NEWTON has called meteoroids—discrete particles of varying and uncertain size. It might be claimed that the very fact that the parts of the comet keep together militates against this view, since such a cloud would speedily be dissipated by the unequal action of disturbing planets, and even of the Sun, upon various parts of its mass. If we admit this, we have to admit the corresponding fact that the action of a planet, such as Jupiter, upon different parts of the comet at any one moment will be very different, and we can not be sure that the nucleus, or other point on which observations are made, corresponds to the mean of all the actions of Jupiter upon the several parts. These considerations are supported by the generally erratic movement of comets, and by the ill agreement of their ephemerides with observations. There is, however, one recent determination in which the conditions seem to be so exceptionally good as to merit a special consideration.

Mass of Jupiter, as derived by Von Haerdtl from motions of Winnecke's comet.—This investigation is contained in two memoirs, published in the *Denkschriften* of the Vienna Academy of Sciences, Vols. 55 and 56, Vienna, 1889. From these papers it would seem that the perturbations produced by Jupiter in the motion of the comet in question amount, if the elements are not first adjusted for perturbations, to more than twenty degrees in the mean anomaly, and more than a quadrant in the geocentric place. Even after an adjustment the perturbations would amount, it would seem, to several degrees in the geocentric place. The result derived in the first memoir is

$$1:m = 1047.1752 \pm 0.0136^{*}$$

In deriving this result, the perturbations by the Earth and Mars were computed with the old values of the masses of those planets—that of the earth being too small by about one-twelfth, and that of Mars too large by about one-eighth. The correction of the equations of condition from the source is, however, a comparatively simple matter, and is undertaken in the second memoir. The result is

$$1:m = 1047.1758 \pm 0.0210$$

* First paper, page 299.

so that the mass is not appreciably altered, though the probable error is increased. The latter circumstance seems to indicate that the representation of the observations is not so good when the correct masses are adopted.

I have not been able to give Dr. VON HAERDTL'S work the critical examination which it merits, but it impresses one as being executed with great numerical accuracy. Still, several possible sources of minute error are to be considered. The rigorous perturbations from 1858 to 1886 were computed with the mass of Jupiter 1 : 1047.54, and the reduction to the final provisional mass 1:1047.1752 seems to have been made by increasing all the perturbations computed with the former mass, in the ratio of one of these quantities to the other. The error arising from such an assumed proportionality of the perturbations to the masses might not be inconsiderable; its uncertainty can not be estimated with precision, but I think it might well amount to as much as one-tenth of the correction. We might therefore assign to the reciprocal of the mass a possible error of ± 0.03 from this cause alone.

In view of the very large systematic errors by which the observations of faint comets are always affected, arising from the impossibility of seeing any definite point which is known to coincide with the center of gravity of the comet, and of the erratic way in which these bodies always seem to vary from the ephemerides, it is impossible to regard the small mean error assigned to this value as complete. At the same time, in view of the magnitude of the perturbations, and of the satisfactory way in which the normal places are represented, this result would actually seem entitled to greater weight than any other, that from Polyhymnia excepted, notwithstanding the uncertainties just alluded to.

Mass of Jupiter from its action on Saturn.—Mr. GEORGE W. HILL has worked up the observations of Saturn from 1751 until the present time. An abstract of this work will be found in the introduction to his tables of that planet. The masses of Jupiter and Uranus were determined from the observations of Saturn. For Jupiter the result is

$$1 : m = 1047.378 \pm 0.121$$

As the observations of Saturn show systematic, though small deviations from theory, the possible error of this determination must be considered as somewhat greater than would follow from the probable error here assigned.

Mass of Jupiter from its action on Faye's comet.—In MÖLLER'S well-known discussion of the motions of FAYE'S comet, he derived the mass of Jupiter,

$$1 : 1047.88$$

Although the observations are fairly well represented, there are still, as is usual in such cases, marks of systematic error. In the paper already alluded to, VON HAERDTL has subjected MÖLLER'S results to a careful examination, with a view of determining whether the motions of this comet are reconcilable with the mass which he himself has deduced. The result is given on page 303 of this paper. The increased mass leads to quite large systematic errors, especially during the apparition of 1865–66, when the errors are almost always negative, and amount at the maximum to 18″. I do not think this result at all conclusive against the larger mass. It simply proves the

correctness of the remark already made, that we should not expect the motion of that part of the comet which observers actually determine the position of to be coincident with the mean of all the parts on which the Sun and planets may act. VON HAERDTL also suggests that his omission of the effect of the changed mass of Jupiter on the inclination and node of the comet may have increased the discrepancy.

Krüger's investigation of the motion of Themis.—I have remarked in the early part of this paper that the planet Themis is one of those which are well adapted to determine the mass of Jupiter, and have alluded to Professor KRÜGER'S work on that subject. The final result of his investigation is found in the *Astronomische Nachrichten, Band 81, S. 331–336*. The observations extend from 1853 to 1870, between which times sixteen normal places are formed. The final result for the mass of Jupiter is

$$1 : 1047.538 \pm 0.192$$

There are a number of other well-known determinations, which may be passed over quite rapidly. The exhaustive researches on the motion of ENCKE'S comet, made first by ENCKE, and then continued by Von ASTEN and BACKLUND at Pulkowa, do not seem well calculated to give a good mass of Jupiter. The irregular though persistent acceleration to which the comet seems to be subject would alone be fatal to an accurate determination.

HANSEN, in his tables of Egeria, found the reciprocal of the mass to be

$$1051.12 \pm 0.81$$

Quite apart from the large value of this denominator the result must be subject to doubt from the subsequent deviations of HANSEN'S tables of Egeria from observation. This deviation shows that something must be wrong in the theory on which they are founded, and leads us to omit this determination from those which may safely be utilized in determining the value of the mass.

BECKER, from the motion of Amphitrite, found

$$1047.37 \pm 1.31$$

The value thus reached is very close to that which we must conceive to be the most probable, but the very large probable error renders it unnecessary to introduce the result.

Another result quoted by KEMPF is that of DUBJAGO, from the motions of Diana. The result is

$$1045.25 \pm 0.46$$

It seems hardly necessary to consider this value further.

On the whole, I think that the most probable value of the mass of Jupiter will be obtained from the following determinations, by assigning weights as I do below. As is usual in such cases, these weights can not be determined by any numerical process,

but only by a general judgment of the character of the several results. The probable error arising from the uncertainty of such judgments must be included among the possible unavoidable sources of error.

From all observations upon the satellites	1047.82	*wt.* 1
Möller's investigation of Faye's comet	1047.79	1
Krüger's investigation of the motion of Themis . .	1047.54	5
Hill's investigation of the motion of Saturn . . .	1047.38	7
Preceding investigation of the motion of Polyhymnia	1047.34	20
Von Haerdtl's investigation of Winnecke's comet .	1047.17	10
General mean	1047.35 ± 0.065	

The mean error $\pm .065$ is that derived from the discordance of the results *inter se*, and not from the probable errors assigned to the separate results.

Had we used the latter probable errors, the result would, of course, have been much smaller, but entirely illusory.

The fact that when the values are arranged in the order of magnitude, the result to which the smallest probable error was assigned by the original investigator stands at one end of the series, shows that the value to be finally deduced must depend very largely on the relative weight which we assign to the result in question. According as we assign a greater weight to Von Haerdtl's result we shall get a smaller denominator in the expression for the mass. A clearer view of the relation of the mass finally derived to the weights assigned to the different methods will be gained if we begin by considering the results that might be obtained from the action of Jupiter upon the three planets. By one extreme method of treatment we should take the indiscriminate mean of these three results, which would give

$$1 : m = 1047.42$$

At the other extreme we should have the result derived by assigning weights as they result from the probable errors given by the three investigations. Numbers sufficiently near for the weights would then be as follows:

Themis,	weight	2
Saturn,	"	5
Polyhymnia,	"	80

These weights would give

$$1 : m = 1047.347$$

The general principles of the theory of probabilities, and the fact that every method must be considered as subject to constant errors peculiar to itself, leads us to assign a system of weights intermediate between the two systems given above. Taking the intermediate system, which I consider the most probable, namely, that given above, the result is

$$\mu = 1047.38 \pm .05$$

From these three intermediate values which result from the action on planets, we have on one side the results from measures of the satellites and from FAYE'S comet, and on the other side the result from WINNECKE'S comet.

Quite possibly I have erred in giving to the latter result only ten times the weight assigned to each of the first two results. Possibly, also, the results from planets are not subject to constant errors peculiar to each, as I have supposed; certainly the three results do not differ more widely than one would expect them to from their probable errors. Excepting, however, for the doubt which must depend upon the weight to be assigned to VON HAERDTL'S result, I do not conceive that any reassignment of weights would greatly change the result 1047.35.

It is however to be noticed that the mean error which we assign to this result is much larger than that which would result from having considered the small planets alone. In other words, we have increased the seeming uncertainty of the result by adding to the data on which it depends. This however arises from the fact just mentioned, that the three results from the planets agree as well as one would expect from their probable errors, and therefore better than the relative weights which I have assigned would presuppose. Were it allowable in so small a number of results to change the weights assigned *a priori* on account of discrepancies in the extreme of the series, then we should assign less relative weights to the results from the two comets, and thereby reach a smaller probable error for the original result. In other words, the larger probable error assigned to the final result depends upon the very fact that VON HAERDTL by so excellent an investigation as he made upon WINNECKE'S comet obtained a result with so small a probable error, and yet deviating widely from what would otherwise be the most likely result. This legitimately increases uncertainty as to the final value which should be adopted for the mass. On the whole, I consider that the mass of Jupiter

$$1 : 1047.35 \pm 0.065$$

is the one that, from a sane and well-weighed consideration of all the results, we should consider the most probable, and that the mean error here assigned, corresponding to a probable error ± 0.045, may have a sound basis in a sane expectation.

APPENDIX.

EXHIBIT OF MATERIALS AND RESULTS IN TABULAR FORM.

Table I gives the summation of the differential variations of the perturbations by Jupiter. The formulæ employed were those of OPPOLZER, as given in his *Lehrbuch zur Bahnbestimmung der Cometen und Planeten*, Vol. II.

The dates given in the first column of the table are those for which the differential variations were computed. The unit interval is the usual one of 40 days. A double summation being necessary for the mean motion, the numbers of this double summation will correspond to the dates themselves. In the case of all the other elements the summations correspond to the dates half way between those of computation. Hence, for perspicuity the lines are broken so as to show where the summed numbers fall in. The differential variations for 40 days may be found by taking the difference between any two consecutive numbers.

The computations were made for Berlin noon of the dates in the first column, and the adopted mass of Jupiter is

$$1 \div 1047.879$$

Table II contains the summation of the perturbations by Saturn and Mars, made in the same way as in the case of Jupiter. Those by Jupiter are summed separately, in order that they may be available for computing the coefficients for correcting the mass of Jupiter in the equations of condition.

The adopted masses are

Saturn; $1 \div 3501.6$
Mars; $1 \div 3\,093\,500$

Table III contains the summation of the perturbations by Uranus for the principal epochs at or near the dates of the normal places. The method of computation has been already described on page 389. I have deemed it unnecessary to give the summation in full, believing that the numbers given for the dates of the normal places will answer every purpose of examination and control.

Table IV contains the summation of the perturbations of the mean longitude, the longitude of the perihelion, and the eccentricity, produced by the Earth.

The action of Uranus and the Earth on the inclination and node has been omitted, as probably too small to affect the results of the present investigation.

From the summations in these four tables, the perturbations of the elements at any moment are determined by well-known formulæ.

Table V. In this table is given, with great fullness, the computation of osculating elements for dates near those of the normal places, as derived from the perturbations given in the four preceding tables. The computations were originally made for an unnecessary number of dates, in order to obtain the greatest number of points of comparison between the new computations and the older ones previously described. The difficulty of selecting the particular set of elements necessary to be published has led to the inclusion of all computed.

It will be noticed that the computations are made sometimes for Greenwich noon and sometimes for Greenwich midnight. The computations of each element are arranged in seven numbered lines. Line (1) contains only the value of L found by carrying the original mean longitude forward to the date with the undisturbed mean motion. It will be noted that this motion is not the osculating motion at the epoch, this motion being corrected for the mean effect of the Earth's perturbations during the thirty-four years of observation in a way to be explained presently.

Line (2) contains the perturbations by Jupiter, Saturn, and Mars derived from the numbers given in Tables I and II. The reduction from the finite integrals given in these tables to the integrated perturbations at the date, are made by the well-known formulæ found in Vol. II of OPPOLZER'S *Lehrbuch* and in WATSON'S *Theoretical Astronomy*.

It will be noticed that the same values of these perturbations are generally used for several dates in succession. This course was deemed allowable for the reason that, by a well-known property of osculating elements, they give not only the rigorous place of the planet, but its rigorous motion. Consequently the deviations in the position of the planet due to changes in the osculating elements are, for dates near those of osculation, of the second order, and may therefore be regarded as evanescent.

Line (3) contains the perturbations by Uranus and line (4) those by the Earth, as determined from Tables III and IV. The latter perturbations are not, however, those found directly from Table IV, a modification being made to bring them as nearly as possible into accord with the general perturbations which had been used in first carrying through the investigation. The modification consists in applying to L the correction

$$+1''.27-0''.024963\,t,$$

t being the time, in days, from the epoch 1873, July 17. So far as the result is concerned, it is a matter of indifference whether we apply this correction to the perturbations themselves, or apply to the elliptic elements the corrections

$$\begin{aligned}\delta L_0 &= +1''.27\\ \delta\mu &= -0\ .024963\end{aligned}$$

to reduce them to their osculating values. The number in question was, however, applied to the perturbations rather than the elements, in order to make the change from one system to the other as small as possible.

A comparison of the perturbations of the mean longitude throughout the period of computation makes evident the equation of long period in the action of the Earth on Polyhymnia, having as its argument

$$5g'-g$$

and having a coefficient of about 3''. The existence of this term renders the action of the Earth of fundamental importance in the discussion of the motions of the planet and the derivation of results from them.

Line (5) gives the precession, including the effect of motion of the ecliptic, from the fundamental epoch 1870.0 to the beginning of the year of computation. This precession is throughout that of STRUVE and PETERS.

Line (6) contains the precession from the beginning of the year to the date, together with the nutation at the date.

The sum of the six quantities thus formed, the value of each Element V being understood as if given in line (1), gives the osculating elements found in line (7).

Table VI. From the osculating elements in Table V geocentric places of the planet are computed for the dates shown in the table, with the results there given. For the sake of uniformity the rectangular coordinates of the Sun are, throughout, those from LEVERRIER's tables, the numbers given in the *British Nautical Almanac* being used after the introduction of those tables in 1864. I may remark that the computations of the osculating elements given in Table V were made in duplicate by two different computers, one of them Dr. HERZ, and in slightly different ways. In computing the geocentric places Dr. HERZ used the osculating elements computed by himself, while those given in Table V are those of the other computer. As, however, the two sets differ by only insignificant amounts, generally only a few hundredths of a second, I have not thought it necessary to take any account of this difference.

The geocentric places as given are geometric places uncorrected for aberration.

Table VII contains the mean corrections to the provisional ephemerides as given by observations. It was not generally deemed necessary to form a complete ephemeris from Elements V. The positions given by observation were first compared with such ephemeris as happened to be available. This was in some cases that of the *Astronomisches Jahrbuch*, in some cases an ephemeris found in the *Astronomische Nachrichten*, and in others an ephemeris computed by Professor COMSTOCK from Elements IV. In some cases the ephemeris was computed from the definitive Elements V. To the mean corrections thus found from observation were applied the differences between the two ephemerides.

In order to make the results of observations as homogeneous as possible, Professor COMSTOCK entered upon a careful determination of the positions of the various comparison stars, and corrected the published results of observation accordingly. Notwithstanding the care with which this work was done, it can not, in view of the better material likely to be available for determining the positions of faint stars, be regarded as definitive. I have therefore deemed it unnecessary to print the rather voluminous mass of material necessary to show the result of each separate observation.

Tables VIII and IX give the equations of condition in two forms, the unknowns of the one being the corrections of the elements, those of the other convenient functions of those corrections chosen to facilitate the solution.

TABLE I.—*Summation of Perturbations by Jupiter.*

Date.	$\Sigma^2\Delta\mu$	$\Sigma\Delta u$	$\Sigma\Delta L_1$	$\Sigma\Delta\varphi$	$\Sigma\Delta\pi$	$\Sigma\Delta i$	$\Sigma\Delta\Omega$
	″	″	″	″	″	″	″
1854, Aug. 26	+8053.651						
		− 25.6082	+3269.048	− 525.403	−5259.277	+30.788	+1149.009
Oct. 5	+8028.043						
		− 26.3328	+3273.796	− 524.752	−5246.934	+30.755	+1148.637
Nov. 14	+8001.710						
		− 27.7245	+3277.567	− 522.816	−5230.023	+30.708	+1147.678
Dec. 24	+7973.986						
		− 29.5218	+3280.038	− 520.486	−5209.263	+30.660	+1146.164
1855, Feb. 2	+7944.464						
		− 31.5405	+3281.063	− 518.323	−5185.617	+30.619	+1144.237
Mar. 14	+7912.924						
		− 33.6584	+3280.603	− 516.647	−5160.007	+30.589	+1142.097
Apr. 23	+7879.266						
		− 35.7975	+3278.683	− 515.620	−5133.195	+30.571	+1139.944
June 2	+7843.469						
		− 37.9091	+3275.366	− 515.310	−5105.780	+30.564	+1137.963
July 12	+7805.560						
		− 39.9639	+3270.737	− 515.733	−5078.211	+30.565	+1136.311
Aug. 21	+7765.596						
		− 41.9446	+3264.889	− 516.878	−5050.823	+30.570	+1135.113
Sept. 30	+7723.651						
		− 43.8418	+3257.921	− 518.720	−5023.878	+30.575	+1134.459
Nov. 9	+7679.809						
		− 45.6506	+3249.930	− 521.231	−4997.577	+30.576	+1134.405
Dec. 19	+7634.158						
		− 47.3694	+3241.013	− 524.378	−4972.085	+30.567	+1134.975
1856, Jan. 28	+7586.789						
		− 48.9980	+3231.265	− 528.127	−4947.543	+30.544	+1136.161
Mar. 8	+7537.791						
		− 50.5374	+3220.775	− 532.444	−4924.079	+30.503	+1137.927
Apr. 17	+7487.254						
		− 51.9893	+3209.630	− 537.292	−4901.805	+30.440	+1140.213
May 27	+7435.265						
		− 53.3553	+3197.914	− 542.632	−4880.826	+30.351	+1142.935
July 6	+7381.910						
		− 54.6372	+3185.706	− 548.420	−4861.240	+30.235	+1145.993
Aug. 15	+7327.273						
		− 55.8365	+3173.081	− 554.611	−4843.136	+30.090	+1149.270
Sept. 24	+7271.437						
		− 56.9545	+3160.113	− 561.152	−4826.589	+29.915	+1152.639
Nov. 3	+7214.483						
		− 57.9920	+3146.871	− 567.989	−4811.661	+29.710	+1155.968
Dec. 13	+7156.491						
		− 58.9493	+3133.422	− 575.063	−4798.395	+29.476	+1159.119
1857, Jan. 22	+7097.542						
		− 59.8261	+3119.830	− 582.309	−4786.816	+29.215	+1161.957
Mar. 3	+7037.716						
		− 60.6214	+3106.155	− 589.662	−4776.921	+28.930	+1164.353
Apr. 12	+6977.095						
		− 61.3333	+3092.459	− 597.052	−4768.680	+28.625	+1166.186
May 22	+6915.762						
		− 61.9588	+3078.800	− 604.411	−4762.028	+28.303	+1167.350
July 1	+6853.803						
		− 62.4938	+3065.234	− 611.671	−4756.868	+27.970	+1167.753
Aug. 10	+6791.309						
		− 62.9326	+3051.822	− 618.767	−4753.059	+27.632	+1167.326
Sept. 19	+6728.376						
		− 63.2678	+3038.622	− 625.642	−4750.424	+27.294	+1166.022
Oct. 29	+6665.108						
		− 63.4898	+3025.698	− 632.246	−4748.737	+26.964	+1163.820
Dec. 8	+6601.618						
		− 63.5867	+3013.117	− 638.547	−4747.736	+26.647	+1160.729
1858, Jan. 17	+6538.031						
		− 63.5432	+3000.957	− 644.528	−4747.109	+26.350	+1156.788
Feb. 26	+6474.488						
		− 63.3413	+2989.303	− 650.197	−4746.520	+26.080	+1152.072
Apr. 7	+6411.147						
		− 62.9588	+2978.255	− 655.595	−4745.609	+25.843	+1146.692
May 17	+6348.188						
		− 62.3702	+2967.930	− 660.798	−4744.023	+25.644	+1140.797

TABLE I.—*Summation of Perturbations by Jupiter*—Continued.

Date.	$\Sigma^2\Delta\mu$	$\Sigma\Delta\mu$	$\Sigma\Delta L_1$	$\Sigma\Delta\varphi$	$\Sigma\Delta\pi$	$\Sigma\Delta i$	$\Sigma\Delta\Omega$
	″	″	″	″	″	″	″
1858, June 26	+6285.818						
		− 61.5471	+2958.461	− 665.925	−4741.452	+25.488	+1134.574
Aug. 5	+6224.271						
		− 60.4607	+2950.005	− 671.136	−4737.687	+25.378	+1128.249
Sept. 14	+6163.810						
		− 59.0871	+2942.739	− 676.619	−4732.705	+25.316	+1122.084
Oct. 24	+6104.723						
		− 57.4179	+2936.854	− 682.554	−4726.754	+25.300	+1116.372
Dec. 3	+6047.305						
		− 55.4777	+2932.542	− 689.041	−4720.414	+25.326	+1111.411
1859, Jan. 12	+5991.827						
		− 53.3484	+2929.979	− 695.987	−4714.517	+25.384	+1107.477
Feb. 21	+5938.479						
		− 51.1928	+2929.289	− 702.995	−4709.810	+25.461	+1104.757
Apr. 2	+5887.286						
		− 49.2585	+2930.516	− 709.353	−4706.343	+25.538	+1103.272
May 12	+5838.028						
		− 47.8445	+2933.579	− 714.210	−4702.875	+25.595	+1102.797
June 21	+5790.184						
		− 47.2412	+2938.220	− 716.884	−4696.793	+25.612	+1102.806
July 31	+5742.943						
		− 47.6763	+2943.962	− 717.116	−4684.653	+25.573	+1102.462
Sept. 9	+5695.267						
		− 49.2984	+2950.073	− 715.104	−4662.946	+25.470	+1100.654
Oct. 19	+5645.969						
		− 52.1891	+2955.553	− 711.389	−4628.622	+25.305	+1096.054
Nov. 28	+5593.780						
		− 56.3843	+2959.136	− 706.703	−4579.282	+25.091	+1087.184
1860, Jan. 7	+5537.396						
		− 61.8912	+2959.279	− 701.842	−4513.143	+24.854	+1072.495
Feb. 16	+5475.505						
		− 68.6950	+2954.162	− 697.585	−4428.961	+24.633	+1050.470
Mar. 27	+5406.810						
		− 76.7574	+2941.685	− 694.642	−4325.950	+24.478	+1019.749
May 6	+5330.053						
		− 86.0087	+2919.479	− 693.609	−4203.772	+24.451	+ 979.317
June 15	+5244.044						
		− 96.3358	+2884.940	− 694.911	−4062.552	+24.619	+ 928.703
July 25	+5147.708						
		−107.5689	+2835.327	− 698.753	−3902.880	+25.048	+ 868.222
Sept. 3	+5040.139						
		−119.4637	+2767.913	− 705.049	−3725.904	+25.791	+ 799.219
Oct. 13	+4920.675						
		−131.6924	+2680.202	− 713.352	−3533.282	+26.875	+ 724.197
Nov. 22	+4788.983						
		−143.8369	+2570.241	− 722.811	−3327.172	+28.286	+ 646.854
1861, Jan. 1	+4645.146						
		−155.3991	+2436.984	− 732.185	−3110.067	+29.953	+ 571.870
Feb. 10	+4489.747						
		−165.8246	+2280.660	− 739.910	−2884.632	+31.741	+ 504.388
Mar. 22	+4323.922						
		−174.5453	+2103.099	− 744.266	−2653.490	+33.460	+ 449.321
May 1	+4149.377						
		−181.0405	+1907.833	− 743.615	−2418.972	+34.880	+ 410.440
June 10	+3968.337						
		−184.9019	+1699.986	− 736.684	−2182.985	+35.769	+ 389.606
July 20	+3783.435						
		−185.8902	+1485.872	− 722.813	−1947.012	+35.933	+ 386.328
Aug. 29	+3597.545						
		−183.9697	+1272.341	− 702.100	−1712.215	+35.256	+ 397.809
Oct. 8	+3413.575						
		−179.3084	+1066.041	− 675.405	−1479.632	+33.722	+ 419.530
Nov. 17	+3234.267						
		−172.2480	+ 872.753	− 644.200	−1250.399	+31.415	+ 446.186
Dec. 27	+3062.019						
		−163.2514	+ 696.905	− 610.331	−1025.881	+28.501	+ 472.648
1862, Feb. 5	+2898.768						
		−152.8387	+ 541.370	− 575.730	− 807.703	+25.195	+ 494.754
Mar. 17	+2745.929						
		−141.5312	+ 407.475	− 542.173	− 597.666	+21.723	+ 509.780

TABLE 1.—*Summation of Perturbations by Jupiter*—Continued.

Date.	$\Sigma^2\Delta\mu$	$\Sigma\Delta\mu$	$\Sigma\Delta L_1$	$\Sigma\Delta\varphi$	$\Sigma\Delta\pi$	$\Sigma\Delta i$	$\Sigma\Delta\Omega$
	″	″	″	″	″	″	″
1862, Apr. 26	+2604.398	—129.8092	+ 295.244	— 511.119	— 397.641	+18.293	+ 516.508
June 5	+2474.588	—118.0855	+ 203.680	— 483.622	— 209.403	+15.074	+ 515.033
July 15	+2356.503	—106.6943	+ 131.088	— 460.321	— 34.518	+12.185	+ 506.425
Aug. 24	+2249.809	— 95.8917	+ 75.360	— 441.478	+ 125.761	+ 9.698	+ 492.332
Oct. 3	+2153.917	— 85.8638	+ 34.208	— 427.042	+ 270.533	+ 7.640	+ 474.634
Nov. 12	+2068.053	— 76.7391	+ 5.327	— 416.723	+ 399.230	+ 6.004	+ 455.175
Dec. 22	+1991.314	— 68.5998	— 13.485	— 410.055	+ 511.616	+ 4.758	+ 435.581
1863, Jan. 31	+1922.714	— 61.4944	— 24.272	— 406.453	+ 607.741	+ 3.854	+ 417.164
Mar. 12	+1861.220	— 55.4486	— 28.881	— 405.259	+ 687.917	+ 3.237	+ 400.882
April 21	+1805.771	— 50.4715	— 28.952	— 405.762	+ 752.705	+ 2.847	+ 387.346
May 31	+1755.299	— 46.5613	— 25.924	— 407.225	+ 802.934	+ 2.628	+ 376.845
July 10	+1708.738	— 43.7081	— 21.042	— 408.903	+ 839.722	+ 2.526	+ 369.391
Aug. 19	+1665.030	— 41.8896	— 15.367	— 410.078	+ 864.576	+ 2.495	+ 364.759
Sept. 28	+1623.130	— 41.0629	— 9.773	— 410.113	+ 879.468	+ 2.497	+ 362.532
Nov. 7	+1582.067	— 41.1496	— 4.950	— 408.544	+ 886.855	+ 2.501	+ 362.146
Dec. 17	+1540.917	— 42.0153	— 1.398	— 405.208	+ 889.541	+ 2.485	+ 362.941
1864, Jan. 26	+1498.902	— 43.4600	+ 0.576	— 400.327	+ 890.250	+ 2.438	+ 364.227
Mar. 6	+1455.442	— 45.2305	+ 0.842	— 394.479	+ 890.995	+ 2.357	+ 365.361
Apr. 15	+1410.212	— 47.0742	— 0.603	— 388.362	+ 892.589	+ 2.247	+ 365.824
May 25	+1363.138	— 48.7830	— 3.651	— 382.544	+ 894.778	+ 2.118	+ 365.262
July 4	+1314.355	— 50.2293	— 8.134	— 377.295	+ 896.745	+ 1.980	+ 363.492
Aug. 13	+1264.126	— 51.3594	— 13.848	— 372.610	+ 897.682	+ 1.846	+ 360.474
Sept. 22	+1212.767	— 52.1701	— 20.584	— 368.318	+ 897.075	+ 1.724	+ 356.263
Nov. 1	+1160.597	— 52.6853	— 28.135	— 364.190	+ 894.760	+ 1.621	+ 350.984
Dec. 11	+1107.912	— 52.9389	— 36.330	— 360.007	+ 890.837	+ 1.544	+ 344.791
1865, Jan. 20	+1054.973	— 52.9663	— 45.015	— 355.588	+ 885.581	+ 1.495	+ 337.860
Mar. 1	+1002.007	— 52.8002	— 54.065	— 350.797	+ 879.357	+ 1.478	+ 330.366
Apr. 10	+ 949.207	— 52.4683	— 63.377	— 345.547	+ 872.564	+ 1.493	+ 322.491
May 20	+ 896.739	— 51.9941	— 72.870	— 339.785	+ 865.603	+ 1.539	+ 314.412
June 29	+ 844.745	— 51.3962	— 82.471	— 333.487	+ 858.856	+ 1.616	+ 306.271
Aug. 8	+ 793.349	— 50.6899	— 92.125	— 326.661	+ 852.676	+ 1.722	+ 298.224
Sept. 17	+ 742.659	— 49.8874	— 101.784	— 319.330	+ 847.383	+ 1.854	+ 290.410
Oct. 27	+ 692.772	— 48.9977	— 111.408	— 311.530	+ 843.256	+ 2.010	+ 282.946
Dec. 6	+ 643.774	— 48.0282	— 120.959	— 303.316	+ 840.542	+ 2.186	+ 275.937
1866, Jan. 15	+ 595.746	— 46.9844	— 130.403	— 294.746	+ 839.451	+ 2.378	+ 269.471

TABLE I.—*Summation of Perturbations by Jupiter*—Continued.

Date.	$\Sigma^2\Delta\mu$	$\Sigma\Delta\mu$	$\Sigma\Delta L_1$	$\Sigma\Delta\varphi$	$\Sigma\Delta\pi$	$\Sigma\Delta i$	$\Sigma\Delta\Omega$
	″	″	″	″	″	″	″
1866, Feb. 24	+ 548.762						
		− 45.8699	− 139.705	− 285.888	+ 840.160	+ 2.581	+ 263.615
Apr. 5	+ 502.892						
		− 44.6876	− 148.832	− 276.813	+ 842.812	+ 2.792	+ 258.422
May 15	+ 458.204						
		− 43.4392	− 157.747	− 267.597	+ 847.518	+ 3.006	+ 253.923
June 24	+ 414.765						
		− 42.1257	− 166.410	− 258.323	+ 854.360	+ 3.218	+ 250.131
Aug. 3	+ 372.639						
		− 40.7473	− 174.784	− 249.068	+ 863.390	+ 3.424	+ 247.039
Sept. 12	+ 331.892						
		− 39.3040	− 182.823	− 239.914	+ 874.636	+ 3.620	+ 244.622
Oct. 22	+ 292.588						
		− 37.7955	− 190.474	− 230.943	+ 888.089	+ 3.802	+ 242.836
Dec. 1	+ 254.793						
		− 36.2212	− 197.685	− 222.237	+ 903.721	+ 3.967	+ 241.619
1867, Jan. 10	+ 218.572						
		− 34.5808	− 204.399	− 213.879	+ 921.477	+ 4.112	+ 240.894
Feb. 19	+ 183.991						
		− 32.8740	− 210.552	− 205.947	+ 941.266	+ 4.235	+ 240.572
Mar. 31	+ 151.117						
		− 31.1016	− 216.073	− 198.520	+ 962.972	+ 4.334	+ 240.551
May 10	+ 120.015						
		− 29.2650	− 220.891	− 191.673	+ 986.444	+ 4.409	+ 240.721
June 19	+ 90.750						
		− 27.3674	− 224.930	− 185.477	+1011.490	+ 4.460	+ 240.970
July 29	+ 63.383						
		− 25.4146	− 228.105	− 180.001	+1037.872	+ 4.488	+ 241.186
Sept. 7	+ 37.968						
		− 23.4158	− 230.334	− 175.304	+1065.293	+ 4.495	+ 241.264
Oct. 17	+ 14.552						
		− 21.3859	− 231.538	− 171.430	+1093.390	+ 4.485	+ 241.115
Nov. 26	− 6.834						
		− 19.3479	− 231.637	− 168.410	+1121.700	+ 4.462	+ 240.672
1868, Jan. 5	− 26.182						
		− 17.3366	− 230.568	− 166.238	+1149.641	+ 4.430	+ 239.901
Feb. 14	− 43.519						
		− 15.4049	− 228.289	− 164.854	+1176.484	+ 4.396	+ 238.816
Mar. 25	− 58.924						
		− 13.6321	− 224.793	− 164.110	+1201.327	+ 4.365	+ 237.490
May 4	− 72.556						
		− 12.1365	− 220.144	− 163.718	+1223.077	+ 4.343	+ 236.073
June 13	− 84.692						
		− 11.0902	− 214.519	− 163.178	+1240.507	+ 4.332	+ 234.809
July 23	− 95.782						
		− 10.7364	− 208.270	− 161.682	+1252.410	+ 4.331	+ 234.040
Sept. 1	− 106.518						
		− 11.3940	− 202.041	− 158.061	+1257.957	+ 4.330	+ 234.191
Oct. 11	− 117.912						
		− 13.4208	− 196.916	− 150.881	+1257.264	+ 4.305	+ 235.679
Nov. 20	− 131.333						
		− 17.0841	− 194.611	− 138.858	+1251.815	+ 4.212	+ 238.681
Dec. 30	− 148.417						
		− 22.2282	− 197.576	− 121.981	+1243.812	+ 3.984	+ 242.621
1869, Feb. 8	− 170.645						
		− 27.9659	− 208.477	− 102.420	+1233.333	+ 3.545	+ 245.578
Mar. 20	− 198.611						
		− 32.5981	− 228.686	− 84.494	+1215.059	+ 2.861	+ 244.157
Apr. 29	− 231.209						
		− 34.4020	− 256.155	− 72.158	+1180.000	+ 1.996	+ 235.016
June 8	− 265.611						
		− 32.7810	− 285.563	− 66.237	+1123.352	+ 1.101	+ 217.304
July 18	− 298.392						
		− 28.4925	− 311.313	− 64.470	+1049.271	+ 0.327	+ 193.279
Aug. 27	− 326.884						
		− 22.8219	− 330.136	− 64.226	+ 967.063	− 0.249	+ 166.501
Oct. 6	− 349.706						
		− 16.8280	− 341.349	− 63.821	+ 885.312	− 0.621	+ 139.994
Nov. 15	− 366.534						
		− 11.1314	− 345.722	− 62.643	+ 809.032	− 0.823	+ 115.504

TABLE I.—*Summation of Perturbations by Jupiter*—Continued.

Date.	$\Sigma^2\Delta u$	$\Sigma\Delta\mu$	$\Sigma\Delta L_1$	$\Sigma\Delta\varphi$	$\Sigma\Delta\pi$	$\Sigma\Delta i$	$\Sigma\Delta\Omega$
	″	″	″	″	″	″	″
1869, Dec. 25	− 377.665						
		− 6.0094	− 344.492	− 60.690	+ 740.606	− 0.897	+ 93.889
1870, Feb. 3	− 383.674						
		− 1.5411	− 338.899	− 58.188	+ 680.489	− 0.881	+ 75.391
Mar. 15	− 385.215						
		+ 2.2914	− 329.975	− 55.395	+ 628.097	− 0.806	+ 59.881
Apr. 24	− 382.924						
		+ 5.5465	− 318.546	− 52.521	+ 582.490	− 0.696	+ 47.096
June 3	− 377.377						
		+ 8.2919	− 305.248	− 49.716	+ 542.633	− 0.568	+ 36.715
July 13	− 369.085						
		+ 10.5945	− 290.569	− 47.068	+ 507.528	− 0.434	+ 28.401
Aug. 22	− 358.490						
		+ 12.5136	− 274.887	− 44.622	+ 476.300	− 0.303	+ 21.838
Oct. 1	− 345.976						
		+ 14.1000	− 258.497	− 42.387	+ 448.205	− 0.180	+ 16.740
Nov. 10	− 331.876						
		+ 15.3963	− 241.639	− 40.355	+ 422.628	− 0.071	+ 12.852
Dec. 20	− 316.480						
		+ 16.4378	− 224.500	− 38.498	+ 399.069	+ 0.026	+ 9.950
1871, Jan. 29	− 300.042						
		+ 17.2539	− 207.244	− 36.786	+ 377.134	+ 0.109	+ 7.841
Mar. 10	− 282.788						
		+ 17.8679	− 190.011	− 35.185	+ 356.515	+ 0.177	+ 6.362
Apr. 19	− 264.920						
		+ 18.2995	− 172.923	− 33.665	+ 336.974	+ 0.230	+ 5.372
May 29	− 246.620						
		+ 18.5642	− 156.090	− 32.201	+ 318.326	+ 0.269	+ 4.758
July 8	− 228.056						
		+ 18.6747	− 139.617	− 30.771	+ 300.429	+ 0.295	+ 4.418
Aug. 17	− 209.381						
		+ 18.6408	− 123.601	− 29.362	+ 283.170	+ 0.309	+ 4.270
Sept. 26	− 190.740						
		+ 18.4700	− 108.138	− 27.969	+ 266.453	+ 0.312	+ 4.244
Nov. 5	− 172.270						
		+ 18.1680	− 93.316	− 26.593	+ 250.185	+ 0.305	+ 4.280
Dec. 15	− 154.102						
		+ 17.7387	− 79.226	− 25.241	+ 234.274	+ 0.290	+ 4.328
1872, Jan. 24	− 136.363						
		+ 17.1845	− 65.953	− 23.920	+ 218.614	+ 0.268	+ 4.345
Mar. 4	− 119.179						
		+ 16.5068	− 53.584	− 22.635	+ 203.092	+ 0.240	+ 4.297
Apr. 13	− 102.672						
		+ 15.7058	− 42.199	− 21.388	+ 187.573	+ 0.208	+ 4.158
May 23	− 86.966						
		+ 14.7808	− 31.881	− 20.173	+ 171.912	+ 0.173	+ 3.910
July 2	− 72.185						
		+ 13.7314	− 22.705	− 18.970	+ 155.964	+ 0.138	+ 3.548
Aug. 11	− 58.454						
		+ 12.5570	− 14.745	− 17.746	+ 139.590	+ 0.104	+ 3.079
Sept. 20	− 45.897						
		+ 11.2582	− 8.069	− 16.448	+ 122.682	+ 0.073	+ 2.522
Oct. 30	− 34.639						
		+ 9.8385	− 2.732	− 15.005	+ 105.187	+ 0.047	+ 1.912
Dec. 9	− 24.800						
		+ 8.3055	+ 1.215	− 13.336	+ 87.147	+ 0.026	+ 1.295
1873, Jan. 18	− 16.495						
		+ 6.6747	+ 3.740	− 11.359	+ 68.733	+ 0.012	+ 0.726
Feb. 27	− 9.820						
		+ 4.9725	+ 4.836	− 9.005	+ 50.257	+ 0.004	+ 0.265
Apr. 8	− 4.848						
		+ 3.2420	+ 4.528	− 6.258	+ 32.208	+ 0.001	− 0.032
May 18	− 1.606						
		+ 1.5498	+ 2.887	− 3.186	+ 15.210	+ 0.001	− 0.127
June 27	− 0.056						
		− 0.0101	+ 0.042	+ 0.008	− 0.089	0.000	− 0.008
Aug. 6	− 0.066						
		− 1.3175	− 3.815	+ 2.980	− 13.248	− 0.004	+ 0.296
Sept. 15	− 1.383						
		− 2.2418	− 8.429	+ 5.279	− 24.334	− 0.016	+ 0.709

TABLE I.—*Summation of Perturbations by Jupiter*—Continued.

Date.	$\Sigma^2\Delta\mu$	$\Sigma\Delta\mu$	$\Sigma\Delta L_1$	$\Sigma\Delta\varphi$	$\Sigma\Delta\pi$	$\Sigma\Delta i$	$\Sigma\Delta\Omega$
	″	″	″	″	″	″	″
1873, Oct. 25	− 3.625						
		− 2.6745	− 13.499	+ 6.498	− 34.111	− 0.037	+ 1.115
Dec. 4	− 6.300						
		− 2.5688	− 18.698	+ 6.480	− 43.876	− 0.068	+ 1.373
1874, Jan. 13	− 8.869						
		− 1.9561	− 23.698	+ 5.365	− 54.951	− 0.107	+ 1.348
Feb. 22	− 10.825						
		− 0.9290	− 28.196	+ 3.533	− 68.155	− 0.152	+ 0.936
Apr. 3	− 11.754						
		+ 0.3956	− 31.935	+ 1.422	− 83.655	− 0.199	+ 0.077
May 13	− 11.358						
		+ 1.9064	− 34.713	− 0.608	− 101.127	− 0.245	− 1.243
June 22	− 9.452						
		+ 3.5135	− 36.391	− 2.323	− 120.016	− 0.285	− 2.999
Aug. 1	− 5.939						
		+ 5.1504	− 36.888	− 3.607	− 139.743	− 0.318	− 5.138
Sept. 10	− 0.789						
		+ 6.7712	− 36.164	− 4.429	− 159.813	− 0.341	− 7.592
Oct. 20	+ 5.982						
		+ 8.3452	− 34.216	− 4.805	− 179.849	− 0.352	− 10.280
Nov. 29	+ 14.327						
		+ 9.8524	− 31.063	− 4.781	− 199.592	− 0.352	− 13.121
1875, Jan. 8	+ 24.179						
		+ 11.2799	− 26.741	− 4.411	− 218.876	− 0.340	− 16.039
Feb. 17	+ 35.459						
		+ 12.6199	− 21.295	− 3.752	− 237.613	− 0.317	− 18.960
Mar. 29	+ 48.079						
		+ 13.8670	− 14.782	− 2.854	− 255.769	− 0.284	− 21.821
May 8	+ 61.946						
		+ 15.0179	− 7.262	− 1.764	− 273.342	− 0.242	− 24.554
June 17	+ 76.964						
		+ 16.0701	+ 1.202	− 0.520	− 290.356	− 0.193	− 27.129
July 27	+ 93.034						
		+ 17.0211	+ 10.544	+ 0.842	− 306.848	− 0.138	− 29.494
Sept. 5	+ 110.055						
		+ 17.8685	+ 20.699	+ 2.292	− 322.865	− 0.079	− 31.617
Oct. 15	+ 127.923						
		+ 18.6096	+ 31.600	+ 3.797	− 338.464	− 0.019	− 33.472
Nov. 24	+ 146.533						
		+ 19.2401	+ 43.180	+ 5.326	− 353.708	+ 0.040	− 35.040
1876, Jan. 3	+ 165.773						
		+ 19.7555	+ 55.371	+ 6.848	− 368.675	+ 0.096	− 36.308
Feb. 12	+ 185.528						
		+ 20.1495	+ 68.103	+ 8.328	− 383.457	+ 0.146	− 37.275
Mar. 23	+ 205.678						
		+ 20.4143	+ 81.306	+ 9.728	− 398.166	+ 0.187	− 37.947
May 2	+ 226.092						
		+ 20.5405	+ 94.903	+ 11.008	− 412.948	+ 0.216	− 38.344
June 11	+ 246.633						
		+ 20.5160	+ 108.814	+ 12.127	− 427.979	+ 0.230	− 38.498
July 21	+ 267.149						
		+ 20.3258	+ 122.950	+ 13.041	− 443.483	+ 0.225	− 38.457
Aug. 30	+ 287.475						
		+ 19.9516	+ 137.212	+ 13.705	− 459.741	+ 0.197	− 38.291
Oct. 9	+ 307.427						
		+ 19.3713	+ 151.483	+ 14.080	− 477.096	+ 0.143	− 38.093
Nov. 18	+ 326.798						
		+ 18.5575	+ 165.626	+ 14.126	− 495.979	+ 0.058	− 37.985
Dec. 28	+ 345.355						
		+ 17.4769	+ 179.472	+ 13.819	− 516.910	− 0.061	− 38.126
1877, Feb. 6	+ 362.832						
		+ 16.0899	+ 192.811	+ 13.150	− 540.512	− 0.218	− 38.715
Mar. 18	+ 378.922						
		+ 14.3487	+ 205.373	+ 12.133	− 567.528	− 0.416	− 40.003
Apr. 27	+ 393.271						
		+ 12.1983	+ 216.810	+ 10.815	− 598.811	− 0.657	− 42.299
June 6	+ 405.469						
		+ 9.5777	+ 226.670	+ 9.288	− 635.310	− 0.941	− 45.978
July 16	+ 415.047						
		+ 6.4244	+ 234.359	+ 7.703	− 678.005	− 1.265	− 51.490

TABLE I.—*Summation of Perturbations by Jupiter*—Continued.

Date.	$\Sigma^2\Delta\mu$	$\Sigma\Delta\mu$	$\Sigma\Delta L_1$	$\Sigma\Delta\varphi$	$\Sigma\Delta\pi$	$\Sigma\Delta i$	$\Sigma\Delta\Omega$
	″	″	″	″	″	″	″
1877, Aug. 25	+ 421.471						
		+ 2.6881	+ 239.114	+ 6.268	− 727.757	− 1.621	− 59.342
Oct. 4	+ 424.159						
		− 1.6408	+ 239.980	+ 5.227	− 785.010	− 1.995	− 70.074
Nov. 13	+ 422.518						
		− 6.4750	+ 235.839	+ 4.777	− 849.247	− 2.363	− 84.153
Dec. 23	+ 416.043						
		− 11.5334	+ 225.605	+ 4.873	− 918.162	− 2.689	− 101.757
1878, Feb. 1	+ 404.510						
		− 16.2118	+ 208.726	+ 4.857	− 986.744	− 2.922	− 122.377
Mar. 13	+ 388.298						
		− 19.5272	+ 186.076	+ 3.080	−1047.342	− 3.013	− 144.380
Apr. 22	+ 368.771						
		− 20.3675	+ 160.837	− 2.893	−1092.326	− 2.941	− 164.813
June 1	+ 348.404						
		− 18.2157	+ 138.032	− 14.691	−1119.329	− 2.742	− 180.596
July 11	+ 330.188						
		− 13.7852	+ 122.005	− 31.098	−1133.483	− 2.500	− 190.276
Aug. 20	+ 316.403						
		− 8.6634	+ 114.061	− 48.311	−1142.227	− 2.295	− 194.757
Sept. 29	+ 307.740						
		− 4.2782	+ 112.766	− 62.637	−1149.008	− 2.161	− 196.113
Nov. 8	+ 303.462						
		− 1.3147	+ 115.750	− 72.445	−1152.923	− 2.094	− 196.173
Dec. 18	+ 302.147						
		+ 0.1809	+ 120.936	− 78.053	−1151.856	− 2.072	− 196.008
1879, Jan. 27	+ 302.328						
		+ 0.4685	+ 126.867	− 80.671	−1144.672	− 2.075	− 196.045
Mar. 8	+ 302.796						
		− 0.1283	+ 132.622	− 81.593	−1131.455	− 2.087	− 196.363
Apr. 17	+ 302.668						
		− 1.3281	+ 137.653	− 81.833	−1113.081	− 2.099	− 196.835
May 27	+ 301.340						
		− 2.9190	+ 141.657	− 82.085	−1090.706	− 2.106	− 197.281
July 6	+ 298.421						
		− 4.7519	+ 144.485	− 82.777	−1065.458	− 2.108	− 197.515
Aug. 15	+ 293.669						
		− 6.7242	+ 146.085	− 84.156	−1038.329	− 2.107	− 197.376
Sept. 24	+ 286.945						
		− 8.7666	+ 146.464	− 86.349	−1010.141	− 2.107	− 196.737
Nov. 3	+ 278.178						
		− 10.8332	+ 145.665	− 89.411	− 981.560	− 2.111	− 195.511
Dec. 13	+ 267.345						
		− 12.8928	+ 143.751	− 93.352	− 953.126	− 2.124	− 193.650
1880, Jan. 22	+ 254.452						
		− 14.9252	+ 140.796	− 98.152	− 925.274	− 2.151	− 191.145
Mar. 2	+ 239.527						
		− 16.9177	+ 136.883	− 103.775	− 898 367	− 2.196	− 188.018
Apr. 11	+ 222.609						
		− 18.8614	+ 132.091	− 110.173	− 872.707	− 2.264	− 184.321
May 21	+ 203.748						
		− 20.7515	+ 126.503	− 117.288	− 848.548	− 2.357	− 180.131
June 30	+ 182.996						
		− 22.5846	+ 120.199	− 125.058	− 826.107	− 2.479	− 175.549
Aug. 9	+ 160.411						
		− 24.3595	+ 113.253	− 133.415	− 805.572	− 2.631	− 170.688
Sept. 18.	+ 136.052						
		− 26.0757	+ 105.737	− 142.283	− 787.100	− 2.815	− 165.677
Oct. 28	+ 109.976						
		− 27.7332	+ 97.719	− 151.587	− 770.819	− 3.030	− 160.653
Dec. 7	+ 82.243						
		− 29.3325	+ 89.261	− 161.243	− 756.831	− 3.276	− 155.752
1881, Jan. 16	+ 52.911						
		− 30.8740	+ 80.422	− 171.164	− 745.214	− 3.551	− 151.114
Feb. 25	+ 22.037						
		− 32.3580	+ 71.253	− 181.261	− 736.012	− 3.852	− 146.871
Apr. 6	− 10.321						
		− 33.7846	+ 61.806	− 191.441	− 729.240	− 4.177	− 143.148
May 16	− 44.106						
		− 35.1535	+ 52.123	− 201.610	− 724.879	− 4.521	− 140.058

TABLE I.—*Summation of Perturbations by Jupiter*—Continued.

Date.	$\Sigma^2\Delta\mu$	$\Sigma\Delta\mu$	$\Sigma\Delta L_1$	$\Sigma\Delta\varphi$	$\Sigma\Delta\pi$	$\Sigma\Delta i$	$\Sigma\Delta\Omega$
	″	″	″	″	″	″	″
1881, June 25	− 79.259						
		− 36.4637	+ 42.244	− 211.674	− 722.875	− 4.881	− 137.697
Aug. 4	− 115.723						
		− 37.7134	+ 32.205	− 221.541	− 723.135	− 5.250	− 136.149
Sept. 13	− 153.436						
		− 38.8998	+ 22.041	− 231.119	− 725.526	− 5.624	− 135.473
Oct. 23	− 192.336						
		− 40.0187	+ 11.785	− 240.322	− 729.872	− 5.997	− 135.707
Dec. 2	− 232.355						
		− 41.0643	+ 1.469	− 249.072	− 735.955	− 6.363	− 136.866
1882, Jan. 11	− 273.419						
		− 42.0288	− 8.872	− 257.298	− 743.505	− 6.716	− 138.938
Feb. 20	− 315.448						
		− 42.9020	− 19.206	− 264.945	− 752.209	− 7.050	− 141.889
Apr. 1	− 358.350						
		− 43.6706	− 29.490	− 271.974	− 761.706	− 7.359	− 145.655
May 11	− 402.021						
		− 44.3175	− 39.675	− 278.373	− 771.589	− 7.639	− 150.147
June 20	− 446.338						
		− 44.8208	− 49.697	− 284.155	− 781.409	− 7.884	− 155.250
July 30	− 491.159						
		− 45.1535	− 59.479	− 289.370	− 790.692	− 8.090	− 160.822
Sept. 8	− 536.313						
		− 45.2824	− 68.919	− 294.117	− 798.959	− 8.255	− 166.693
Oct. 18	− 581.595						
		− 45.1682	− 77.893	− 298.551	− 805.767	− 8.376	− 172.668
Nov. 27	− 626.763						
		− 44.7666	− 86.249	− 302.877	− 810.768	− 8.452	− 178.529
1883, Jan. 6	− 671.530						
		− 44.0327	− 93.800	− 307.360	− 813.818	− 8.485	− 184.043
Feb. 15	− 715.563						
		− 42.9311	− 100.331	− 312.280	− 815.077	− 8.478	− 188.966
Mar. 27	− 758.494						
		− 41.4537	− 105.610	− 317.860	− 815.113	− 8.437	− 193.072
May 6	− 799.948						
		− 39.6463	− 109.409	− 324.145	− 814.848	− 8.371	− 196.180
June 15	− 839.594						
		− 37.6333	− 111.529	− 330.883	− 815.223	− 8.292	− 198.205
July 25	− 877.227						
		− 35.6208	− 111.826	− 337.517	− 816.598	− 8.215	− 199.202
Sept. 3	− 912.848						
		− 33.8621	− 110.239	− 343.361	− 818.215	− 8.153	− 199.407
Oct. 13	− 946.710						
		− 32.6006	− 106.832	− 347.885	− 818.171	− 8.116	− 199.212
Nov. 22	− 979.311						
		− 32.0259	− 101.806	− 350.920	− 813.914	− 8.110	− 199.125
1884, Jan. 1	−1011.337						
		− 32.2621	− 95.507	− 352.678	− 802.878	− 8.134	− 199.699
Feb. 10	−1043.599						
		− 33.3791	− 88.414	− 353.672	− 782.879	− 8.183	− 201.465
Mar. 21	−1076.978						
		− 35.4112	− 81.117	− 354.592	− 752.256	− 8.246	− 204.867
Apr. 30	−1112.389						
		− 38.3736	− 74.293	− 356.222	− 709.855	− 8.310	− 210.214
June 9	−1150.765						
		− 42.2713	− 68.698	− 359.390	− 654.966	− 8.359	− 217.632
July 19	−1193.036						
		− 47.1054	− 65.164	− 364.943	− 587.242	− 8.377	− 227.024
Aug. 28	−1240.141						
		− 52.8743	− 64.588	− 373.726	− 506.672	− 8.353	− 238.034
Oct. 7	−1293.015						
		− 59.5718	− 67.929	− 386.569	− 413.582	− 8.280	− 250.027
Nov. 16	−1352.587						
		− 67.1848	− 76.208	− 404.271	− 308.634	− 8.162	− 262.074
Dec. 26	−1419.772						
		− 75.6920	− 90.507	− 427.570	− 192.817	− 8.015	− 272.975
1885, Feb. 4	−1495.464						
		− 85.0589	− 111.964	− 457.113	− 67.456	− 7.872	− 281.299
Mar. 16	−1580.523						
		− 95.2322	− 141.754	− 493.416	+ 65.800	− 7.784	− 285.476

TABLE I.—*Summation of Perturbations by Jupiter*—Continued.

Date.	$\Sigma^2\Delta\mu$	$\Sigma\Delta\mu$	$\Sigma\Delta L_{\prime}$	$\Sigma\Delta\varphi$	$\Sigma\Delta\pi$	$\Sigma\Delta i$	$\Sigma\Delta\Omega$
	″	″	″	″	″	″	″
1885, Apr. 30	−1675.755						
		−106.1340	− 181.075	− 356.811	+ 205.009	− 7.824	− 283.912
June 4	−1781.889						
		−117.6560	− 231.126	− 587.392	+ 348.011	− 8.086	− 275.156
July 14	−1899.545						
		−129.6510	− 293.043	− 644.937	+ 492.535	− 8.685	− 258.086
Aug. 23	−2029.196						
		−141.9235	− 367.816	− 708.945	+ 636.301	− 9.748	− 232.139
Oct. 2	−2171.120						
		−154.2255	− 456.205	− 778.400	+ 777.245	−11.408	− 197.515
Nov. 11	−2325.345						
		−166.2509	− 558.577	− 851.945	+ 913.704	−13.789	− 155.366
Dec. 21	−2491.596						
		−177.6361	− 674.751	− 927.792	+1044.631	−16.988	− 107.896
1886, Jan. 30	−2669.232						
		−187.9689	− 803.767	−1003.794	+1169.766	−21.056	− 58.320
Mar. 11	−2857.201						
		−196.8129	− 943.732	−1077.599	+1289.755	−25.978	− 10.710
Apr. 20	−3054.014						
		−203.7415	−1091.709	−1146.840	+1406.108	−31.656	+ 30.466
May 30	−3257.755						
		−208.3917	−1243.703	−1209.406	+1520.942	−37.905	+ 60.926
July 9	−3466.147						
		−210.5186	−1394.928	−1263.722	+1636.560	−44.465	+ 77.313
Aug. 18	−3676.666						
		−210.0479	−1540.226	−1308.978	+1754.878	−51.027	+ 77.828
Sept. 27	−3886.704						
		−207.1021	−1674.689	−1345.230	+1876.894	−57.275	+ 62.624
Nov. 6	−4093.816						
		−201.9956	−1794.260	−1373.332	+2002.302	−62.939	+ 33.764
Dec. 16	−4295.812						
		−195.1916	−1896.220	−1394.711	+2129.448	−67.825	− 5.236
1887, Jan. 25	−4491.004						
		−187.2359	−1979.408	−1411.056	+2255.619	−71.832	− 50.100
Mar. 6	−4678.240						
		−178.6865	−2044.124	−1424.007	+2377.516	−74.953	− 96.588
Apr. 15	−4856.926						
		−170.0562	−2091.830	−1434.923	+2491.765	−77.257	− 141.093
May 25	−5026.982						
		−161.7751	−2124.758	−1444.761	+2595.397	−78.859	− 180.970
July 4	−5188.757						
		−154.1789	−2145.509	−1454.046	+2686.104	−79.896	− 214.618
Aug. 13	−5342.936						
		−147.5125	−2156.741	−1462.908	+2762.379	−80.507	− 241.372
Sept. 22	−5490.448						
		−141.9421	−2160.959	−1471.158	+2823.601	−80.820	− 261.320
Nov. 1	−5632.390						
		−137.5714	−2160.390	−1478.375	+2870.003	−80.941	− 275.074
Dec. 11	−5769.961						
		−134.4494	−2156.930	−1484.009	+2902.715	−80.957	− 283.595
1888, Jan. 20	−5904.410						
		−132.5725	−2152.123	−1487.501	+2923.756	−80.931	− 288.016
Feb. 29	−6036.982						
		−131.8734	−2147.165	−1488.452	+2935.978	−80.908	− 289.549
Apr. 9	−6168.855						
		−132.2066	−2142.920	−1486.793	+2942.749	−80.913	− 289.364
May 19	−6301.062						
		−133.3467	−2139.954	−1482.885	+2947.323	−80.958	− 288.504
June 28	−6434.409						
		−135.0096	−2138.580	−1477.461	+2952.083	−81.041	− 287.811
Aug. 7	−6569.419						
		−136.9138	−2138.924	−1471.352	+2958.074	−81.153	− 287.875
Sept. 16	−6706.333						
		−138.8304	−2140.978	−1465.229	+2965.195	−81.282	− 289.038
Oct. 26	−6845.163						
		−140.6076	−2144.656	−1459.459	+2972.754	−81.416	− 291.439
Dec. 5	−6985.771						
		−142.1659	−2149.804	−1454.148	+2979.988	−81.542	− 295.058

TABLE II.—*Summation of Perturbations by Saturn and Mars.*

Date.	$\Sigma^2\Delta\mu$	$\Sigma\Delta\mu$	$\Sigma\Delta L_1$	$\Sigma\Delta\varphi$	$\Sigma\Delta\pi$	$\Sigma\Delta i$	$\Sigma\Delta\Omega$
	″	″	″	″	″	″	″
1854, Aug. 26	—45. 841						
		+0. 1218	+53. 761	—12. 453	—17. 602	+2. 016	+55. 157
Oct. 5	—45. 719						
		—0. 1191	+53. 633	—11. 753	—16. 279	+1. 990	+54. 863
Nov. 14	—45. 838						
		—0. 3874	+53. 184	—10. 961	—14. 743	+1. 956	+54. 174
Dec. 24	—46. 225						
		—0. 6427	+52. 365	—10. 146	—13. 289	+1. 918	+52. 976
1855, Feb. 2	—46. 868						
		—0. 8480	+51. 189	— 9. 340	—12. 261	+1. 881	+51. 221
Mar. 14	—47. 716						
		—0. 9773	+49. 726	— 8. 536	—11. 938	+1. 849	+48. 939
Apr. 23	—48. 693						
		—1. 0186	+48. 090	— 7. 703	—12. 463	+1. 826	+46. 232
June 2	—49. 712						
		—0. 9732	+46. 413	— 6. 807	—13. 823	+1. 815	+43. 244
July 12	—50. 685						
		—0. 8527	+44. 823	— 5. 824	—15. 881	+1. 817	+40. 137
Aug. 21	—51. 538						
		—0. 6744	+43. 426	— 4. 752	—18. 426	+1. 831	+37. 066
Sept. 30	—52. 212						
		—0. 4574	+42. 283	— 3. 603	—21. 229	+1. 855	+34. 154
Nov. 9	—52. 669						
		—0. 2179	+41. 421	— 2. 394	—24. 095	+1. 887	+31. 489
Dec. 19	—52. 887						
		+0. 0301	+40. 846	— 1. 151	—26. 889	+1. 924	+29. 123
1856, Jan. 28	—52. 857						
		+0. 2769	+40. 551	+ 0. 099	—29. 514	+1. 964	+27. 074
Mar. 8	—52. 580						
		+0. 5149	+40. 511	+ 1. 330	—31. 906	+2. 004	+25. 341
Apr. 17	—52. 065						
		+0. 7382	+40. 700	+ 2. 520	—34. 032	+2. 044	+23. 910
May 27	—51. 327						
		+0. 9427	+41. 081	+ 3. 647	—35. 879	+2. 082	+22. 756
July 6	—50. 384						
		+1. 1257	+41. 615	+ 4. 695	—37. 449	+2. 116	+21. 851
Aug. 15	—49. 258						
		+1. 2852	+42. 261	+ 5. 648	—38. 756	+2. 146	+21. 165
Sept. 24	—47. 973						
		+1. 4194	+42. 986	+ 6. 490	—39. 829	+2. 172	+20. 666
Nov. 3	—46. 554						
		+1. 5273	+43. 762	+ 7. 205	—40. 712	+2. 193	+20. 323
Dec. 13	—45. 027						
		+1. 6080	+44. 572	+ 7. 784	—41. 470	+2. 209	+20. 103
1857, Jan. 22	—43. 419						
		+1. 6616	+45. 404	+ 8. 227	—42. 178	+2. 221	+19. 976
Mar. 3	—41. 757						
		+1. 6899	+46. 251	+ 8. 549	—42. 901	+2. 229	+19. 912
Apr. 12	—40. 067						
		+1. 6944	+47. 098	+ 8. 768	—43. 685	+2. 233	+19. 888
May 22	—38. 373						
		+1. 6772	+47. 929	+ 8. 903	—44. 558	+2. 234	+19. 884
July 1	—36. 696						
		+1. 6406	+48. 725	+ 8. 973	—45. 534	+2. 233	+19. 886
Aug. 10	—35. 055						
		+1. 5859	+49. 466	+ 8. 993	—46. 625	+2 230	+19. 882
Sept. 19	—33. 469						
		+1. 5151	+50. 140	+ 8. 978	—47. 831	+2. 225	+19. 863
Oct. 29	—31. 954						
		+1. 4295	+50. 731	+ 8. 940	—49. 155	+2. 219	+19. 823
Dec. 8	—30. 524						
		+1. 3311	+51. 233	+ 8. 895	—50. 586	+2. 212	+19. 758
1858, Jan. 17	—29. 193						
		+1. 2221	+51. 641	+ 8. 856	—52. 107	+2. 205	+19. 664
Feb. 26	—27. 971						
		+1. 1047	+51. 957	+ 8. 840	—53. 691	+2. 198	+19. 538
Apr. 7	—26. 866						
		+0. 9801	+52. 182	+ 8. 855	—55. 322	+2. 191	+19. 379
May. 17	—25. 886						
		+0. 8496	+52. 308	+ 8. 902	—56. 983	+2. 185	+19. 191

TABLE II.—*Summation of Perturbations by Saturn and Mars*—Continued.

Date.	$\Sigma^2\Delta\mu$	$\Sigma\Delta\mu$	$\Sigma\Delta L_1$	$\Sigma\Delta\varphi$	$\Sigma\Delta\pi$	$\Sigma\Delta i$	$\Sigma\Delta\Omega$
	″	″	″	″	″	″	″
1858, June 26	−25.036						
		+0.7195	+52.313	+ 8.975	−58.608	+2.180	+18.990
Aug. 5	−24.317						
		+0.6012	+52.191	+ 9.058	−60.091	+2.177	+18.789
Sept. 14	−23.716						
		+0.5050	+51.961	+ 9.129	−61.351	+2.175	+18.587
Oct. 24	−23.211						
		+0.4404	+51.647	+ 9.158	−62.353	+2.174	+18.394
Dec. 3	−22.771						
		+0.4146	+51.283	+ 9.118	−63.109	+2.175	+18.219
1859, Jan. 12	−22.356						
		+0.4328	+50.904	+ 8.981	−63.677	+2.177	+18.075
Feb. 21	−21.923						
		+0.4962	+50.553	+ 8.733	−64.144	+2.180	+17.970
Apr. 2	−21.427						
		+0.5996	+50.268	+ 8.384	−64.615	+2.183	+17.908
May 12	−20.827						
		+0.7294	+50.085	+ 7.972	−65.161	+2.186	+17.883
June 21	−20.098						
		+0.8665	+50.027	+ 7.553	−65.774	+2.188	+17.884
July 31	−19.231						
		+0.9912	+50.104	+ 7.176	−66.364	+2.189	+17.897
Sept. 9	−18.240						
		+1.0872	+50.309	+ 6.868	−66.786	+2.190	+17.908
Oct. 19	−17.153						
		+1.1427	+50.621	+ 6.632	−66.885	+2.190	+17.907
Nov. 28	−16.010						
		+1.1513	+51.005	+ 6.448	−66.539	+2.190	+17.888
1860, Jan. 4	−14.859						
		+1.1102	+51.415	+ 6.287	−65.663	+2.189	+17.852
Feb. 16	−13.749						
		+1.0191	+51.798	+ 6.117	−64.209	+2.189	+17.808
Mar. 27	−12.730						
		+0.8795	+52.096	+ 5.903	−62.164	+2.189	+17.770
May 6	−11.851						
		+0.6946	+52.247	+ 5.617	−59.540	+2.189	+17.760
June 15	−11.156						
		+0.4685	+52.195	+ 5.230	−56.371	+2.189	+17.804
July 25	−10.688						
		+0.2070	+51.887	+ 4.725	−52.720	+2.188	+17.932
Sept. 3	−10.481						
		−0.0822	+51.280	+ 4.090	−48.681	+2.186	+18.169
Oct. 13	−10.563						
		−0.3893	+50.338	+ 3.332	−44.371	+2.181	+18.537
Nov. 22	−10.952						
		−0.7030	+49.028	+ 2.476	−39.913	+2.172	+19.048
1861, Jan. 1	−11.655						
		−1.0118	+47.328	+ 1.561	−35.410	+2.157	+19.706
Feb. 10	−12.667						
		−1.3040	+45.227	+ 0.635	−30.942	+2.136	+20.509
Mar. 22	−13.971						
		−1.5685	+42.734	− 0.252	−26.559	+2.107	+21.444
May 1	−15.539						
		−1.7949	+39.881	− 1.051	−22.288	+2.069	+22.491
June 10	−17.334						
		−1.9741	+36.722	− 1.719	−18.130	+2.021	+23.611
July 20	−19.308						
		−2.0989	+33.336	− 2.219	−14.074	+1.963	+24.769
Aug. 29	−21.407						
		−2.1643	+29.812	− 2.526	−10.089	+1.896	+25.910
Oct. 8	−23.571						
		−2.1678	+26.249	− 2.628	− 6.129	+1.820	+26.981
Nov. 17	−25.739						
		−2.1092	+22.745	− 2.522	− 2.150	+1.738	+27.929
Dec. 27	−27.848						
		−1.9912	+19.404	− 2.220	+ 1.888	+1.652	+28.711
1862, Feb. 5	−29.839						
		−1.8192	+16.309	− 1.751	+ 6.018	+1.564	+29.297
Mar. 17	−31.658						
		−1.6016	+13.534	− 1.154	+10.245	+1.477	+29.673

TABLE II.—*Summation of Perturbations by Saturn and Mars*—Continued.

Date.	$\Sigma^2\Delta\mu$	$\Sigma\Delta\mu$	$\Sigma\Delta L_1$	$\Sigma\Delta\varphi$	$\Sigma\Delta\pi$	$\Sigma\Delta i$	$\Sigma\Delta\Omega$
	″	″	″	″	″	″	″
1862, Apr. 26	−33.260	−1.3481	+11.123	− 0.477	+14.557	+1.394	+29.837
June 5	−34.608	−1.0698	+ 9.096	+ 0.226	+18.919	+1.316	+29.801
July 15	−35.678	−0.7793	+ 7.460	+ 0.904	+23.251	+1.245	+29.590
Aug. 24	−36.457	−0.4898	+ 6.215	+ 1.516	+27.435	+1.183	+29.237
Oct. 3	−36.947	−0.2113	+ 5.336	+ 2.038	+31.379	+1.130	+28.783
Nov. 12	−37.158	+0.0456	+ 4.795	+ 2.460	+34.968	+1.087	+28.273
Dec. 22	−37.112	+0.2746	+ 4.546	+ 2.789	+38.148	+1.054	+27.749
1863, Jan. 31	−36.837	+0.4681	+ 4.536	+ 3.033	+40.839	+1.030	+27.255
Mar. 12	−36.369	+0.6219	+ 4.705	+ 3.211	+43.012	+1.014	+26.825
Apr. 21	−35.747	+0.7315	+ 4.994	+ 3.347	+44.635	+1.004	+26.487
May 31	−35.015	+0.7945	+ 5.342	+ 3.469	+45.710	+0.999	+26.256
July 10	−34.221	+0.8111	+ 5.694	+ 3.606	+46.279	+0.997	+26.136
Aug. 19	−33.410	+0.7817	+ 6.004	+ 3.789	+46.399	+0.997	+26.126
Sept. 28	−32.628	+0.7115	+ 6.234	+ 4.037	+46.179	+0.997	+26.207
Nov. 7	−31.916	+0.6095	+ 6.359	+ 4.356	+45.756	+0.996	+26.353
Dec. 17	−31.307	+0.4895	+ 6.367	+ 4.726	+45.273	+0.993	+26.529
1864, Jan 26	−30.818	+0.3697	+ 6.263	+ 5.106	+44.837	+0.987	+26.695
Mar. 6	−30.448	+0.2693	+ 6.067	+ 5.439	+44.484	+0.979	+26.813
Apr. 15	−30.179	+0.2029	+ 5.806	+ 5.682	+44.157	+0.969	+26.856
May 25	−29.976	+0.1787	+ 5.516	+ 5.814	+43.755	+0.958	+26.809
July 4	−29.797	+0.1975	+ 5.234	+ 5.843	+43.177	+0.948	+26.670
Aug. 13	−29.600	+0.2547	+ 4.993	+ 5.795	+42.355	+0.938	+26.443
Sept. 22	−29.345	+0.3422	+ 4.825	+ 5.702	+41.284	+0.929	+26.136
Nov. 1	−29.003	+0.4487	+ 4.758	+ 5.595	+40.024	+0.922	+25.761
Dec. 11	−28.554	+0.5600	+ 4.795	+ 5.502	+38.689	+0.917	+25.349
1865, Jan. 20	−27.994	+0.6679	+ 4.910	+ 5.443	+37.349	+0.914	+24.937
Mar. 1	−27.326	+0.7710	+ 5.092	+ 5.420	+36.020	+0.913	+24.541
Apr. 10	−26.555	+0.8687	+ 5.346	+ 5.425	+34.719	+0.914	+24.167
May 20	−25.686	+0.9589	+ 5.677	+ 5.446	+33.476	+0.916	+23.828
June 29	−24.727	+1.0398	+ 6.085	+ 5.473	+32.313	+0.919	+23.535
Aug. 8	−23.687	+1.1093	+ 6.564	+ 5.492	+31.251	+0.922	+23.300
Sept. 17	−22.578	+1.1658	+ 7.104	+ 5.493	+30.297	+0.925	+23.133
Oct. 27	−21.412	+1.2079	+ 7.691	+ 5.463	+29.457	+0.927	+23.043
Dec. 6	−20.204	+1.2337	+ 8.309	+ 5.386	+28.738	+0.927	+23.036
1866, Jan. 15	−18.970	+1.2414	+ 8.940	+ 5.246	+28.134	+0.925	+23.117

TABLE II.—*Summation of Perturbations by Saturn and Mars*—Continued.

Date.	$\Sigma^2\Delta\mu$	$\Sigma\Delta\mu$	$\Sigma\Delta L_1$	$\Sigma\Delta\varphi$	$\Sigma\Delta\pi$	$\Sigma\Delta i$	$\Sigma\Delta\Omega$
	"	"	"	"	"	"	"
1866, Feb. 24	−17.729						
		+1.2291	+ 9.568	+ 5.024	+27.632	+0.919	+23.288
Apr. 5	−16.500						
		+1.1956	+10.182	+ 4.701	+27.199	+0.909	+23.541
May 15	−15.304						
		+1.1394	+10.775	+ 4.262	+26.784	+0.893	+23.870
June 24	−14.165						
		+1.0607	+11.342	+ 3.703	+26.322	+0.871	+24.255
Aug. 3	−13.104						
		+0.9607	+11.871	+ 3.032	+25.753	+0.843	+24.675
Sept. 12	−12.143						
		+0.8407	+12.342	+ 2.260	+25.035	+0.808	+25.106
Oct. 22	−11.302						
		+0.7026	+12.727	+ 1.403	+24.141	+0.766	+25.521
Dec. 1	−10.599						
		+0.5488	+12.997	+ 0.478	+23.055	+0.716	+25.889
1867, Jan. 10	−10.050						
		+0.3812	+13.121	− 0.504	+21.776	+0.658	+26.177
Feb. 19	− 9.669						
		+0.2039	+13.074	− 1.520	+20.315	+0.593	+26.349
Mar. 31	− 9.465						
		+0.0212	+12.834	− 2.553	+18.693	+0.521	+26.365
May 10	− 9.444						
		−0.1611	+12.389	− 3.580	+16.955	+0.442	+26.186
June 19	− 9.605						
		−0.3356	+11.736	− 4.577	+15.164	+0.359	+25.778
July 29	− 9.941						
		−0.4941	+10.883	− 5.523	+13.406	+0.273	+25.115
Sept. 7	−10.435						
		−0.6260	+ 9.845	− 6.401	+11.803	+0.188	+24.189
Oct. 17	−11.061						
		−0.7191	+ 8.662	− 7.199	+10.506	+0.107	+23.014
Nov. 26	−11.780						
		−0.7612	+ 7.394	− 7.914	+ 9.666	+0.034	+21.627
1868, Jan. 5	−12.541						
		−0.7442	+ 6.125	− 8.556	+ 9.389	−0.029	+20.095
Feb. 14	−13.285						
		−0.6652	+ 4.945	− 9.150	+ 9.709	−0.079	+18.505
Mar. 25	−13.950						
		−0.5298	+ 3.940	− 9.721	+10.559	−0.114	+16.969
May 4	−14.480						
		−0.3515	+ 3.167	−10.292	+11.783	−0.136	+15.589
June 13	−14.831						
		−0.1515	+ 2.662	−10.857	+13.161	−0.146	+14.455
July 23	−14.983						
		+0.0440	+ 2.417	−11.389	+14.464	−0.147	+13.622
Sept. 1	−14.939						
		+0.2079	+ 2.386	−11.838	+15.510	−0.144	+13.095
Oct. 11	−14.731						
		+0.3206	+ 2.495	−12.154	+16.230	−0.140	+12.830
Nov. 20	−14.410						
		+0.3692	+ 2.665	−12.287	+16.667	−0.138	+12.755
Dec. 30	−14.041						
		+0.3565	+ 2.823	−12.242	+16.955	−0.139	+12.775
1869, Feb. 8	−13.685						
		+0.2960	+ 2.914	−12.052	+17.225	−0.144	+12.809
Mar. 20	−13.389						
		+0.2075	+ 2.905	−11.776	+17.546	−0.152	+12.792
Apr. 29	−13.182						
		+0.1103	+ 2.782	−11.468	+17.907	−0.163	+12.680
June 8	−13.071						
		+0.0184	+ 2.545	−11.162	+18.256	−0.174	+12.460
July 18	−13.053						
		−0.0605	+ 2.202	−10.868	+18.539	−0.185	+12.129
Aug. 27	−13.113						
		−0.1230	+ 1.769	−10.583	+18.726	−0.194	+11.696
Oct. 6	−13.236						
		−0.1687	+ 1.266	−10.302	+18.813	−0.201	+11.179
Nov. 15	−13.405						
		−0.1988	+ 0.713	−10.017	+18.814	−0.206	+10.598

TABLE II.—*Summation of Perturbations by Saturn and Mars*—Continued.

Date.	$\Sigma^2\Delta\iota$	$\Sigma\Delta\mu$	$\Sigma\Delta L_1$	$\Sigma\Delta\varphi$	$\Sigma\Delta\pi$	$\Sigma\Delta i$	$\Sigma\Delta\Omega$
	″	″	″	″	″	″	″
1869, Dec. 25	−13.604						
		−0.2145	+ 0.140	− 9.730	+18.743	−0.208	+ 9.978
1870, Feb. 3	−13.818						
		−0.2164	− 0.430	− 9.439	+18.607	−0.207	+ 9.336
Mar. 15	−14.034						
		−0.2043	− 0.971	− 9.142	+18.392	−0.204	+ 8.687
Apr. 24	−14.238						
		−0.1783	− 1.470	− 8.830	+18.086	−0.198	+ 8.038
June 3	−14.416						
		−0.1388	− 1.915	− 8.495	+17.689	−0.190	+ 7.397
July 13	−14.555						
		−0.0871	− 2.304	− 8.126	+17.214	−0.180	+ 6.768
Aug. 22	−14.642						
		−0.0250	− 2.627	− 7.720	+16.687	−0.168	+ 6.160
Oct. 1	−14.667						
		+0.0455	− 2.882	− 7.277	+16.137	−0.154	+ 5.583
Nov. 10	−14.622						
		+0.1224	− 3.066	− 6.800	+15.595	−0.139	+ 5.045
Dec. 20	−14.500						
		+0.2035	− 3.180	− 6.294	+15.089	−0.123	+ 4.553
1871, Jan. 29	−14.296						
		+0.2871	− 3.229	− 5.766	+14.643	−0.106	+ 4.112
Mar. 10	−14.009						
		+0.3724	− 3.218	− 5.217	+14.275	−0.088	+ 3.722
Apr. 19	−13.637						
		+0.4587	− 3.146	− 4.652	+13.993	−0.070	+ 3.385
May 29	−13.178						
		+0.5451	− 3.019	− 4.078	+13.805	−0.052	+ 3.104
July 8	−12.633						
		+0.6304	− 2.838	− 3.502	+13.716	−0.035	+ 2.879
Aug. 17	−12.003						
		+0.7133	− 2.610	− 2.935	+13.730	−0.019	+ 2.708
Sept. 26	−11.290						
		+0.7915	− 2.342	− 2.395	+13.837	−0.004	+ 2.590
Nov. 5	−10.499						
		+0.8622	− 2.042	− 1.900	+14.013	+0.009	+ 2.518
Dec. 15	− 9.637						
		+0.9221	− 1.712	− 1.467	+14.210	+0.020	+ 2.484
1872, Jan. 24	− 8.715						
		+0.9674	− 1.350	− 1.110	+14.362	+0.028	+ 2.478
Mar. 4	− 7.748						
		+0.9958	− 0.952	− 0.827	+14.407	+0.034	+ 2.488
Apr. 13	− 6.752						
		+1.0047	− 0.520	− 0.613	+14.286	+0.037	+ 2.503
May 23	− 5.747						
		+0.9929	− 0.065	− 0.453	+13.966	+0.038	+ 2.512
July 2	− 4.754						
		+0.9579	+ 0.392	− 0.341	+13.402	+0.037	+ 2.502
Aug. 11	− 3.796						
		+0.8981	+ 0.824	− 0.266	+12.568	+0.034	+ 2.460
Sept. 20	− 2.898						
		+0.8127	+ 1.199	− 0.214	+11.446	+0.029	+ 2.372
Oct. 30	− 2.085						
		+0.7016	+ 1.480	− 0.172	+10.027	+0.023	+ 2.224
Dec. 9	− 1.383						
		+0.5670	+ 1.630	− 0.122	+ 8.331	+0.016	+ 2.005
1873, Jan. 18	− 0.816						
		+0.4158	+ 1.617	− 0.061	+ 6.431	+0.009	+ 1.707
Feb. 27	− 0.400						
		+0.2599	+ 1.421	+ 0.015	+ 4.445	+0.003	+ 1.332
Apr. 8	− 0.140						
		+0.1200	+ 1.050	+ 0.088	+ 2.569	−0.001	+ 0.898
May 18	− 0.020						
		+0.0227	+ 0.544	+ 0.110	+ 1.017	−0.002	+ 0.436
June 27	+ 0.003						
		−0.0037	− 0.003	+ 0.008	− 0.023	0.000	− 0.003
Aug. 6	− 0.001						
		+0.0585	− 0.467	− 0.289	− 0.506	+0.005	− 0.366
Sept. 15	+ 0.058						
		+0.2047	− 0.723	− 0.788	− 0.537	+0.012	− 0.608

TABLE II.—***Summation of Perturbations by Saturn and Mars*—Continued.**

Date.	$\Sigma^2\Delta n$	$\Sigma\Delta\mu$	$\Sigma\Delta L_1$	$\Sigma\Delta\varphi$	$\Sigma\Delta\pi$	$\Sigma\Delta i$	$\Sigma\Delta\Omega$
	″	″	″	″	″	″	″
1873, Oct. 25	+ 0.263	+0.4015	− 0.677	− 1.432	− 0.290	+0.017	− 0.700
Dec. 4	+ 0.664	+0.5935	− 0.304	− 2.084	+ 0.143	+0.016	− 0.691
1874, Jan. 13	+ 1.257	+0.7062	+ 0.286	− 2.531	+ 0.807	+0.009	− 0.695
Feb. 22	+ 1.963	+0.7311	+ 0.850	− 2.744	+ 1.617	+0.003	− 0.750
Apr. 3	+ 2.694	+0.7034	+ 1.300	− 2.822	+ 2.509	0.000	− 0.810
May 13	+ 3.397	+0.6439	+ 1.633	− 2.830	+ 3.499	−0.002	− 0.873
June 22	+ 4.041	+0.5645	+ 1.850	− 2.804	+ 4.587	−0.004	− 0.951
Aug. 1	+ 4.605	+0.4729	+ 1.945	− 2.764	+ 5.753	−0.006	− 1.050
Sept. 10	+ 5.078	+0.3769	+ 1.918	− 2.723	+ 6.952	−0.007	− 1.168
Oct. 20	+ 5.455	+0.2813	+ 1.770	− 2.685	+ 8.154	−0.008	− 1.300
Nov. 29	+ 5.736	+0.1900	+ 1.504	− 2.648	+ 9.325	−0.008	− 1.441
1875, Jan. 8	+ 5.926	+0.1055	+ 1.126	− 2.608	+10.453	−0.007	− 1.583
Feb. 17	+ 6.031	+0.0291	+ 0.642	− 2.560	+11.533	−0.006	− 1.720
Mar. 29	+ 6.060	−0.0383	+ 0.058	− 2.500	+12.571	−0.005	− 1.845
May 8	+ 6.022	−0.0966	− 0.618	− 2.426	+13.582	−0.003	− 1.951
June 17	+ 5.925	−0.1462	− 1.378	− 2.341	+14.586	−0.001	− 2.033
July 27	+ 5.779	−0.1878	− 2.209	− 2.251	+15.600	0.000	− 2.086
Sept. 5	+ 5.591	−0.2221	− 3.089	− 2.165	+16.626	+0.001	− 2.108
Oct. 15	+ 5.369	−0.2489	− 3.996	− 2.087	+17.652	+0.001	− 2.101
Nov. 24	+ 5.120	−0.2673	− 4.904	− 2.016	+18.655	0.000	− 2.071
1876, Jan. 3	+ 4.853	−0.2760	− 5.793	− 1.942	+19.618	−0.002	− 2.028
Feb. 12	+ 4.577	−0.2738	− 6.653	− 1.852	+20.539	−0.005	− 1.977
Mar. 23	+ 4.303	−0.2600	− 7.477	− 1.738	+21.431	−0.008	− 1.924
May 2	+ 4.043	−0.2346	− 8.259	− 1.593	+22.314	−0.012	− 1.872
June 11	+ 3.808	−0.1975	− 8.994	− 1.417	+23.208	−0.017	− 1.823
July 21	+ 3.611	−0.1490	− 9.674	− 1.209	+24.134	−0.022	− 1.780
Aug. 30	+ 3.462	−0.0897	−10.292	− 0.974	+25.110	−0.028	− 1.745
Oct. 9	+ 3.372	−0.0206	−10.836	− 0.716	+26.142	−0.034	− 1.721
Nov. 18	+ 3.351	+0.0575	−11.305	− 0.443	+27.246	−0.041	− 1.712
Dec. 28	+ 3.408	+0.1439	−11.700	− 0.162	+28.434	−0.048	− 1.721
1877, Feb. 6	+ 3.552	+0.2388	−12.024	+ 0.121	+29.726	−0.055	− 1.749
Mar. 18	+ 3.791	+0.3422	−12.274	+ 0.399	+31.131	−0.063	− 1.798
Apr. 27	+ 4.133	+0.4531	−12.448	+ 0.663	+32.646	−0.070	− 1.869
June 6	+ 4.586	+0.5697	−12.546	+ 0.900	+34.251	−0.077	− 1.963
July 16	+ 5.156	+0.6885	−12.571	+ 1.095	+35.895	−0.084	− 2.082

Table II.—*Summation of Perturbations by Saturn and Mars*—Continued.

Date.	$\Sigma^2\Delta\mu$	$\Sigma\Delta u$	$\Sigma\Delta L_1$	$\Sigma\Delta\varphi$	$\Sigma\Delta\pi$	$\Sigma\Delta i$	$\Sigma\Delta\Omega$
	"	"	"	"	"	"	"
1877, Aug. 25	+ 5.844	+0.8036	−12.525	+ 1.239	+37.492	−0.090	− 2.219
Oct. 4	+ 6.648	+0.9078	−12.408	+ 1.336	+38.946	−0.095	− 2.365
Nov. 13	+ 7.556	+0.9923	−12.224	+ 1.400	+40.155	−0.099	− 2.505
Dec. 23	+ 8.548	+1.0480	−11.987	+ 1.458	+41.034	−0.101	− 2.618
1878, Feb. 1	+ 9.596	+1.0645	−11.724	+ 1.548	+41.514	−0.102	− 2.682
Mar. 13	+10.660	+1.0320	−11.480	+ 1.711	+41.554	−0.102	− 2.675
Apr. 22	+11.692	+0.9442	−11.318	+ 1.987	+41.168	−0.102	− 2.581
June 1	+12.636	+0.8065	−11.305	+ 2.390	+40.453	−0.104	− 2.400
July 11	+13.442	+0.6410	−11.505	+ 2.880	+39.547	−0.110	− 2.157
Aug. 20	+14.083	+0.4926	−11.938	+ 3.342	+38.569	−0.121	− 1.912
Sept. 29	+14.576	+0.4124	−12.552	+ 3.633	+37.508	−0.137	− 1.746
Nov. 8	+14.988	+0.4283	−13.231	+ 3.681	+36.233	−0.156	− 1.729
Dec. 18	+15.416	+0.5295	−13.833	+ 3.527	+34.683	−0.175	− 1.872
1879, Jan. 27	+15.945	+0.6773	−14.247	+ 3.275	+32.993	−0.191	− 2.132
Mar. 8	+16.622	+0.8320	−14.431	+ 3.014	+31.372	−0.203	− 2.442
Apr. 17	+17.454	+0.9651	−14.394	+ 2.782	+30.038	−0.210	− 2.733
May 27	+18.419	+1.0637	−14.172	+ 2.575	+29.112	−0.214	− 2.955
July 6	+19.483	+1.1246	−13.807	+ 2.369	+28.627	−0.215	− 3.074
Aug. 15	+20.608	+1.1508	−13.342	+ 2.138	+28.538	−0.215	− 3.075
Sept. 24	+21.759	+1.1484	−12.822	+ 1.867	+28.753	−0.215	− 2.963
Nov. 3	+22.907	+1.1229	−12.283	+ 1.548	+29.189	−0.216	− 2.746
Dec. 13	+24.030	+1.0791	−11.755	+ 1.182	+29.771	−0.218	− 2.443
1880, Jan. 22	+25.109	+1.0205	−11.260	+ 0.769	+30.448	−0.222	− 2.064
Mar. 2	+26.129	+0.9497	−10.814	+ 0.310	+31.180	−0.228	− 1.617
Apr. 11	+27.079	+0.8685	−10.439	− 0.191	+31.945	−0.237	− 1.117
May 21	+27.948	+0.7791	−10.152	− 0.728	+32.720	−0.249	− 0.577
June 30	+28.727	+0.6846	− 9.965	− 1.292	+33.475	−0.264	− 0.016
Aug. 9	+29.412	+0.5875	− 9.884	− 1.873	+34.191	−0.282	+ 0.556
Sept. 18	+30.000	+0.4901	− 9.907	− 2.464	+34.849	−0.303	+ 1.121
Oct. 28	+30.490	+0.3939	−10.037	− 3.057	+35.447	−0.327	+ 1.670
Dec. 7	+30.884	+0.3003	−10.273	− 3.648	+35.987	−0.353	+ 2.192
1881, Jan. 16	+31.184	+0.2101	−10.617	− 4.235	+36.476	−0.382	+ 2.678
Feb. 25	+31.394	+0.1237	−11.070	− 4.821	+36.922	−0.413	+ 3.120
Apr. 6	+31.518	+0.0409	−11.626	− 5.412	+37.323	−0.447	+ 3.507
May 16	+31.559	−0.0386	−12.272	− 6.014	+37.664	−0.483	+ 3.831

TABLE II.—*Summation of Perturbations by Saturn and Mars*—Continued.

Date.	$\Sigma^2\Delta\mu$	$\Sigma\Delta\mu$	$\Sigma\Delta L_{,,}$	$\Sigma\Delta\varphi$	$\Sigma\Delta\pi$	$\Sigma\Delta i$	$\Sigma\Delta\Omega$
	″	″	″	″	″	″	″
1881, June 25	+31.520						
		−0.1148	−12.988	− 6.627	+37.922	−0.521	+ 4.079
Aug. 4	+31.405						
		−0.1866	−13.751	− 7.240	+38.077	−0.560	+ 4.242
Sept. 13	+31.218						
		−0.2524	−14.540	− 7.839	+38.130	−0.599	+ 4.312
Oct. 23	+30.966						
		−0.3102	−15.344	− 8.411	+38.101	−0.638	+ 4.288
Dec. 2	+30.656						
		−0.3579	−16.154	− 8.944	+38.020	−0.676	+ 4.168
1882, Jan. 11	+30.298						
		−0.3937	−16.956	− 9.430	+37.923	−0.712	+ 3.954
Feb. 20	+29.904						
		−0.4158	−17.744	− 9.868	+37.849	−0.746	+ 3.650
Apr. 1	+29.488						
		−0.4224	−18.507	−10.259	+37.836	−0.778	+ 3.261
May 11	+29.066						
		−0.4119	−19.230	−10.605	+37.920	−0.807	+ 2.794
June 20	+28.654						
		−0.3828	−19.895	−10.913	+38.137	−0.833	+ 2.262
July 30	+28.271						
		−0.3334	−20.481	−11.190	+38.520	−0.855	+ 1.676
Sept. 8	+27.938						
		−0.2627	−20.962	−11.445	+39.097	−0.873	+ 1.052
Oct. 18	+27.675						
		−0.1718	−21.305	−11.686	+39.877	−0.886	+ 0.403
Nov. 27	+27.503						
		−0.0675	−21.481	−11.914	+40.811	−0.894	− 0.245
1883, Jan. 6	+27.436						
		+0.0377	−21.502	−12.131	+41 778	−0.898	− 0.835
Feb. 15	+27.474						
		+0.1413	−21.420	−12.351	+42.705	−0.897	− 1.307
Mar. 27	+27.615						
		+0.2410	−21.243	−12.587	+43.583	−0.894	− 1.640
May 6	+27.856						
		+0.3202	−20.982	−12.786	+44.356	−0.890	− 1.832
June 15	+28.176						
		+0.3517	−20.660	−12.862	+45.008	−0.888	− 1.889
July 25	+28.528						
		+0.3125	−20.336	−12.738	+45.630	−0.891	− 1.853
Sept. 3	+28.840						
		+0.1927	−20.100	−12.381	+46.368	−0.902	- 1.818
Oct. 13	+29.033						
		+0.0052	−20.060	−11.825	+47.313	−0.923	− 1.926
Nov. 22	+29.038						
		−0.2208	−20.319	−11.141	+48.392	−0.953	− 2.342
1884, Jan. 1	+28.817						
		−0.4431	−20.945	−10.405	+49.364	−0.990	− 3.211
Feb. 10	+28.374						
		−0.6224	−21.941	− 9.666	+49.914	−1.029	− 4.615
Mar. 21	+27.752						
		−0.7312	−23.249	− 8.928	+49.781	−1.065	− 6.556
Apr. 30	+27.021						
		−0.7565	−24.759	− 8.169	+48.837	−1.094	− 8.947
June 9	+26.265						
		−0.6995	−26.342	− 7.357	+47.102	−1.112	−11.655
July 19	+25.565						
		−0.5723	−27.862	− 6.474	+44.732	−1.118	−14.506
Aug. 28	+24.993						
		−0.3920	−29.220	− 5.512	+41.933	−1.112	−17.353
Oct. 7	+24.601						
		−0.1779	−30.345	− 4.485	+38.937	−1.096	−20.061
Nov. 16	+24.423						
		+0.0523	−31.202	− 3.418	+35.948	−1.072	−22.539
Dec. 26	+24.475						
		+0.2840	−31.775	− 2.348	+33.117	−1.043	−24.729
1885, Feb. 4	+24.759						
		+0.5062	−32.064	− 1.313	+30.534	−1.011	−26.602
Mar. 16	+25.265						
		+0.7123	−32.083	− 0.342	+28.226	−0.978	−28.162

TABLE II.—*Summation of Perturbations by Saturn and Mars*—Continued.

Date.	$\Sigma^2\Delta\mu$	$\Delta\Sigma\mu$	$\Sigma\Delta L_1$	$\Sigma\Delta\varphi$	$\Sigma\Delta\varpi$	$\Sigma\Delta i$	$\Sigma\Delta\Omega$
	″	″	″	″	″	″	″
1885, Apr. 25	+25.977						
		+0.8988	−31.852	+ 0.547	+26.178	−0.946	−29.431
June 4	+26.876						
		+1.0639	−31.400	+ 1.347	+24.356	−0.916	−30.439
July 14	+27.940						
		+1.2069	−30.764	+ 2.057	+22.729	−0.889	−31.220
Aug. 23	+29.147						
		+1.3278	−29.982	+ 2.680	+21.266	−0.865	−31.807
Oct. 2	+30.475						
		+1.4267	−29.091	+ 3.217	+19.940	−0.845	−32.232
Nov. 11	+31.902						
		+1.5038	−28.123	+ 3.669	+18.722	−0.828	−32.526
Dec. 21	+33.406						
		+1.5596	−27.106	+ 4.039	+17.581	−0.815	−32.715
1886, Jan. 30	+34.966						
		+1.5949	−26.066	+ 4.332	+16.489	−0.806	−32.825
Mar. 11	+36.561						
		+1.6100	−25.031	+ 4.548	+15.422	−0.801	−32.877
Apr. 20	+38.171						
		+1.6062	−24.027	+ 4.697	+14.362	−0.799	−32.893
May 30	+39.777						
		+1.5851	−23.081	+ 4.785	+13.295	−0.800	−32.890
July 9	+41.362						
		+1.5492	−22.210	+ 4.824	+12.216	−0.803	−32.882
Aug. 18	+42.911						
		+1.5002	−21.432	+ 4.823	+11.117	−0.808	−32.882
Sept. 27	+44.411						
		+1.4398	−20.762	+ 4.786	+ 9.993	−0.815	−32.899
Nov. 6	+45.851						
		+1.3689	−20.218	+ 4.715	+ 8.834	−0.824	−32.943
Dec. 16	+47.220						
		+1.2882	−19.808	+ 4.611	+ 7.618	−0.834	−33.022
1887, Jan. 25	+48.508						
		+1.1976	−19.535	+ 4.476	+ 6.316	−0.845	−33.142
Mar. 6	+49.706						
		+1.0975	−19.393	+ 4.320	+ 4.899	−0.856	−33.302
Apr. 15	+50.803						
		+0.9890	−19.374	+ 4.160	+ 3.359	−0.866	−33.498
May 25	+51.792						
		+0.8751	−19.464	+ 4.014	+ 1.727	−0.875	−33.723
July 4	+52.667						
		+0.7596	−19.655	+ 3.895	+ 0.049	−0.883	−33.970
Aug. 13	+53.427						
		+0.6480	−19.937	+ 3.810	− 1.605	−0.889	−34.227
Sept. 22	+54.075						
		+0.5462	−20.295	+ 3.757	− 3.166	−0.893	−34.485
Nov. 1	+54.621						
		+0.4614	−20.715	+ 3.721	− 4.565	−0.895	−34.733
Dec. 11	+55.082						
		+0.4018	−21.172	+ 3.682	− 5.744	−0.895	−34.961
1888, Jan. 20	+55.484						
		+0.3751	−21.637	+ 3.609	− 6.675	−0.894	−35.157
Feb. 29	+55.859						
		+0.3874	−22.078	+ 3.471	− 7.377	−0.892	−35.312
Apr. 9	+56.246						
		+0.4396	−22.458	+ 3.250	− 7.914	−0.889	−35.421
May 19	+56.686						
		+0.5257	−22.743	+ 2.952	− 8.376	−0.886	−35.485
June 28	+57.212						
		+0.6315	−22.906	+ 2.611	− 8.822	−0.883	−35.511
Aug. 7	+57.843						
		+0.7373	−22.927	+ 2.278	− 9.229	−0.881	−35.510
Sept. 16	+58.580						
		+0.8237	−22.799	+ 1.993	− 9.489	−0.879	−35.495
Oct. 26	+59.404						
		+0.8758	−22.530	+ 1.772	− 9.458	−0.878	−35.480
Dec. 5	+60.280						
		+0.8849	−22.145	+ 1.604	− 9.010	−0.878	−35.475
	+61.165						

TABLE III.—*Summation of Perturbations by Uranus.*

Date.	$\Sigma^2 \Delta \mu$	$\Sigma \Delta \Sigma \mu$	$\Sigma \Delta L_1$	$\Sigma \Delta \varphi$	$\Sigma \Delta \pi$
	″	″	″	″	″
1854, Nov. 9.7	—.2361				
		—.0072	+0.494	—0.426	+1.505
Dec. 28.9	.2433				
		.0072	0.478	0.419	1.483
1855, Feb. 16.1	.2505				
		—.0052	+0.462	—0.414	+1.448
Apr. 6.3	—.2557				
1855, Dec. 8.4	—.2275				
		+.0149	+0.461	—0.389	+1.205
1856, Jan. 26.7	.2126				
		.0163	0.475	0.389	1.183
Mar. 15.9	.1963				
		+.0169	+0.490	—0.391	+1.167
May 4.1	—.1794				
1857, Feb. 23.4	—.1025				
		+.0045	+0.558	—0.465	+1.098
Apr. 13.7	.0980				
		+.0015	0.556	0.480	1.084
June 1.9	.0965				
		—.0015	+0.551	—0.495	+1.068
July 21.1	—.0980				
1858, Mar. 24.2	—.1399				
		—.0115	+0.468	—0.565	+0.986
May 12.4	.1514				
		.0107	0.452	0.573	0.986
June 30.7	.1621				
		—.0090	+0.438	—0.581	+0.993
Aug. 18.9	—.1711				
1859, Sept. 16.7	—.1677				
		—.0076	+0.419	—0.583	+1.118
Nov. 4.9	.1753				
		.0109	0.404	0.572	1.124
Dec. 24.1	.1862				
		—.0124	+0.385	—0.562	+1.117
1860, Feb. 11.3	—.1986				
1860, Dec. 2.7	—.2221				
		+.0098	+0.314	—0.477	+0.842
1861, Jan. 20.9	.2123				
		.0130	0.324	0.467	0.808
Mar. 11.1	.1993				
		+.0156	+0.337	—0.459	+0.779
Apr. 29.3	—.1837				
1862, Feb. 18.7	—.0798				
		+.0124	+0.444	—0.477	+0.629
Apr. 8.9	.0674				
		.0096	0.451	0.488	0.603
May 28.1	.0578				
		+.0065	+0.455	—0.499	+0.575
July 16.3	—.0513				

Table III.—*Summation of Perturbations by Uranus*—Continued.

Date.	$\Sigma^2\Delta u$	$\Sigma\Delta\mu$	$\Sigma\Delta L_1$	$\Sigma\Delta\varphi$	$\Sigma\Delta\pi$
	″	″	″	″	″
1864, Oct. 30.1	−.0957				
		−.0128	+0.343	−0.573	+0.528
Dec. 18.3	.1085				
		.0164	0.320	0.560	0.547
1865, Feb. 5.6	.1249				
		−.0183	+0.292	−0.546	+0.553
Mar. 26.8	−.1432				
1868, May 1.9	−.0145				
		−.0051	+0.280	−0.355	−0.086
June 20.1	.0196				
		.0055	0.267	0.360	0.106
Aug. 8.3	.0251				
		−.0044	+0.255	−0.366	−0.117
Sept. 26.6	−.0295				
1869, Oct. 25.3	−.0043				
		−.0106	+0.243	−0.379	+0.054
Dec. 13.6	.0149				
		.0161	0.222	0.370	0.104
1870, Jan. 31.8	.0310				
		−.0206	+0.193	−0.360	+0.147
Mar. 22.0	−.0516				
1870, Nov. 23.1	−.1690				
		−.0161	−0.033	−0.237	+0.227
1871, Jan. 11.3	.1851				
		.0113	0.058	0.208	0.226
Mar. 1.6	.1964				
		−.0061	−0.077	−0.178	+0.226
Apr. 19.8	−.2025				
1872, Feb. 9.1	−.1410				
		+.0209	−0.039	−0.024	+0.261
Mar. 29.3	.1201				
		.0216	0.022	0.015	0.258
May 17.6	.0985				
		+.0212	−0.006	−0.009	+0.248
July 5.8	−.0773				
1873, Apr. 27.1	.0015				
		+.0015	+0.013	+0.001	+0.029
June 15.3	0				
		0	+0.002	0	+0.003
Aug. 3.6	0				
		+.0003	−0.009	−0.003	−0.015
Sept. 21.8	+.0003				
1874, Oct. 20.6	−.0702				
		−.0022	−0.002	−0.040	+0.128
Dec. 8.8	.0680				
		.0088	0.015	0.040	0.199
1875, Jan. 27.0	.0592				
		−.0153	−0.038	−0.040	+0.273
Mar. 17.2	−.0439				

TABLE III.—*Summation of Perturbations by Uranus*—Continued.

Date.	$\Sigma^2\Delta\mu$	$\Sigma\Delta\mu$	$\Sigma\Delta L_1$	$\Sigma\Delta\varphi$	$\Sigma\Delta\pi$
	″	″	″	″	″
1876, Jan. 6.6	—.1263				
		—.0277	—0.361	+0.031	+0.660
Feb. 24.8	.1540				
		.0238	0.406	0.055	0.702
Apr. 14.0	.1778				
		—.0189	—0.444	+0.081	+0.744
June 2.2	—.1967				
1877, Mar. 24.6	—.1894				
		+.0183	—0.514	+0.244	+1.049
May 12.8	.1711				
		.0209	0.491	0.256	1.075
July 1.0	.1502				
		+.0221	—0.477	+0.265	+1.090
Aug. 19.2	—.1281				
1878, July 29.8	—.0322				
		+.0038	—0.470	+0.301	+0.920
Sept. 17.0	.0284				
		.0054	0.480	0.297	0.902
Nov. 5.2	.0230				
		+.0087	—0.487	+0.289	+0.881
Dec. 24.4	—.0143				
1879, Dec. 4.0	+.0952				
		+.0053	—0.443	+0.232	+1.006
1880, Jan. 22.2	.1005				
		—.0009	0.450	0.220	1.076
Mar. 11.4	.0996				
		—.0075	—0.465	+0.207	+1.150
Apr. 29.7	+.0921				
1883, Sept. 12.2	—.0715				
		+.0051	—1.033	+0.317	+2.256
Oct. 31.4	.0664				
		.0078	1.042	0.313	2.230
Dec. 19.7	.0586				
		+.0115	—1.047	+0.307	+2.199
1884, Feb. 6.9	—.0471				
1884, Nov. 28.2	+.0714				
		+.0173	—0.982	+0.255	+2.175
1885, Jan. 16.4	.0887				
		.0134	0.973	0.239	2.213
Mar. 6.7	.1021				
		+.0084	—0.969	+0.220	+2.258
Apr. 24.9	+.1105				
1886, Feb. 14.2	+.0393				
		—.0289	—1.148	+0.045	+2.597
Apr. 4.4	+.0104				
		.0311	1.201	0.026	2.644
May 23.7	—.0207				
		—.0318	—1.255	+0.011	+2.693
July 11.9	—.0525				

TABLE III.—*Summation of Perturbations by Uranus*—Continued.

Date.	$\Sigma^2\Delta u$	$\Sigma\Delta\mu$	$\Sigma\Delta L_1$	$\Sigma\Delta\varphi$	$\Sigma\Delta\pi$
	″	″	″	″	″
1887, Mar. 15.0	—.1686				
		—.0065	—1.491	—0.011	+3.086
May 3.2	.1751				
		+.0001	1.502	0.010	3.157
June 21.4	.1750				
		+.0061	—1.505	—0.010	+3.221
Aug. 9.7	—.1689				
1888, Sept. 6.4	—.0926				
		+.0016	—1.505	+0.033	+3.271
Oct. 25.7	.0910				
		.0034	1.516	0.033	3.243
Dec. 13.9	—.0876				
		+.0064	—1.524	+0.032	+3.209

TABLE IV.—*Summation of Perturbations by the Earth.*

Date.	$\Sigma^2\Delta\mu$	$\Sigma\Delta\mu$	$\Sigma\Delta L_1$	$\Sigma\Delta\varphi$	$\Sigma\Delta\pi$
	″	″	″	″	″
1854, Oct. 16.1	—144.9049				
		+.0314	—31.532	—0.354	+10.631
Nov. 9.7	144.8735				
		.1017	30.293	1.051	11.660
Dec. 4.3	144.7718				
		.1967	29.274	1.774	11.769
Dec. 28.9	144.5751				
		.3172	28.532	2.453	10.845
1855, Jan. 22.5	144.2579				
		.4535	28.082	3.002	9.035
Feb. 16.1	143.8044				
		+.5884	—27.938	—3.344	+ 6.690
Mar. 12.7	—143.2160				
1856, Jan. 2.1	—135.4266				
		+.3026	—29.779	—4.641	+10.436
Jan. 26.7	135.1240				
		.3273	28.529	4.958	9.237
Feb. 20.3	134.7967				
		+.3801	—27.387	—4.969	+ 7.530
Mar. 15.9	—134.4166				
1857, Feb. 23.4	—125.5890				
		+.3474	—28.493	—5.405	+ 2.726
Mar. 20.1	125.2416				
		.3432	27.222	5.218	1.394
Apr. 13.7	124.8984				
		.3674	26.079	4.731	0.535
May 8.3	124.5310				
		+.4168	—25.193	—4.035	+ 0.326
June 1.9	—124.1142				
1858, May 12.4	—115.5152				
		+.1964	—26.275	—2.782	— 4.030
June 6.1	115.3189				
		.1746	25.193	2.230	3.774
June 30.7	115.1443				
		+.1896	—24.257	—1.778	— 2.723
July 25.3	—114.9547				

TABLE IV.—*Summation of Perturbations by the Earth*—Continued.

Date.	$\Sigma^2\Delta\mu$	$\Sigma\Delta\mu$	$\Sigma\Delta L_1$	$\Sigma\Delta\varphi$	$\Sigma\Delta\pi$
	"	"	"	"	"
1859, Nov. 4.9	−104.9259				
		+.0727	−22.711	−1.150	+12.552
Nov. 29.5	104.8532				
		.1430	21.550	1.839	12.719
Dec. 24.1	104.7102				
		+.2391	−20.557	−2.493	+11.993
1860, Jan. 17.7	−104.4711				
1860, Dec. 27.3	− 95.9115				
		+.3018	−22.022	−4.515	+ 9.859
1861, Jan. 20.9	95.6097				
		.3042	20.774	4.922	8.890
Feb. 14.5	95.3055				
		+.3373	−19.535	−5.020	+ 7.379
Mar. 11.1	− 94.9682				
1862, Mar. 15.3	− 86.0807				
		+.3129	−19.447	−5.053	+ 0.924
Apr. 8.9	85.7678				
		.3180	18.233	4.664	− 0.062
May 3.5	85.4498				
		.3514	17.211	4.051	− 0.376
May 28.1	85.0984				
		.4096	16.484	3.320	+ 0.054
June 21.7	84.6888				
		+.4861	−16.107	−2.586	+ 1.175
July 16.3	− 84.2027				
1864, Nov. 23.7	− 65.9388				
		+.1337	−14.032	−2.136	+13.815
Dec. 18.3	65.8051				
		.2015	12.902	2.749	13.316
1865, Jan. 11.9	65.6036				
		+.2957	−11.945	−3.245	+11.941
Feb. 5.6	− 65.3079				
1868, May 26.5	− 37.8756				
		+.1020	− 9.902	−1.373	− 2.817
June 20.1	37.7736				
		.0406	8.731	0.704	2.253
July 14.7	37.7330				
		+.0044	− 7.610	−0.156	− 1.172
Aug. 8.3	− 37.7286				
1869, Oct. 25.3	− 28.1882				
		+.1571	− 7.181	−1.928	+12.875
Nov. 18.9	28.0311				
		.1535	6.135	2.458	13.648
Dec. 13.6	27.8776				
		.1897	4.976	3.033	13.470
1870, Jan. 7.2	27.6879				
		+.2582	− 3.867	−3.518	+12.381
Jan. 31.8	− 27.4297				
1871, Jan. 11.3	− 19.0302				
		+.3182	− 5.223	−4.931	+ 8.494
Feb. 4.9	18.1720				
		.3107	3.949	5.240	7.266
Mar. 1.6	18.4013				
		+.3327	− 2.680	−5.195	+ 5.769
Mar. 26.2	− 18.0686				

TABLE IV.—*Summation of Perturbations by the Earth*—**Continued.**

Date.	$\Sigma^2\Delta\mu$	$\Sigma\Delta\mu$	$\Sigma\Delta L_{\prime}$	$\Sigma\Delta\varphi$	$\Sigma\Delta\pi$
	$''$	$''$	$''$	$''$	$''$
1872, Mar. 4.7	— 9.6609				
		+.3157	— 3.855	—4.795	+ 0.840
Mar. 29.3	9.3452				
		.2796	2.654	4.580	— 0.548
Apr. 22.9	9.0656				
		.2731	1.474	4.128	1.259
May 17.6	8.7925				
		+.2976	— 0.458	—3.531	— 1.153
June 11.2	— 8.4949				
1873, June 15.3	- 0.0461				
		+.0484	— 1.003	—0.503	— 1.120
July 9.9	+ 0.0023				
		—.0124	+ 0.281	+0.136	+ 0.335
Aug. 3.6	— 0.0101				
		.0613	1.570	0.674	2.129
Aug. 28.2	0.0714				
		—.0783	+ 2.692	+0.963	+ 4.022
Sept. 21.8	— 0.1497				
1874, Nov. 14.2	+ 7.9188				
		+.1380	+ 2.765	—2.229	+13.450
Dec. 8.8	8.0568				
		.1421	3.882	2.797	13.667
1875, Jan. 2.4	8.1989				
		.1831	5.065	3.292	12.916
Jan. 27.0	8.3820				
		+.2537	+ 6.146	—3.598	+11.346
Feb. 20.6	+ 8.6357				
1876, Jan. 31.2	+16.2268				
		+.2602	+ 4.952	—4.755	+ 7.188
Feb. 24.8	16.4870				
		.2630	6.246	4.821	5.752
Mar. 20.4	16.7500				
		+.2934	+ 7.440	—4.535	+ 4.365
Apr. 14.0	+17.0434				
1877, Feb. 27.9	+24.3388				
		+.2804	+ 5.121	—4.084	+ 1.159
Mar. 24.6	24.6192				
		.2234	6.235	3.937	— 0.474
Apr. 18.2	24.8426				
		.1943	7.421	3.548	1.454
May 12.8	25.0369				
		+.1963	+ 8.522	—3.013	— 1.622
June 6.4	+25.2332				
1878, July 29.8	+32.9713				
		—.0589	+ 9.921	+0.821	+ 4.082
Aug. 23.4	32.9124				
		.0747	11.421	0.997	6.834
Sept. 17.0	32.8377				
		.0819	12.817	1.036	9.446
Oct. 11.6	32.7558				
		—.0514	+13.953	+0.803	+11.488
Nov. 5.2	+32.7044				

TABLE IV.—*Summation of Perturbations by the Earth*—Continued.

Date.			$\Sigma^2 \Delta\mu$	$\Sigma\Delta\mu$	$\Sigma\Delta L_1$	$\Sigma\Delta\varphi$	$\Sigma\Delta\pi$
			″	″	″	″	″
1879,	Dec.	28. 6	+ 40. 5011				
				+. 1650	+14. 908	−3. 331	+15. 403
1880,	Jan.	22. 2	40. 6661				
				. 2119	16. 101	3. 679	14. 139
	Feb.	15. 8	40. 8780				
				+. 2847	+17. 121	−3. 757	+12. 268
	Mar.	11. 4	+ 41. 1627				
1880,	June	17. 9	+ 43. 1908				
				+. 6802	+17. 068	−1. 034	+ 5. 016
	July	12. 5	43. 8710				
				. 7044	16. 152	0. 535	5. 387
	Aug.	6. 1	44. 5754				
				+. 6998	+15. 133	−0. 291	+ 6. 340.
	Aug.	30. 7	+ 45. 2752				
1883,	Oct.	6. 8	+ 64. 2823				
				−. 0022	+22. 679	+0. 167	+13. 611
	Oct.	31. 4	64. 2801				
				+. 0574	23. 921	−0. 410	15. 233
	Nov.	25. 1	64. 3375				
				+. 1499	+24. 890	−1. 100	+15. 843
	Dec.	19. 7	+ 64. 4874				
1884,	Dec.	22. 8	+ 73. 3002				
				+. 2631	+24. 190	−4. 135	+16. 116
1885,	Jan.	16. 4	73. 5633				
				. 2856	25. 424	4. 543	15. 155
	Feb.	10. 1	73. 8489				
				. 3383	26. 588	4. 681	13. 525
	Mar.	6. 7	74. 1872				
				+. 4130	+27. 511	−4. 508	+11. 564
	Mar.	31. 3	+ 74. 6002				
1886,	Feb.	14. 2	+ 82. 7006				
				+. 3299	+25. 415	−5. 277	+ 8. 768
	Mar.	10. 8	83. 0305				
				. 3229	26. 700	5. 209	7. 338
	Apr.	4. 4	83. 3534				
				+. 3442	+27. 890	−4. 812	+ 6. 241
	Apr.	29. 1	+ 83. 6976				
1887,	Apr.	8. 6	+ 92. 1561				
				+. 2533	+26. 599	−3. 612	+ 1. 953
	May	3. 2	92. 4094				
				. 2039	27. 724	3. 130	1. 141
	May	27. 8	92. 6133				
				+. 1852	+28. 817	−2. 590	+ 1. 164
	June	21. 4	+ 92. 7985				
1888,	Sept.	6. 4	+102. 1664				
				−. 0218	+29. 500	+0. 350	+13. 115
	Oct.	1. 1	102. 1446				
				−. 0035	30. 590	−0. 043	14. 698
	Oct.	25. 7	102. 1411				
				+. 0489	31. 827	0. 666	15. 946
	Nov.	19. 3	102. 1900				
				. 1229	33. 011	1. 374	16. 584
	Dec.	13. 9	102. 3129				
				+. 2222	+33. 993	−2. 084	+16. 305
1889,	Jan.	7. 5	+102. 5351				

TABLE V.—*Osculating Elements of Polyhymnia for dates of Ephemerides.*

Elements V.

	°	′	″
L_0 =	318	53	30.00
M =	336	33	22.5
π =	342	20	7.5
φ =	19	52	54.2
☊ =	9	2	44.9
i =	1	56	20.0

$\mu = 732''.81750$

Mean Equinox. 1870.0.

Epoch July 17.0. 1873. Berlin Mean Time.

Form of Computation.

(1) L_0
(2) Pert. by ♃ + ♄ − ♂
(3) " ♀
(4) " ⊕
(5) Prec. to beg. of year.
(6) Nut. and Prec. to date.
(7) Osculating Elements.

Greenwich noon.		L_0	π	☊	i	φ	μ
1854.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	9 23 45.01					
	(2)	+ 3 8 5.84	− 1 27 35.04	+0 20 2.82	+0 0 32.71	− 0 8 55.34	− 0.67937
	(3)	+ 0.26	+ 1.51			− 0.43	− 13
Nov. 8	(4)	− 4.22	+ 4.51			+ 1.69	− 2257
	(5)	− 13 24.12	− 13 24.12	− 12 22.39	− 7.34		
	(6)	+ 28.07	+ 28.07	+ 24.78	+ 0.40		
	(7)	12 18 50.84	340 39 42.43	9 10 50.11	1 56 45.77	19 44 0.12	732.11543
1854.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	9 48 10.65					
	(2)	+ 3 8 5.84	− 1 27 35.04	+0 20 2.82	+0 0 32.71	− 0 8 55.34	− 0.67937
	(3)	+ 0.26	+ 1.51			− 0.43	− 13
Nov. 10	(4)	− 4.15	+ 4.61			+ 1.63	− 2233
	(5)	− 13 24.12	− 13 24.12	− 12 22.39	− 7.34		
	(6)	+ 28.37	+ 28.37	+ 25.06	+ 0.40		
	(7)	12 43 16.85	340 39 42.83	9 10 50.39	1 56 45.77	19 44 0.06	732.11567
1854.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	10 12 36.28					
	(2)	+ 3 8 5.84	− 1 27 35.04	+0 20 2.82	+0 0 32.71	− 0 8 55.34	− 0.67937
	(3)	+ 0.26	+ 1.51			− 0.43	− 13
Nov 12	(4)	− 4.09	+ 4.69			+ 1.57	− 2209
	(5)	− 13 24.12	− 13 24.12	− 12 22.39	+ 7.34		
	(6)	+ 28.68	+ 28.68	+ 25.34	+ 0.41		
	(7)	13 7 42.85	340 39 43.22	9 10 50.67	1 56 45.78	19 44 0.00	732.11591
1854.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	10 37 1.92					
	(2)	+ 3 8 5.84	− 1 27 35.04	+0 20 2.82	+0 0 32.71	− 0 8 55.34	− 0.67937
	(3)	+ 0.26	+ 1.51			− 0.43	− 13
Nov. 14	(4)	− 4.04	+ 4.76			+ 1.51	− 2183
	(5)	− 13 24.12	− 13 24.12	− 12 22.39	− 7.34		
	(6)	+ 29.00	+ 29.00	+ 25.64	+ 0.41		
	(7)	13 32 8.86	340 39 43.61	9 10 50.97	1 56 45.78	19 43 59.94	732.11617
1854.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	12 26 57.28					
	(2)	+ 3 7 58.56	− 1 27 29.32	+0 20 2.30	+0 0 32.69	− 0 8 54.45	− 0.69242
	(3)	+ 0.25	+ 1.50			− 0.43	− 14
Nov. 23	(4)	− 3.79	+ 5.05			+ 1.25	− 2064
	(5)	− 13 24.12	− 13 24.12	− 12 22.39	− 7.34		
	(6)	+ 30.50	+ 30.50	+ 27.05	+ 0.42		
	(7)	15 21 58.68	340 39 51.11	9 10 51.86	1 56 45.77	19 44 0.57	732.10430

TABLE V.—*Osculating Elements of Polyhymnia for dates of Ephemerides*—Continued.

Greenwich noon.		L_0	π	☊	i	φ	μ
1854.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	12 39 10.09					
	(2)	+ 3 7 58.56	− 1 27 29.32	+0 20 2.30	+0 0 32.69	− 0 8 54.45	− 0.69242
	(3)	+ 0.26	+ 1.50			− 0.43	− 14
Nov. 24	(4)	− 3.76	+ 5.07			+ 1.22	+ 2050
	(5)	− 13 24.12	− 13 24.12	− 12 22.39	− 7.34		
	(6)	+ 30.68	+ 30.68	+ 27.22	+ 0.42		
	(7)	15 34 11.71	340 39 51.31	9 10 52.03	1 56 45.77	19 44 0.54	732.10444
1854.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	12 51 22.91					
	(2)	+ 3 7 58.56	− 1 27 29.32	+0 20 2.30	+0 0 32.69	− 0 8 54.45	− 0.69242
	(3)	+ 0.25	+ 1.50			− 0.43	− 14
Nov. 25	(4)	− 3.73	+ 5.09			+ 1.19	− 2035
	(5)	− 13 24.12	− 13 24.12	− 12 22.39	− 7.34		
	(6)	+ 30.85	+ 30.85	+ 27.38	+ 0.42		
	(7)	15 46 24.72	340 39 51.50	9 10 52.19	1 56 45.77	19 44 0.51	732.10459
1854.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	15 54 35.17					
	(2)	+ 3 7 48.68	− 1 27 21.49	+0 20 1.46	+0 0 32.66	− 0 8 53.32	− 0.71030
	(3)	+ 0.25	+ 1.50			− 0.43	− 15
Dec. 10	(4)	− 3.40	+ 5.19			+ 0.74	− 1798
	(5)	− 13 24.12	− 13 24.12	− 12 22.39	− 7.34		
	(6)	+ 33.62	+ 33.62	+ 29.99	+ 0.44		
	(7)	18 49 30.20	340 40 2.20	9 10 53.96	1 56 45.76	19 44 1.19	732.08907
1854.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	16 19 0.81					
	(2)	+ 3 7 48.68	− 1 27 21.49	+0 20 1.46	+0 0 32.66	− 0 8 53.32	− 0.71030
	(3)	+ 0.25	+ 1.50			− 0.43	− 15
Dec. 12	(4)	− 3.34	+ 5.18			+ 0.68	− 1763
	(5)	− 13 24.12	− 13 24.12	− 12 22.39	− 7.34		
	(6)	+ 34.01	+ 34.01	+ 30.36	+ 0.44		
	(7)	19 13 56.29	340 40 2.58	9 10 54.33	1 56 45.76	19 44 1.13	732.08942
1854.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	16 43 26.44					
	(2)	+ 3 7 48.68	− 1 27 21.49	+0 20 1.46	+0 0 32.66	− 0 8 53.32	− 0.71030
	(3)	+ 0.25	+ 1.50			− 0.43	− 15
Dec. 14	(4)	− 3.32	+ 5.14			+ 0.62	− 1728
	(5)	− 13 24.12	− 13 24.12	− 12 22.39	− 7.34		
	(6)	+ 34.40	+ 34.40	+ 30.73	+ 0.44		
	(7)	19 38 22.33	340 40 2.93	9 10 54.70	1 56 45.76	19 44 1.07	732.08977
1854.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	17 7 52.08					
	(2)	+ 3 7 48.68	− 1 27.21.49	+0 20 1.46	+0 0 32.66	+ 0 8 53.32	− 0.71030
	(3)	+ 0.25	+ 1.50			− 0.43	− 15
Dec. 16	(4)	− 3.29	+ 5.13			+ 0.56	− 1692
	(5)	− 13 24.12	− 13 24.12	− 12 22.39	− 7.34		
	(6)	+ 34.78	+ 34.78	+ 31.09	+ 0.45		
	(7)	20 2 48.38	340 40 3.30	9 10 55.06	1 56 45.77	19 44 1.01	732.09013
1855.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	22 0 59.70					
	(2)	+ 3 7 26.66	− 1 27 1.25	+0 19 58.94	+0 0 32.58	− 0 8 50.49	− 0.75702
	(3)	+ 0.23	+ 1.48			− 0.42	− 15
Jan. 9	(4)	− 2.88	+ 4.26			− 0.09	− 1225
	(5)	− 12 33.86	− 12 33.86	− 11 36.01	− 6.88		
	(6)	− 10.84	− 10.84	− 10.94	+ 0.01		
	(7)	24 55 39.01	340 40 27.29	9 10 56.89	1 56 45.71	19 44 3.20	732.04808

TABLE V.—*Osculating Elements of Polyhymnia for dates of Ephemerides*—Continued.

Greenwich noon.		L_0	π	Ω	i	φ	μ
1855.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	22. 25 25.33					
	(2)	+ 3 7 26.66	− 1 27 1.25	+0 19 58.94	+0 0 32.58	− 0 8 50.49	− 0.75702
	(3)	+ 0.23	+ 1.48			− 0.42	− 15
Jan. 11	(4)	− 2.85	+ 4.14			− 0.15	- 1181
	(5)	− 12 33.86	− 12 33.86	− 11 36.01	− 6.88		
	(6)	− 10.48	− 10.48	− 10.60	+ 0.01		
	(7)	25 20 5.03	340 40 27.53	9 10 57.23	1 56 45.71	19 44 3.14	732.04852
1855.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	22 49 50.97					
	(2)	+ 3 7 26.66	− 1 27 1.25	+0 19 58.94	−0 0 32.58	− 0 8 50.49	− 0.75702
	(3)	+ 0.23	+ 1.48.			− 0.42	− 14
Jan. 13	(4)	− 2.82	+ 4.02			− 0.20	− 1136
	(5)	− 12 33.86	− 12 33.86	− 11 36.01	− 6.88		
	(6)	− 10.13.	− 10.13	− 10.27	+ 0.02		
	(7)	25 44 31.05	340 40 27.76	9 10 57.56	1 56 45.72	19 44 0.39	732.04898
1855.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	23 14 16.60					
	(2)	+ 3 7 26.66	− 1 27 1.25	+0 19 58.94	+0 0 32.58	− 0 8 50.49	− 0.75702
	(3)	+ 0.23	+ 1.48			− 0.42	− 14
Jan. 15	(4)	− 2.80	+ 3.87			− 0.24.	− 1091
	(5)	− 12 33.86	− 12 33.86	− 11 36.01	− 6.88		
	(6)	− 9.78	− 9.78	− 9.94	+ 0.02		
	(7)	26 8 57.05	340 40 27.96	9 10 57.89	1 56 45.72	19 44 3.05	732.04943
1856.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	95 30 6.82					
	(2)	+ 3 0 42.35	− 1 23 20.12	+0 19 24.21	+0 0 32.49	− 0 8 45.44	− 1.18161
	(3)	+ 0.24	+ 1.19			− 0.39	+ 31
Jan. 5	(4)	− 4.51	+ 4.04			− 2.11	− 1262
	(5)	− 11 43.61	− 11 43.61	− 10 49.63	− 6.42		
	(6)	− 7.24	− 7.24	− 7.30	+ 0.00		
	(7)	98 18 54.05	340 45 1.76	9 11 12.18	1 56 46.07	19 44 6.26	731.62358
1856.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	95 42 19.63					
	(2)	+ 3 0 42.35	− 1 23 20.12	+0 19 24.21	+0 0 32.49	− 0 8 45.44	− 1.18161
	(3)	+ 0.24	+ 1.19			− 0.39	+ 31
Jan. 6	(4)	− 4.48	+ 4.02			− 2.15	− 1262
	(5)	− 11 43.61	− 11 43.61	− 10 49.63	− 6.42		
	(6)	− 7.05	− 7.05	− 7.12	+ 0.01		
	(7)	98 31 7.08	340 45 1.93	9 11 12.36	1 56 46.08	19 44 6.22	731.62358
1856.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	95 54 32.45					
	(2)	+ 3 0 42.35	− 1 23 20.12	+0 19 24.21	+0 0 32.49	− 0 8 45.44	− 1.18161
	(3)	+ 0.24	+ 1.19			− 0.39	+ 31
Jan. 7	(4)	− 4.44	+ 4.00			− 2.15	− 1262
	(5)	− 11 43.61	− 11 43.61	− 10 49.63	− 6.42		
	(6)	− 6.87	− 6.87	− 6.95	+ 0.01		
	(7)	98 43 20.12	340 45 2.09	9 11 12.53	1 56 46.08	19 44 6.22	731.62358
1856.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	100 35 27.25					
	(2)	+ 3 0 6.28	− 1 23 6.29	+0 19 23.53	+0 0 32.50	− 0 8 46.92	− 1.20355
	(3)	− 0.25	+ 1.19			− 0.39	+ 32
Jan. 30	(4)	− 3.60	+ 3.14			− 2.53	− 1213
	(5)	− 11 43.61	− 11 43.61	− 10 49.63	− 6.42		
	(6)	− 2.83	− 2.83	− 3.15	+ 0.04		
	(7)	103 23 43.74	340 45 19.10	9 11 15.65	1 56 46.12	19 44 4.36	731.60214

TABLE V.—*Osculating Elements of Polyhymnia for dates of Ephemerides*—Continued.

Greenwich noon.		L_0	π	Ω	i	φ	μ
1856.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	100 47 40.07					
	(2)	+ 3 0 6.28	− 1 23 6.29	+0 19 23.53	+0 0 32.50	− 0 8 46.92	− 1.20355
	(3)	+ 0.25	+ 1.19			− 0.39	+ 32
Jan. 31	(4)	− 3.56	+ 3.09			− 2.54	− 1209
	(5)	− 11 43.61	− 11 43.61	− 10 49.63	− 6.42		
	(6)	− 2.67	− 2.67	− 3.00	+ 0.04		
	(7)	103 35 56.76	340 45 19.21	9 11 15.80	1 56 46.12	19 44 4.35	731.60218
1856.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	100 59 52.89					
	(2)	+ 3 0 6.28	− 1 23 6.29	+0 19 23.53	+0 0 32.50	− 0 8 46.92	− 1.20355
	(3)	+ 0.25	+ 1.19			− 0.39	+ 32
Feb. 1	(4)	− 3.51	+ 3.03			− 2.55	− 1205
	(5)	− 11 43.61	− 11 43.61	− 10 49.63	− 6.42		
	(6)	− 2.52	− 2.52	− 2.86	+ 0.04		
	(7)	103 48 9.78	340 45 19.30	9 11 15.94	1 56 46.12	19 44 4.34	731.60222
1857.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	184 27 28.06					
	(2)	+ 2 48 42.27	− 1 20 20.05	+0 19 44.24	+0 0 31.16	− 0 9 40.99	− 1.47288
	(3)	+ 0.46	+ 1.10			− 0.46	+ 9
Mar. 17	(4)	− 3.10	− 4.57			− 3.01	− 1102
	(5)	− 10 53.35	− 10 53.35	− 10 3.20	− 5.96		
	(6)	+ 8.62	+ 8.62	+ 7.82	+ 0.10		
	(7)	187 5 22.96	340 48 59.25	9 12 33.76	1 56 45.30	19 43 9.74	231.33369
1857.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	184 51 53.70					
	(2)	+ 2 48 42.27	− 1 20 20.05	+0 19 44.24	+0 0 31.16	− 0 9 40.99	− 1.47288
	(3)	+ 0.46	+ 1.10			− 0.46	+ 9
Mar. 19	(4)	− 3.02	− 4.57			− 3.01	− 1102
	(5)	− 10 53.35	− 10 53.35	− 10 3.20	− 5.96		
	(6)	+ 8.84	+ 8.84	+ 8.02	+ 0.10		
	(7)	187 29 48.90	340 48 59.47	9 12 33.96	1 56 45.30	19 43 9.74	731.33369
1857.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	185 16 19.33					
	(2)	+ 2 48 42.27	− 1 20 20.05	+0 19 44.24	+0 0 31.16	− 0 9 40.99	− 1.47288
	(3)	+ 0.46	+ 1.10			− 0.46	+ 9
Mar. 21	(4)	− 2.94	− 4.57			− 3.01	− 1102
	(5)	− 10 53.35	− 10 53.35	− 10 3.20	− 5.96		
	(6)	+ 9.06	+ 9.06	+ 8.22	+ 0.10		
	(7)	187 54 14.83	340 48 59.69	9 12 34.16	1 56 45.30	19 43 9.74	731.33369
1857.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	185 40 44.97					
	(2)	+ 2 48 42.27	− 1 20 20.05	+0 19 44.24	+0 0 31.16	− 0 9 40.99	− 1.47288
	(3)	+ 0.46	+ 1.10			− 0.46	+ 9
Mar. 23	(4)	− 2.90	− 4.57			− 3.01	− 1102
	(5)	− 10 53.35	− 10 53.35	− 10 3.20	− 5.96		
	(6)	+ 9.27	+ 9.27	+ 8.41	+ 0.10		
	(7)	188 18 40.72	340 48 59.90	9 12 34.35	1 56 45.30	19 43 9.74	731.33369
'57.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	189 57 14.14					
	(2)	+ 2 47 55.28	− 1 20 15.17	+0 19 45.40	+0 0 30.98	− 0 9 45.45	− 1.48412
	(3)	+ 0.46	+ 1.09			− 0.47	+ 6
Apr. 13	(4)	− 2.07	− 5.64			− 2.69	− 1063
	(5)	− 10 53.35	− 10 53.35	− 10 3.20	− 5.96		
	(6)	+ 11.68	+ 11.68	+ 10.60	+ 0.13		
	(7)	192 34 26.14	340 49 6.11	9 12 37.70	1 56 45.15	19 43 5.59	731.32281

TABLE V.—*Osculating Elements of Polyhymnia for dates of Ephemerides*—Continued.

Greenwich noon.		L_0	π	Ω	i	φ	μ
1857.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	190 9 26.95					
	(2)	+ 2 47 55.28	− 1 20 15.17	+0 19 45.40	+0 0 30.98	− 0 9 45.45	− 1.48412
	(3)	+ 0.46	+ 1.09			− 0.47	+ 6
Apr. 14	(4)	− 2.03	− 5.68			− 2.67	− 1060
	(5)	− 10 53.35	− 10 53.35	− 10 3.20	− 5.96		
	(6)	+ 11.81	+ 11.81	+ 10.72	+ 0.13		
	(7)	192 46 39.12	340 49 6.20	9 12 37.82	1 56 45.15	19 43 5.61	731.32284
1857.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	190 21 39.77					
	(2)	+ 2 47 55.28	− 1 20 15.17	+0 19 45.40	+0 0 30.98	− 0 9 45.45	− 1.48412
	(3)	+ 0.46	+ 1.09			− 0.47	+ 6
Apr. 15	(4)	− 1.99	− 5.72			− 2.65	− 1057
	(5)	− 10 53.35	− 10 53.35	− 10 3.20	− 5.96		
	(6)	+ 11.94	+ 11.94	+ 10.84	+ 0.13		
	(7)	192 58 52.11	340 49 6.29	9 12 37.94	1 56 45.15	19 43 5.63	731.32287
1857.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	190 33 52.59					
	(2)	+ 2 47 55.28	− 1 20 15.17	+0 19 45.40	+0 0 30.98	− 0 9 45.45	− 1.48412
	(3)	+ 0.46	+ 1.09			− 0.47	+ 6
Apr. 16	(4)	− 1.95	− 5.76			− 2.63	− 1054
	(5)	− 10 53 35	− 10 53.35	− 10 3.20	− 5.96		
	(6)	+ 12.06	+ 12.06	+ 10.94	+ 0.14		
	(7)	193 11 5.09	340 49 6.37	9 12 38.04	1 56 45.16	19 43 5.65	731.32289
1858.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	275 14 44.67					
	(2)	+ 2 35 10.89	− 1 20 1.00	+0 19 19.90	+0 0 27.83	− 0 10 51.96	− 1.53759
	(3)	+ 0.30	+ 0.99			− 0.57	− 22
June 6.0	(4)	− 1.97	− 10.57			− 0.17	− 1753
	(5)	− 10 3.10	− 10 3.10	− 9 16.78	− 5.50		
	(6)	+ 25.85	+ 25.85	+ 24.19	+ 0.20		
	(7)	277 40 16.64	340 50 19.67	9 13 12.21	1 56 42.53	19 42 1.50	731.26216
1858.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	275 20 51.08					
	(2)	+ 2 35 10.89	− 1 20 1.00	+0 19 19.90	+0 0 27.83	− 0 10 51.96	− 1.53759
	(3)	+ 0.30	+ 0.99			− 0.57	− 22
June 6.5	(4)	− 1.97	− 10.57			− 0.17	− 1753
	(5)	− 10 3.10	− 10 3.10	− 9 16.78	− 5.50		
	(6)	+ 25.95	+ 25.95	+ 24.29	+ 0.20		
	(7)	277 46 23.15	340 50 19.77	9 13 12.31	1 56 42.53	19 42 1.50	731.26216
1858.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	275 26 57.49					
	(2)	+ 2 35 10.89	− 1 20 1.00	+0 19 19.90	+0 0 27.83	− 0 10 51.96	− 1.53759
	(3)	+ 0.30	+ 0.99			− 0.57	− 22
June 7.0	(4)	− 1.97	− 10.57			− 0.17	− 1753
	(5)	− 10 3.10	− 10 3.10	− 9 16.78	− 5.50		
	(6)	+ 26.05	+ 26.05	+ 24.38	+ 0.20		
	(7)	277 52 29.66	340 50 19.87	9 13 12.40	1 56 42.53	19 42 1.50	731.26216
1859.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	22 31 19.49					
	(2)	+ 2 23 21.06	− 1 18 7.00	+0 18 32.49	+0 0 27.46	− 0 11 43.98	− 1.29400
	(3)	+ 0.23	+ 1.12			− 0.58	− 20
Nov. 15	(4)	− 1.79	+ 5.88			+ 1.23	− 2215
	(5)	− 9 12.83	− 9 12.83	− 8 30.41	− 5.05		
	(6)	+ 54.56	+ 54.56	+ 51.19	+ 0.41		
	(7)	24 46 20.72	340 53 49.23	9 13 38.17	1 56 42.82	19 41 10.87	731.50115

TABLE V.—*Osculating Elements of Polyhymnia for dates of Ephemerides*—Continued.

Greenwich noon.		L_0	π	☊	i	φ	μ
		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
1859.	(1)	23 20 10.76					
	(2)	+ 2 23 21.06	− 1 18 7.00	+0 18 32.49	+0 0 27.46	− 0 11 43.98	− 1.29400
	(3)	+. 0.23	+ 1.12			− 0.58	− 21
Nov. 19	(4)	− 1.69	+ 5.96			+ 1.13	− 2177
	(5)	− 9 12.83	− 9 12.83	− 8 30.41	− 5.05		
	(6)	+ 55.22	+ 55.22	+ 51.81	+ 0.41		
	(7)	25 35 12.75	340 53 49.97	9 13 38.79	1 56 42.82	19 41 10.77	731.50152
		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
1859.	(1)	24 9 2.03					
	(2)	+ 2 23 5.49	− 1 17 51 73	+0 18 29.88	+0 0 27.39	− 0 11 42.53	− 1.32632
	(3)	+ 0.22	+ 1.12			− 0.57	− 21
Nov. 23	(4)	− 1.58	+ 6.03			+ 1.02	− 2135
	(5)	− 9 12.83	− 9 12.83	− 8 30.41	− 5.05		
	(6)	+ 55.89	+ 55.89	+ 52.44	+ 0.42		
	(7)	26 23 49.22	340 54 5.98	9 13 36.81	1 56 42.76	19 41 12.12	731.46962
		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
1859.	(1)	24 57 53.30					
	(2)	+ 2 23 5.49	− 1 17 51.73	+0 18 29.88	+0 0 27.39	− 0 11 42.53	− 1.32632
	(3)	+ 0.22	+ 1.12			− 0.57	− 22
Nov. 27	(4)	− 1.46	+ 6.08			+ 0.90	− 2091
	(5)	− 9 12.83	− 9 12.83	− 8 30.41	− 5.05		
	(6)	+ 56.58	+ 56.58	+ 53.09	+ 0.42		
	(7)	27 12 41.30	340 54 6.72	9 13 37.46	1 56 42.76	19 41 12.00	731.47005
		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
1859.	(1)	25 46 44.57					
	(2)	+ 2 22 59.61	− 1 17 45.70	+0 18 28.79	+0 0 27.36	− 0 11 41.99	− 1.33901
	(3)	+ 0.22	+ 1.12			− 0.57	− 22
Dec. 1	(4)	− 1.36	+ 6.11			+ 0.78	− 2043
	(5)	− 9 12.83	− 9 12.83	− 8 30.41	− 5.05		
	(6)	+ 57.29	+ 57.29	+ 53.75	+ 0.43		
	(7)	28 1 27.50	340 54 13.49	9 13 37.03	1 56 42.74	19 41 12.42	731.45784
		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
1859.	(1)	26 35 35.84					
	(2)	+ 2 22 59.61	− 1 17 45.70	+0 18 28.79	+0 0 27.36	− 0 11 41.99	− 1.33901
	(3)	+ 0.22	+ 1.12			− 0.57	− 22
Dec. 5	(4)	− 1.27	+ 6.11			+ 0.67	− 1992
	(5)	− 9 12.83	− 9 12.83	− 8 30.41	− 5.05		
	(6)	+ 58.04	+ 58.04	+ 54.46	+ 0.43		
	(7)	28 50 19.61	340 54 14.24	9 13 37.74	1 56 42.74	19 41 12.31	731.45835
		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
1859.	(1)	27 24 27.11					
	(2)	+ 2 22 47.89	− 1 17 33.31	+0 18 26.47	+0 0 27.31	− 0 11 40.91	− 1.36512
	(3)	+ 0.22	+ 1.12			− 0.57	− 22
Dec. 9	(4)	− 1.18	+ 6.10			+ 0.55	− 1938
	(5)	− 9 12.83	− 9 12.83	− 8 30.41	− 5.05		
	(6)	+ 58.79	+ 58.79	+ 55.17	+ 0.44		
	(7)	29 39 0.00	340 54 27.37	9 18 36.13	1 56 42.70	19 41 13.27	731.43278
		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
1859.	(1)	28 13 18.38					
	(2)	+ 2 22 47.89	− 1 17 33.31	+0 18 26.47	+0 0 27.31	− 0 11 40.91	− 1.36512
	(3)	+ 0.22	+ 1.12			− 0.57	− 23
Dec. 13	(4)	− 1.09	+ 6.07			+ 0.43	− 1881
	(5)	− 9 12.83	− 9 12.83	− 8 30.41	− 5.05		
	(6)	+ 59.55	+ 59.55	+ 55.89	+ 0.45		
	(7)	30 27 52.12	340 54 28.10	9 13 36.85	1 56 42.71	19 41 13.15	731.43334

TABLE V.—*Osculating Elements of Polyhymnia for dates of Ephemerides*—Continued.

Greenwich noon.		L_0	π	☊	i	φ	μ
1861.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	114 7 27.37					
	(2)	+ 1 55 3.08	− 0 50 45.46	+0 9 21.57	+0 0 32.88	− 0 12 14.61	− 4.03044
	(3)	+ 0.12	+ 0.81			− 0.47	+ 25
Feb. 7.0	(4)	− 1.23	+ 1.98			− 2.62	− 1235
	(5)	− 7 32.32	− 7 32.32	− 6 57.96	− 4.13		
	(6)	+ 22.43	+ 22.43	+ 22.03	+ 0.05		
	(7)	115 55 19.45	341 22 14.94	9 5 30.54	1 56 48.80	19 40 36.50	728.77496
1861.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	114 13 33.77					
	(2)	+ 1 55 3.08	− 0 50 45.46	+0 9 21.57	+0 0 32.88	− 0 12 14.61	− 4.03044
	(3)	+ 0.12	+ 0.81			− 0.47	+ 25
Feb. 7.5	(4)	− 1.23	+ 1.95			− 2.62	− 1233
	(5)	− 7 32.32	− 7 32.32	− 6 57.96	− 4.13		
	(6)	+ 22.50	+ 22.50	+ 22.10	+ 0.05		
	(7)	116 1 25.92	341 22 14.98	9 5 30.61	1 56 48.80	19 40 36.50	728.77498
1861.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	114 19 40.18					
	(2)	+ 1 55 3.08	− 0 50 45.46	+0 9 21.57	+0 0 32.88	− 0 12 14.61	− 4.03044
	(3)	+ 0.12	+ 0.81			− 0.47	+ 25
Feb. 8.0	(4)	− 1.23	+ 1.92			− 2.62	− 1231
	(5)	− 7 32.32	− 7 32.32	− 6 57.96	− 4.13		
	(6)	+ 22.56	+ 22.56	+ 22.15	+ 0.05		
	(7)	116 7 32.39	341 22 15.01	9 5 30.66	1 56 48.80	19 40 36.50	728.77500
1862.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	198 11 40.99					
	(2)	+ 0 52 8.91	− 0 10 44.12	+0 8 56.73	+0 0 24.05	− 0 9 11.53	− 3.64762
	(3)	+ 0.38	+ 0.62			− 0.48	+ 23
Mar. 27	(4)	− 1.00	− 5.66			− 2.71	− 1219
	(5)	− 6 42.07	− 6 42.07	− 6 11.56	− 3.67		
	(6)	+ 28.80	+ 28.80	+ 27.90	+ 0.11		
	(7)	198 57 36.01	342 3 5.07	9 5 57.97	1 56 40.49	19 43 39.48	729.15792
1862.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	198 23 53.81					
	(2)	+ 0 52 8.91	− 0 10 44.12	+0 8 56.73	+0 0 24.05	− 0 9 11.53	− 3.64762
	(3)	+ 0.38	+ 0.62			− 0.48	+ 23
Mar. 28	(4)	− 1.00	− 5.66			− 2.71	− 1219
	(5)	− 6 42.07	− 6 42.07	− 6 11.56	− 3.67		
	(6)	+ 28.90	+ 28.90	+ 27.99	+ 0.11		
	(7)	199 9 48.93	342 3 5.17	9 5 58.06	1 56 40.49	19 43 39.48	729.15792
1862.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	198 36 6.63					
	(2)	+ 0 52 8.91	− 0 10 44.12	+0 8 56.73	+0 0 24.05	− 0 9 11.53	− 3.64762
	(3)	+ 0.38	+ 0.62			− 0.48	+ 23
Mar. 29	(4)	− 1.00	− 5.66			− 2.71	− 1219
	(5)	− 6 42.07	− 6 42.07	− 6 11.56	− 3.67		
	(6)	+ 28.99	+ 28.99	+ 28.07	+ 0.12		
	(7)	199 22 1.84	342 3 5.26	9 5 58.14	1 56 40.50	19 43 39.48	729.15792
1864.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	37 28 49.33					
	(2)	+ 0 17 47.39	+ 0 15 33.58	+0 6 15.17	+0 0 2.52	− 0 5 57.60	− 1.30710
	(3)	+ 0.23	+ 0.54			− 0.57	− 25
Nov. 30	(4)	− 0.14	+ 7.18			+ 0.19	− 1947
	(5)	− 5 1.55	− 5 1.55	− 4 38.61	− 0 2.75		
	(6)	+ 55.91	+ 55.91	+ 52.42	+ 0.42		
	(7)	37 42 31.17	342 31 43.16	9 5 13.88	1 56 20.19	19 46 56.22	731.49068

Table V.—*Osculating Elements of Polyhymnia for dates of Ephemerides*—Continued.

Greenwich noon.		L_0	π	☊	i	φ	μ
1864.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	37 53 14.96					
	(2)	+ 0 17 47.39	+ 0 15 33.58	+0 6 15.17	+0 0 2.52	− 0 5 57.60	− 1.30710
	(3)	+ 0.23	+ 0.54			− 0.57	− 26
Dec. 2	(4)	− 0.14	+ 7.18			+ 0.19	− 1947
	(5)	− 5 1.55	− 5 1.55	− 4 38.61	− 2.75		
	(6)	+ 56.22	+ 56.22	+ 52.71	+ 0.43		
	(7)	38 6 57.11	342 31 43.47	9 5 14.17	1 56 20.20	19 46 56.22	731.49067
1864.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	38 17 40.60					
	(2)	+ 0 17 47.39	+ 0 15 33.58	+0 6 15.17	+0 0 2.52	− 0 5 57.60	− 1.30710
	(3)	+ 0.23	+ 0.54			− 0.57	− 26
Dec. 4	(4)	− 0.14	+ 7.18			+ 0.19	− 1947
	(5)	− 5 1.55	− 5 1.55	− 4 38.61	− 2.75		
	(6)	+ 56.55	+ 56.55	+ 53.02	+ 0.43		
	(7)	38 31 23.08	342 31 43.80	9 5 14.48	1 56 20.20	19 46 56.22	731.49067
1864.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	38 42 6.23					
	(2)	+ 0 17 47.39	+ 0 15 33.58	+0 6 15.17	+0 0 2.52	− 0 5 57.60	− 1.30710
	(3)	+ 0 23	+ 0.54			− 0.57	− 26
Dec. 6	(4)	− 0.14	+ 7.18			+ 0.19	− 1947
	(5)	− 5 1.55	− 5 1.55	− 4 38.61	− 2.75		
	(6)	+ 56.86	+ 56.86	+ 53.31	+ 0.43		
	(7)	38 55 49.02	342 31 44.11	9 5 14.77	1 56 20.20	19 46 56.22	731.49067
1864.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	42 9 44.13					
	(2)	+ 0 17 20.57	+ 0 15 31.27	+0 .6 12.26	+0 0 2.48	− 0 5 55.79	− 1.30876
	(3)	+ 0.23	+ 0.54			− 0.57	− 30
Dec. 23	(4)	+ 0.22	+ 7.04			− 0.12	− 1824
	(5)	− 5 1.55	− 5 1.55	− 4 38.61	− 2.75		
	(6)	+ 59.72	+ 59.72	+ 55.99	+ 0.46		
	(7)	42 23 3.32	342 31 44.52	9 5 14.54	1 56 20.19	19 46 57.72	731.49020
1864.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	42 21 56.95					
	(2)	+ 0 17 20.57	+ 0 15 31.27	+0 6 12.26	+0 0 2.48	− 0 5 55.79	− 1.30876
	(3)	+ 0.23	+ 0.54			− 0.57	− 30
Dec. 24	(4)	+ 0.22	+ 7.04			− 0.12	− 1824
	(5)	− 5 1.55	− 5 1.55	− 4 38.61	− 2.75		
	(6)	+ 59.89	+ 59.89	+ 56.15	+ 0.46		
	(7)	42 35 16.31	342 31 44.69	9 5 14.70	1 56 20.19	19 46 57.72	731.49020
1864.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	42 34 9.76					
	(2)	+ 0 17 20.57	+ 0 15 31.27	+0 6 12.26	+0 0 2.48	− 0 5 55.79	− 1.30876
	(3)	+ 0.23	+ 0.54			− 0.57	− 30
Dec. 25	(4)	+ 0.22	+ 7.04			− 0.12	− 1824
	(5)	− 5 1.55	− 5 1.55	− 4 38.61	− 2.75		
	(6)	+ 1 0.06	+ 1 0.06	+ 56.31	+ 0.46		
	(7)	42 47 29.29	342 31 44.86	9 5 14.86	1 56 20.19	19 46 57.72	731.49020
1868.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	301 5 27.99					
	(2)	− 0 5 15.00	+ 0 20 47.36	+0 4 10.03	+0 0 4.19	− 0 2 54.07	− 0.29308
	(3)	+ 0.25	− 0.10			− 0.36	− 11
June 17	(4)	+ 0.43	− 9.19			+ 1.30	− 2214
	(5)	− 1 40.52	− 1 40.52	− 1 32.88	− 0.92		
	(6)	+ 13.98	+ 13.98	+ 12.23	+ 0.22		
	(7)	300 58 47.13	342 39 19.03	9 5 34.28	1 56 23.49	19 50 1.07	732.50217

TABLE V.—*Osculating Elements of Polyhymnia for dates of Ephemerides*—Continued.

Greenwich noon.		L_0	π	☊	i	φ	μ
1868.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	301 17 40.81					
	(2)	− 0 5 15.00	+ 0 20 47.36	+0 4 10.03	+ 4.19	− 2 54.07	− 0.29308
	(3)	+ 0.25	− 0.10			− 0.36	− 11
June 18	(4)	+ 0.43	− 9.19			+ 1.30	− 2214
	(5)	− 1 40.52	− 1 40.52	− 1 32.88	− 0.92		
	(6)	+ 14.14	+ 14.14	+ 12.38	+ 0.22		
	(7)	301 11 0.11	342 39 19.19	9 5 34.43	1 56 23.49	19 50 1.07	732.50217
1868.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	301 29 53.62					
	(2)	− 0 5 15.00	+ 0 20 47.36	+0 4 10.03	+ 4.19	− 2 54.07	− 0.29308
	(3)	+ 0.25	− 0.10			− 0.36	− 11
June 19	(4)	+ 0.43	− 9.19			+ 1.30	− 2214
	(5)	− 1 40.52	− 1 40.52	− 1 32.88	− 0.92		
	(6)	+ 14.31	+ 14.31	+ 12.54	+ 0.22		
	(7)	301 23 13.09	342 39 19.36	9 5 34.59	1 56 23.49	19 50 1.07	732.50217
1869.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	50 48 36.62					
	(2)	− 0 12 14.40	+ 0 13 39.60	+0 2 3.54	− 1.04	− 1 12.40	− 0.26789
	(3)	+ 0.22	+ 8			− 37	− 26
Dec. 8	(4)	+ 0.57	+ 7.04			− 42	− 1810
	(5)	− 50.26	− 50.26	− 46.45	− 0.46		
	(6)	+ 31.47	+ 31.47	+ 27.92	+ 0.44		
	(7)	50 36 4.22	342 33 35.43	9 4 29.91	1 56 18.94	19 51 41.01	732.53125
1869.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	51 0 49.44					
	(2)	− 0 12 14.40	+ 0 13 39.60	+0 2 3.54	− 1.04	− 1 12.40	− 0.26789
	(3)	+ 0.22	+ 8			− 37	− 26
Dec. 9	(4)	+ 0.57	+ 7.04			− 42	- 1810
	(5)	− 50.26	− 50.26	− 46.45	− 0.46		
	(6)	+ 31.64	+ 31.64	+ 28.08	+ 0.44		
	(7)	50 48 17.21	342 33 35.60	9 4 30.07	1 56 18.94	19 51 41.01	732.53125
1869,		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	51 13 2.26					
	(2)	− 0 12 14.40	+ 0 13 39.60	+0 2 3.54	− 1.04	− 1 12.40	− 0.26789
	(3)	+ 0.22	+ 8			− 37	− 26
Dec. 10	(4)	+ 0.57	+ 7.04			− 42	− 1810
	(5)	− 50.26	− 50.26	− 46.45	− 0.46		
	(6)	+ 31.81	+ 31.81	+ 28.24	+ 0.44		
	(7)	51 0 30.20	342 33 35.77	9 4 30.23	1 56 18.94	19 51 41.01	732.53125
1873.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	318 41 44.45					
	(2)	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	0
	(3)	0	0				0
July 16	(4)	+ 1.27	− 6.60			+ 2.33	− 0.02496
	(5)	+ 2 30.78	+ 2 30.78	+ 2 19.37	+ 1.38		
	(6)	+ 14.89	+ 14.89	+ 12.84	+ 0.25		
	(7)	318 44 31.39	342 22 46.57	9 5 17.11	1 56 21.63	19 52 56.53	732.79254
1873.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	318 53 57.27					
	(2)	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	0
	(3)	0	0			0	0
July 17	(4)	+ 1.27	− 6.60			+ 2.33	− .02496
	(5)	+ 2 30.78	+ 2 30.78	+ 2 19.37	+ 1.38		
	(6)	+ 15.07	+ 15.07	+ 13.01	+ 0.26		
	(7)	318 56 44.39	342 22 46.75	9 5 17.28	1 56 21.64	19 52 56.53	732.79254

Table V.—*Osculating Elements of Polyhymnia for dates of Ephemerides*—Continued.

Greenwich noon.		L_0	π	☊	i	φ	μ
1873.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	319 6 10.09					
	(2)	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	0
	(3)	0	0			0	0
July 18	(4)	+ 1.27	− 6.60			+ 2.33	− 0.02496
	(5)	+ 2 30.78	+ 2 30.78	+ 2 19.37	+ 1.38		
	(6)	+ 15.24	+ 15.24	+ 13.17	+ 0.26		
	(7)	319 8 57.38	342 22 46.92	9 5 17.44	1 56 21.64	19 52 56.53	732.79254
1873.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	320 7 14.17					
	(2)	− 0 0 0.94	− 0 0 3.56	− 0 0 0.04	0 0 0	+ 0 0 0.68	− 0.00840
	(3)	0	0			0	0
July 23	(4)	+ 1.44	− 6.25			+ 2.46	− 2544
	(5)	+ 2 30.78	+ 2 30.78	+ 2 19.37	+ 1.38		
	(6)	+ 16.09	+ 16.09	+ 13.97	+ 0.26		
	(7)	320 10 1.54	342 22 44.56	9 5 18.20	1 56 21.64	19 52 57.34	732.78366
1873.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	320 19 26.99					
	(2)	− 0.94	− 3.56	− 0.04	0	+ 0.68	− 0.00840
	(3)	0	0			0	0
July 24	(4)	+ 1.44	− 6.25			+ 2.46	− 2544
	(5)	+ 2 30.78	+ 2 30.78	+ 2 19.37	+ 1.38		
	(6)	+ 16.25	+ 16.25	+ 14.12	+ 0.26		
	(7)	320 22 14.52	342 22 44.72	9 5 18.35	1 56 21.64	19 52 57.34	732.78366
1873.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	320 31 39.81					
	(2)	− 0.94	− 3.56	− 0.04	0	+ 0.68	− 0.00840
	(3)	0	0			0	0
July 25	(4)	+ 1.44	− 6.25			+ 2.46	− 2544
	(5)	+ 2 30.78	+ 2 30.78	+ 2 19.37	+ 1.38		
	(6)	+ 16.42	+ 16.42	+ 14.28	+ 0.27		
	(7)	320 34 27.51	342 22 44.89	9 5 18.51	1 56 21.65	19 52 57.34	732.78366
1873.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	320 43 52.63					
	(2)	− 0.94	− 3.56	− 0.04	0	+ 0.68	− 0.00840
	(3)	0	0			0	0
July 26	(4)	+ 1.44	− 6.25			+ 2.46	− 2544
	(5)	+ 2 30.78	+ 2 30.78	+ 2 19.37	+ 1.38		
	(6)	+ 16.59	+ 16.59	+ 14.44	+ 0.27		
	(7)	320 46 40.50	342 22 45.06	9 5 18.67	1 56 21.65	19 52 57.34	732.78366
1873.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	321 44 56.71					
	(2)	− 1.62	− 5.43	− 0.05	0	+ 1.07	− 0.01279
	(3)	0	− 1			0	0
July 31	(4)	+ 1.74	− 5.39			+ 2.75	− 2653
	(5)	+ 2 30.78	+ 2 30.78	+ 2 19.37	+ 1.38		
	(6)	+ 17.39	+ 17.39	+ 15.19	+ 0.27		
	(7)	321 47 45.00	342 22 44.84	9 5 19.41	1 56 21.65	19 52 58.02	732.77818
1873.		° ′ ″	° ′ ″	′ ′ ″	° ′ ″	° ′ ″	
	(1)	321 57 9.53					
	(2)	− 1.62	− 5.43	− 0.05	0	+ 1.07	− 0.01279
	(3)	0	− 1			0	0
Aug. 1	(4)	+ 1.74	− 5.39			+ 2.75	− 2653
	(5)	+ 2 30.78	+ 2 30.78	+ 2 19.37	+ 1.38		
	(6)	+ 17.55	+ 17.55	+ 15.34	+ 0.27		
	(7)	321 59 57.98	342 22 45.00	9 5 19.56	1 56 21.65	19 52 58.02	732.77818

TABLE V.—*Osculating Elements of Polyhymnia for dates of Ephemerides*—**Continued.**

Greenwich noon.		L_0	π	☊	i	φ	μ
1873.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	322 9 22.35					
	(2)	— 1.62	— 5.43	— 0.05	0	+ 1.07	— 0.01279
	(3)	0	— 1			0	0
Aug. 2	(4)	+ 1.74	— 5.39			+ 2.75	— 2653
	(5)	+ 2 30.78	+ 2 30.78	+ 2 19.37	+ 1.38		
	(6)	+ 17.70	+ 17.70	+ 15.48	+ 0.27		
	(7)	322 12 10.95	342 22 45.15	9 5 19.70	1 56 21.65	19 52 58.02	732.77818
1875.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ° ″	
	(1)	67 36 1.81					
	(2)	+ 2.00	— 0 3 18.70	—0 0 15.97	—0 0 0.36	— 0 0 7.27	+ 0.26677
	(3)	+ 5	+ 20			— 4	— 18
Jan. 2	(4)	+ 0.62	+ 6.77			— 72	— 1846
	(5)	+ 4 11.30	+ 4 11.30	+ 3 52.29	+ 2.30		
	(6)	— 5.74	— 5.74	— 5.76	0.00		
	(7)	67 40 10.04	342 21 1.33	9 6 15.46	1 56 21.94	19 52 46.17	733.06563
1875.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	68 0 27.45					
	(2)	+ 2.00	— 0 3 18.70	—0 0 15.97	—0 0 0.36	— 0 0 7.27	+ 0.26677
	(3)	+ 5	+ 20			— 4	— 19
Jan. 4	(4)	+ 62	+ 6.77			— 72	— 1846
	(5)	+ 4 11.30	+ 4 11.30	+ 3 52.29	+ 2.30		
	(6)	— 5.35	— 5.35	— 5.39	0.00		
	(7)	68 4 36.07	342 21 1.72	9 6 15.83	1 56 21.94	19 46.17	733.06562
1875.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	68 24 53.08					
	(2)	+ 2.00	— 0 3 18.70	—0 0 15.97	—0 0 0.36	— 0 0 7.27	+ 0.26677
	(3)	+ 5	+ 20			— 4	— 20
Jan. 6	(4)	+ 62	+ 6.77			— 72	— 1846
	(5)	+ 11.30	+ 4 11.30	+ 3 52.29	+ 2.30		
	(6)	— 4.97	— 4.97	— 5.03	+ 0.01		
	(7)	68 29 2.08	342 21 2.10	9 6 16.19	1 56 21.95	19 52 46.17	733.06561
1877.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	237 9 58.79					
	(2)	+ 9 48.76	— 0 9 2.52	—0 0 42.25	—0 0 0.53	+ 0 0 12.32	+ 0.35690
	(3)	— 68	+ 1.05			+ 24	+ 37
Apr. 14	(4)	— 1.28	— 7.62			— 1.43	— 1657
	(5)	+ 5 51.82	+ 5 51.82	+ 5 25.21	+ 3.22		
	(6)	+ 19.57	+ 19.57	+ 18.49	+ 0.13		
	(7)	237 25 56.98	342 17 9.80	9 7 46.35	1 56 22.82	19 53 5.33	733.15820
1877.		° ′ ″	°′ ′ ′	° ′ ″	° ′ ″	° ′ ″	
	(1)	237 22 11.61					
	(2)	+ 9 48.76	— 0 9 2.52	—0 0 42.25	—0 0 0.53	+ 0 0 12.32	+ 0.35690
	(3)	— 68	+ 1.05			+ 24	+ 37
Apr. 15	(4)	— 1.28	— 7.62			— 1.43	— 1657
	(5)	+ 5 51.82	+ 5 51.82	+ 5 25.21	+ 3.22		
	(6)	+ 19.69	+ 19.69	+ 18.60	0.13		
	(7)	237 38 9.92	342 17 9.92	9 7 46.46	1 56 22.82	19 53 5.33	733.15820
1877.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	237 34 24.43					
	(2)	+ 9 48.76	— 0 9 2.52	—0 0 42.25	—0 0 0.53	+ 0 0 12.32	+ 0.35690
	(3)	— 68	+ 1.05			+ 24	+ 37
Apr. 16	(4)	— 1.28	— 7.62			— 1.43	— 1657
	(5)	+ 5 51.82	+ 5 51.82	+ 5 25.21	+ 3.22		
	(6)	+ 19.82	+ 19.82	+ 18.72	+ 0.14		
	(7)	237 50 22.87	342 17 10.05	9 7 46.58	1 56 22.83	19 53 5.33	733.15820

TABLE V.—*Osculating Elements of Polyhymnia for dates of Ephemerides*—Continued.

Greenwich noon.		L_0	π	☊	i	φ	μ
1878.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	339 57 51.63					
	(2)	+ 7 10.87	− 18 22.16	− 3 16.10	−0 0 2.44	− 0 42.26	− 0.22461
	(3)	− 50	+ 92			+ 30	+ 8
Sept. 1	(4)	− 1.27	+ 23			+ 3.32	− 2798
	(5)	+ 6 42.08	+ 6 42.08	+ 6 11.73	+ 0 3.68		
	(6)	+ 47.09	+ 47.09	+ 44.56	+ 0.31		
	(7)	340 12 29.90	342 9 15.66	9 6 25.09	1 56 21.55	19 52 15.56	732.56499
1878.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	340 10 4.45					
	(2)	+ 7 10.87	− 18 22.16	− 3 16.10	− 0 2.44	− 0 42.26	− 0.22461
	(3)	− 50	+ 92			+ 30	+ 8
Sept. 2	(4)	− 1.27	+ 23			+ 3.32	− 2798
	(5)	+ 6 42.08	+ 6 42.08	+ 6 11.73	+ 0 3.68		
	(6)	+ 47.21	+ 47.21	+ 44.66	+ 0.31		
	(7)	340 24 42.84	342 9 15.78	9 6 25.19	1 56 21.55	19 52 15.56	732.56499
1878.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	340 22 17.26					
	(2)	+ 7 10.87	− 0 18 22.16	−0 3 16.10	−0 0 2.44	− 0 0 42.26	− 0.22461
	(3)	− 50	+ 92			+ 30	+ 8
Sept. 3	(4)	− 1.27	+ 23			+ 3.32	− 2798
	(5)	+ 6 42.08	+ 6 42.08	+ 6 11.73	+ 3.68		
	(6)	+ 47.32	+ 47.32	+ 44.76	+ 0.32		
	(7)	340 36 55.76	342 9 15.89	9 6 25.29	1 56 21.59	19 52 15.56	732.56499
1878.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	341 59 59.80					
	(2)	+ 7 7.47	− 18 24.35	− 3 16.77	− 2.41	− 46.20	− 0.19335
	(3)	− 50	+ 91			+ 30	+ 9
Sept. 11	(4)	− 0.89	+ 1.57			+ 3.36	− 2822
	(5)	+ 6 42.08	+ 6 42.08	+ 6 11.73	+ 3.68		
	(6)	+ 48.23	+ 48.23	+ 45.59	+ 0.32		
	(7)	342 14 36.19	342 9 15.94	9 6 25.45	1 56 21.59	19 52 11.66	732.59602
1878.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	342 12 12.62					
	(2)	+ 7 7.47	− 18 24.35	− 3 16.77	− 2.41	− 46.20	− 0.19335
	(3)	− 50	+ 91			+ 30	+ 9
Sept. 12	(4)	− 0.89	+ 1.57			+ 3.36	− 2822
	(5)	+ 6 42.08	+ 6 42.08	+ 6 11.73	+ 3.68		
	(6)	+ 48.33	+ 48.33	+ 45.68	+ 0.32		
	(7)	342 26 49.11	342 9 16.04	9 6 25.54	1 56 21.59	19 52 11.66	732.59602
1878.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	342 24 25.44					
	(2)	+ 7 7.47	− 18 24.35	− 3 16.77	− 2.41	− 46.20	− 0.19335
	(3)	− 50	+ 91			+ 30	+ 9
Sept. 13	(4)	− 0.89	+ 1.57			+ 3.36	− 2822
	(5)	+ 6 42.08	+ 6 42.08	+ 6 11.73	+ 3.68		
	(6)	+ 48.44	+ 48.44	+ 45.78	+ 0.33		
	(7)	342 39 2.04	342 9 16.15	9 6 25.64	1 56 21.60	19 52 11.66	732.59602
1880.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	85 0 5.46					
	(2)	+ 6 46.91	− 15 2.99	− 3 14.08	−0 0 2.36	− 0 1 35.83	− 0.33263
	(3)	− 55	+ 1.06			+ 23	+ 3
Jan. 30	(4)	− 1.58	+ 7.51			− 1.34	− 1630
	(5)	+ 8 22.60	+ 8 22.60	+ 7 44.67	+ 4.63		
	(6)	+ 21.92	+ 21.92	+ 21.61	+ 0.04		
	(7)	85 15 34.76	342 13 57.60	9 7 37.10	1 56 22.31	19 51 17.26	732.46860

Table V.—*Osculating Elements of Polyhymnia for dates of Ephemerides*—Continued.

Greenwich noon.		L_0	π	☊	i	φ	μ
1880.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	85 12 18.28					
	(2)	+ 6 46.91	− 15 2.99	− 3 14.08	− 2.36	− 1 35.83	− 0.33263
	(3)	− 35	+ 1.06			+ 23	+ 3
Jan. 31	(4)	− 1.58	+ 7.51			− 1.34	− 1630
	(5)	+ 8 22.60	+ 8 22.60	+ 7 44.67	+ 4.60		
	(6)	+ 22.07	+ 22.07	+ 21.75	+ 0.04		
	(7)	85 27 47.93	342 13 57.75	9 7 37.24	1 56 22.28	19 51 17.26	732.46860
1880.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	85 24 31.09					
	(2)	+ 6 46.91	− 15 2.99	− 3 14.08	− 2.36	− 1 35.83	− 0.33263
	(3)	− 35	+ 1.06			+ 23	+ 2
Feb. 1	(4)	− 1.58	+ 7.51			− 1.34	− 1630
	(5)	+ 8 22.60	+ 8 22.60	+ 7 44.67	+ 4.60		
	(6)	+ 22.22	+ 22.22	+ 21.89	+ 0.04		
	(7)	85 40 0.89	342 13 57.90	9 7 37.38	1 56 22.28	19 51 17.26	732.46859

Greenwich midnight.		L_0	π	☊	i	φ	μ
1883.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	357 52 27.32					
	(2)	− 17 18.08	− 12 51.77	− 3 21.20	− 9.05	− 5 56.71	− 0.83492
	(3)	− 1.10	+ 2.26			+ 32	+ 10
Oct. 1	(4)	− 5.68	+ 5.93			+ 2.75	− 2602
	(5)	+ 10 53.39	+ 10 53.39	+ 10 4.03	+ 5.97		
	(6)	+ 47.08	+ 47.08	+ 44.24	+ 0.35		
	(7)	357 46 42.93	342 19 4.39	9 10 11.97	1 56 17.27	19 47 0.56	731.95666
1883.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	358 4 40.13					
	(2)	− 17 18.08	− 12 51.77	− 3 21.20	− 9.05	− 5 56.71	− 0.83492
	(3)	− 1.10	+ 2.26			+ 32	+ 10
Oct. 2	(4)	− 5.68	+ 5.93			+ 2.75	− 2602
	(5)	+ 10 53.39	+ 10 53.39	+ 10 4.03	+ 5.97		
	(6)	+ 47.16	+ 47.16	+ 44.31	+ 0.35		
	(7)	357 58 55.82	342 19 4.47	9 10 12.04	1 56 17.27	19 47 0.56	731.95666
1883.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	358 16 52.95					
	(2)	− 17 18.08	− 12 51.77	− 3 21.20	− 9.05	− 5 56.71	− 0.83492
	(3)	− 1.10	+ 2.26			+ 32	+ 10
Oct. 3	(4)	− 5.68	+ 5.93			+ 2.75	− 2602
	(5)	+ 10 53.39	+ 10 53.39	+ 10 4.03	+ 5.97		
	(6)	+ 47.24	+ 47.24	+ 44.37	+ 0.36		
	(7)	358 11 8.72	342 19 4.55	9 10 12.10	1 56 17.28	19 47 0.56	731.95666
1883.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	3 58 51.84					
	(2)	− 17 40.13	− 0 12 50.75	− 3 21.15	− 9.04	− 5 56.71	− 0.81610
	(3)	− 1.10	+ 2.24			+ 32	+ 13
Oct. 31	(4)	− 4.94	+ 7.89			+ 2.75	− 2392
	(5)	+ 10 53.39	+ 10 53.39	+ 10 4.03	+ 0 5.97		
	(6)	+ 49.93	+ 49.93	+ 46.77	+ 0.39		
	(7)	3 52 48.99	342 19 10.20	9 10 14.55	1 56 17.32	19 47 0.56	731.97761

TABLE V.—*Osculating Elements of Polyhymnia for dates of Ephemerides*—Continued.

Greenwich midnight.		L_0	π	☊	i	φ	μ
1883.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	4 11 4.66					
	(2)	− 17 40.13	− 0 12 50.75	−0 3 21.15	−0 0 9.04	− 0 5 59.56	− 0.81610
	(3)	− 1.10	+ 2.24			+ 32	+ 13
Nov. 1	(4)	− 4.94	+ 7.89			+ 2.22	− 2392
	(5)	+ 10 53.39	+ 10 53.39	+ 10 4.03	+ 5.97		
	(6)	+ 50.03	+ 50.03	+ 46.86	+ 0.39		
	(7)	4 5 1.91	342 19 10.30	9 10 14.64	1 56 17.32	19 46 57.18	731.97761
1883.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	4 23 17.48					
	(2)	− 17 40.13	− 0 12 50.75	−0 3 21.15	−0 0 9.04	− 0 5 59.56	0.81610
	(3)	− 1.10	+ 2.24			+ 32	+ 13
Nov. 2	(4)	− 4.94	+ 7.89			+ 2.22	− 2392
	(5)	+ 10 53.39	+ 10 53.39	+ 10 4.03	+ 5.97		
	(6)	+ 50.15	+ 50.15	+ 46.97	+ 0.39		
	(7)	4 17 14.85	342 19 10.42	9 10 14.75	1 56 17.32	19 46 57.18	731.97761
1885.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	96 11 38.17					
	(2)	− 26 22.77	− 0 2 4.95	−0 5 0.86	−0 0 9.01	− 0 7 17.71	− 1.94964
	(3)	− 89	+ 2.20			+ 25	+ 30
Jan. 26	(4)	− 4.75	+ 8.53			− 2.20	− 1329
	(5)	+ 12 33.91	+ 12 33.91	+ 11 37.03	+ 6.89		
	(6)	+ 7.33	+ 7.33	+ 7.06	+ 0.04		
	(7)	95 57 51.00	342 30 54.52	9 9 28.13	1 56 17.92	19 45 34.54	730.85487
1885.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	96 36 3.80					
	(2)	− 26 22.77	− 2 4.95	−0 5 0.86	−0 0 9.01	− 0 7 17.71	− 1.94964
	(3)	− 89	+ 2.20			+ 25	+ 30
Jan. 28	(4)	− 4.75	+ 8.53			− 2.20	− 1329
	(5)	+ 12 33.91	+ 12 33.91	+ 11 37.03	+ 6.89		
	(6)	+ 7.58	+ 7.58	+ 7.29	+ 0.04		
	(7)	96 22 16.88	342 30 54.77	9 9 28.36	1 56 17.92	19 45 34.54	730.85487
1885.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	97 0 29.44					
	(2)	− 26 22.77	− 0 2 4.95	−0 5 0.86	−0 0 9.01	− 0 7 17.71	− 1.94964
	(3)	− 88	+ 2.20			+ 24	+ 29
Jan. 30	(4)	− 4.75	+ 8.53			− 2.20	− 1329
	(5)	+ 12 33.91	+ 12 33.91	+ 11 37.03	+ 6.89		
	(6)	+ 7.84	+ 7.84	+ 7.53	+ 0.04		
	(7)	96 46 42.79	342 30 55.03	9 9 28.60	1 56 17.92	19 45 34.53	730.85486
1885.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	97 24 55.07					
	(2)	− 26 22.77	− 0 2 4.95	−0 5 0.86	−0 0 9.01	− 0 7 17.71	− 1.94964
	(3)	− 88	+ 2.20			+ 24	+ 29
Feb. 1	(4)	− 4.75	+ 8.53			− 2.20	− 1329
	(5)	+ 12 33.91	+ 12 33.91	+ 11 37.03	+ 6.89		
	(6)	+ 8.10	+ 8.10	+ 7.77	+ 0.04		
	(7)	97 11 8.68	342 30 55.29	9 9 28.84	1 56 17.92	19 45 34.53	730.85486
1886.		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
	(1)	178 1 30.80					
	(2)	− 1 1 14.69	+ 0 20 29.50	−0 1 13.60	−0 0 23.58	− 0 17 6.38	− 4.74249
	(3)	− 1.11	+ 2.59			+ 5	− 58
Mar. 4	(4)	− 4.97	+ 1.44			− 2.94	− 1178
	(5)	+ 13 24.17	+ 13 24.17	+ 12 23.13	+ 7.35		
	(6)	+ 5.32	+ 5.32	+ 4.67	+ 0.08		
	(7)	177 13 39.52	342 54 10.52	9 13 59.10	1 56 3.85	19 35 44.93	728.06265

TABLE V.—*Osculating Elements of Polyhymnia for dates of Ephemerides*—Continued.

Greenwich midnight.		L_0	π	☊	i	φ	μ
		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
1886.	(1)	178 13 43.62					
	(2)	− 1 1 14.69	+ 0 20 29.50	−0 1 13.60	−0 0 23.58	− 0 17 6.38	− 4.74249
	(3)	− 1.11	+ 2.59			+ 5	− 58
Mar. 5	(4)	− 4.97	+ 1.44			− 2.94	− 1178
	(5)	+ 13 24.17	+ 13 24.17	+ 12 23.13	+ 7.35		
	(6)	+ 5.40	+ 5.40	+ 4.74	+ 0.08		
	(7)	177 25 52.42	342 54 10.60	9 13 59.17	1 56 3.85	19 35 44.93	728.06265
		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
1886.	(1)	178 25 56.44					
	(2)	− 1 1 14.69	+ 0 20 29.50	−0 1 13.60	−0 0 23.58	− 0 17 6.38	− 4.74249
	(3)	− 1.11	+ 2.59			+ 5	− 58
Mar. 6	(4)	− 4.97	+ 1.44			− 2.94	− 1178
	(5)	+ 13 24.17	+ 13 24.17	+ 12 23.13	+ 7.35		
	(6)	+ 5.49	+ 5.49	+ 4.82	+ 0.08		
	(7)	177 38 5.33	342 54 10.69	9 13 59.25	1 56 3.85	19 35 44.93	728.06265
		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
1887.	(1)	266 46 39.23					
	(2)	− 1 57 28.07	+ 0 41 58.63	−0 3 3.88	−0 1 18.51	− 0 23 53.04	− 4.18009
	(3)	− 1.67	+ 3.14			− 1	− 3
May 14	(4)	− 4.56	− 5.42			− 0.80	− 1662
	(5)	+ 14 14.44	+ 14 14.44	+ 13 9.19	+ 7.81		
	(6)	+ 6.81	+ 6.81	+ 5.42	+ 0.17		
	(7)	265 3 26.18	343 16 25.10	9 12 55.63	1 55 9.47	19 29 0.35	728.62076
		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
1888.	(1)	11 24 27.43					
	(2)	− 2 28 11.08	+ 0 49 16.29	−0 5 24.72	−0 1 22.17	− 0 24 22.82	− 3.45340
	(3)	− 1.60	+ 3.27			+ 3	+ 4
Oct. 9	(4)	− 4.96	+ 8.08			+ 2.28	− 2504
	(5)	+ 15 4.70	+ 15 4.70	+ 13 55.73	+ 8.27		
	(6)	+ 22.60	+ 22.60	+ 19.67	+ 0.36		
	(7)	9 11 37.09	343 25 2.44	9 11 35.58	1 55 6.46	19 28 33.69	729.33910
		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
1888.	(1)	11 36 40.25					
	(2)	− 2 28 11.08	+ 0 49 16.29	−0 5 24.72	−0 1 22.17	− 0 24 22.82	− 3.45340
	(3)	− 1.60	+ 3.27			+ 3	+ 4
Oct. 10	(4)	− 4.96	+ 8.08			+ 2.28	− 2504
	(5)	+ 15 4.70	+ 15 4.70	+ 13 55.73	+ 8.27		
	(6)	+ 22.70	+ 22.70	+ 19.76	+ 0.36		
	(7)	9 23 50.01	343 25 2.54	9 11 35.67	1 55 6.46	19 28 33.69	729.33910
		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
1888.	(1)	15 4 18.14					
	(2)	− 2 29 32.64	+ 0 49 20.64	−0 5 25.99	−0 1 22.25	− 0 24 19.60	− 3.47848
	(3)	− 1.60	+ 3.26			+ 3	+ 5
Oct. 27	(4)	− 4.67	+ 8.76			+ 1.99	− 2413
	(5)	+ 15 4.70	+ 15 4.70	+ 13 55.73	+ 8.27		
	(6)	+ 24.48	+ 24.48	+ 21.36	+ 0.38		
	(7)	12 50 8.41	343 25 9.34	9 11 36.00	1 55 6.40	19 28 36.62	729.31494
		° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
1888.	(1)	15 16 30.96					
	(2)	− 2 29 32.64	+ 0 49 20.64	−0 5 25.99	−0 1 22.25	−0 24 19.60	− 3.47848
	(3)	− 1.60	+ 3.26			+ 3	+ 5
Oct. 28	(4)	− 4.67	+ 8.76		8	+ 1.99	− 2413
	(5)	+ 15 4.70	+ 15 4.70	+ 13 55.73	+ 8.27		
	(6)	+ 24.59	+ 24.59	+ 21.46	+ 0.38		
	(7)	13 2 21.34	343 25 9.45	9 11 36.10	1 55 6.40	19 28 36.62	729.31494

TABLE V.—*Osculating Elements of Polyhymnia for dates of Ephemerides*—Continued.

Greenwich midnight.			L_0		π	Ω		i		φ		μ
1888.			° ′ ″		° ′ ″	° ′ ″		° ′ ″		° ′ ″		
	(1)		17 18 39.13									
	(2)	−	2 29 57.67	+	0 49 21.95	−0	5 26.46	−0	1 22.27	−	0 24 18.65	− 3.48587
	(3)	−	1.60	+	3.25					+	3	+ 6
Nov. 7	(4)	−	4.33	+	9.32					+	1.69	− 2294
	(5)	+	15 4.70	+	15 4.70	+	13 55.75	+	8.27			
	(6)	+	25.84	+	25.84		22.61	+	0.40			
	(7)		15 4 6.08		343 25 12.56		9 11 36.78		1 55 6.40		19 28 37.24	729.30875
1888.			° ′ ″		° ′ ″	° ′ ″		° ′ ″		° ′ ″		
	(1)		17 30 51.95									
	(2)	−	2 29 57.67	+	0 49 21.95	−0	5 26.46	−0	1 22.27	−	0 24 18.65	− 3.48587
	(3)	−	1.60	+	3.25					+	3	+ 6
Nov. 8	(4)	−	4.33	+	9.32					+	1.66	− 2294
	(5)	+	15 4.70	+	15 4.70	+	13 55.73	+	8.27			
	(6)	+	25.98	+	25.98		22.74	+	0.40			
	(7)		15 16 19.03		343 25 12.70		9 11 36.91		1 55 6.40		19 28 37.24	729.30875

TABLE VI.—*Geocentric Places from Elements V.*

Greenwich Mean Time.	Right Ascension.	Declination.	Greenwich Mean Time.	Right Ascension.	Declination.
	h. m. s.	° ′ ″		h. m. s.	° ′ ″
1854, Nov. 8.0	2 24 24.73	+16 23 0.0	1859, Nov. 15.0	4 13 25.61	+23 47 43.6
10.0	2 22 41.84	16 16 3.6	19.0	4 9 13.45	23 39 42.5
12.0	2 21 2.86	16 9 14.8	23.0	4 4 58.24	23 30 35.1
14.0	2 19 28.33	16 2 36.5	27.0	4 0 45.85	23 20 35.8
23.0	2 13 30.94	15 36 17.6	Dec. 1.0	3 56 42.01	23 10 2.0
24.0	2 12 59.09	15 33 50.2	5.0	3 52 51.84	22 59 12.0
25.0	2 12 28.95	15 31 30.0	9.0	3 49 19.68	22 48 24.8
Dec. 10.0	2 8 32.00	15 11 37.3	13.0	3 46 9.02	+22 37 58.1
12.0	2 8 31.13	15 11 16.4	1861, Feb. 6.5	9 16 40.31	+18 3 16.2
14.0	2 8 37.39	15 11 28.6	7.0	9 16 14.61	18 5 4.0
16.0	2 8 50.56	+15 12 13.4	7.5	9 15 48.97	18 6 51.5
1855, Jan. 9.0	2 19 48.02	+16 0 40.3	8.0	9 15 23.35	18 8 38.3
11.0	2 21 18.97	16 7 29.3	8.5	9 14 57.73	+18 10 24.6
13.0	2 22 54.70	16 14 38.9	1862, Mar. 26.0	12 1 49.91	+ 0 10 16.2
15.0	2 24 34.57	+16 22 5.8	27.0	12 1 3.99	0 14 54.2
1856, Jan. 5.0	9 8 7.29	+18 54 42.3	28.0	12 0 18.21	0 19 31.2
6.0	9 7 25.79	18 57 54.7	29.0	11 59 32.62	0 24 6.9
7.0	9 6 43.32	19 1 8.2	30.0	11 58 47.25	+ 0 28 41.0
30.0	8 47 27.75	20 19 48.1	1864, Nov. 30.0	5 19 41.49	+25 59 41.4
31.0	8 46 34.04	20 23 2.2	Dec. 2.0	5 17 31.46	25 58 24.2
Feb. 1.0	8 45 40.46	+20 26 13.8	4.0	5 15 20.20	25 56 50.9
1857, Mar. 17.0	11 39 11.18	+ 2 58 14.8	6.0	5 13 8.41	25 55 1.9
19.0	11 37 39.26	3 7 29.5	23.0	4 55 36.20	25 31 21.8
21.0	11 36 7.79	3 16 39.8	24.0	4 54 42.35	25 29 40.6
23.0	11 34 37.05	3 25 43.7	25.0	4 53 49.79	+25 27 58.6
Apr. 13.0	11 20 39.85	4 47 6.6	1868, June 17.0	17 39 19.71	−26 41 46.9
14.0	11 20 7.79	4 50 6.4	18.0	17 38 18.25	26 41 40.6
15.0	11 19 36.63	4 53 0.3	19.0	17 37 16.72	26 41 31.2
16.0	11 19 6.37	+ 4 55 48.3	20.0	17 36 15.02	−26 41 18.3
1858, June 6.0	15 3 41.88	−19 39 49.4	1869, Dec. 8.0	6 31 35.14	+26 8 10.9
6.5	15 3 18.68	19 38 17.1	9.0	6 30 36.19	26 9 19.2
7.0	15 2 55.71	−19 36 45.1	10.0	6 29 36.35	+26 10 24.0

Table VI.—*Geocentric Places from Elements V*—**Continued.**

Greenwich Mean Time.	Right Ascension.	Declination.
	h. m. s.	° ′ ″
1873, July 16.0	19 49 27.68	−24 49 32.6
17.0	19 48 35.70	24 51 36.9
18.0	19 47 43.43	24 53 37.6
23.0	19 43 20.44	25 2 36.4
24.0	19 42 28.27	25 4 9.8
25.0	19 41 36.46	25 5 37.7
26.0	19 40 45.13	25 6 59.8
31.0	19 36 39.34	25 12 23.3
Aug. 1.0	19 35 53.00	25 13 9.4
2.0	19 35 7.82	−25 13 49.1
1875, Jan. 2.0	7 13 32.26	+25 13 31.8
4.0	7 11 28.44	25 16 55.0
6.0	7 9 24.66	+25 20 5.9
1877, Apr. 14.0	13 33 10.83	−10 33 34.6
15.0	13 32 20.60	10 29 10.0
16.0	13 31 30.30	−10 24 42.9
1878, Sept. 1.0	22 37 51.35	−10 58 15.3
2.0	22 37 10.94	11 0 23.7
3.0	22 36 30.72	11 2 28.2
11.0	22 31 27.03	11 15 59.2
12.0	22 30 52.58	11 17 10.8
13.0	22 30 19.16	−11 18 17.0
1880, Jan. 30.0	7 41 40.40	+24 0 17.6
31.0	7 40 48.58	24 1 52.4
Feb. 1.0	7 39 57.66	+24 3 23.2
1882, Apr. 21.5	14 12 52.05	−14 39 26.8
22.5	14 11 59.88	14 35 29.6
23.5	14 11 7.44	−14 31 29.4
1882, Apr. 24.5	14 10 14.86	−14 27 27.3
25.5	14 9 22.15	14 23 23.0
May 12.5	13 54 54.23	13 12 57.1
13.5	13 54 7.48	13 8 59.0
14.5	13 53 21.50	13 5 3.9
15.5	13 52 36.32	13 1 11.9
16.5	13 51 51.97	−12 57 23.2
1883, Oct. 1.5	1 17 0.44	+ 8 32 32.4
2.5	1 16 14.73	8 29 45.2
3.5	1 15 28.43	8 26 52.4
31.5	0 55 44.44	7 7 49.1
Nov. 1.5	0 55 17.30	7 6 6.8
2.5	0 54 51.74	+ 7 4 32.5
1885, Jan. 26.5	8 28 51.18	+21 33 52.5
28.5	8 27 1.38	21 39 38.7
30.5	8 25 12.66	21 45 12.2
Feb. 1.5	8 23 25.41	+21 50 32.0
1886, Mar. 4.5	11 34 18.64	+ 3 41 22.1
5.5	11 33 33.85	3 45 54.8
6.5	11 32 48.71	+ 3 50 29.8
1887, May 14.5	14 42 58.27	−17 40 5.3
1888, Oct. 9.5	3 4 27.46	+18 52 15.6
10.5	3 3 51.15	18 51 17.0
27.5	2 50 6.03	18 16 9.5
28.5	2 49 9.60	18 13 10.6
Nov. 7.5	2 39 42.75	17 40 21.0
8.5	2 38 47.94	+17 36 55.1

Table VII.—*Mean corrections to normal Ephemeris given by observations.*

Date.	Right Ascension.					Declination.				
	Corr. of first Epheme-ris.	Diff. of Epheme-rides.	Corr. of normal Epheme-ris of El. V.	No. of obs.	Wt.	Corr. of first Epheme-ris.	Diff. of Epheme-rides.	Corr. of normal Epheme-ris of El. V.	No. of obs.	Wt.
	″	″	″			″	″	″		
1854, Nov. 9	− 2.33	+ 2.75	+0.42	14	2.0	− 14.8	+ 12.9	−1.9	18	3.0
24	− 1 55	+ 2.64	+1.09	2	0.2	− 9.5	+ 11.9	+2.4	3	0.6
Dec. 14	+ 0.51	+ 0.50	+1.01	8	0.6	− 1.5	+ 0.5	−1.0	7	1.0
1855, Jan. 12	− 0.09	+ 0.32	+0.23	15	0.8	+ 1.3	+ 0.2	+1.5	13	2.0
1856, Jan. 6	− 0.12	− 0.08	−0.20	3	0.5	− 2.7	− 0.3	−3.0	3	0.6
31	− 0.48	0.00	−0.48	10	1.0	+ 3.7	− 0.8	+2.9	10	2.0
1857, Mar. 17	+ 0.03	− 0.08	−0.05	8	1.0	+ 2.9	+ 0.4	+3.3	6	1.0
Apr. 13	+ 0.43	− 0.03	+0.40	4	0.5	+ 9.3	+ 0.2	+9.5	4	0.6
1858, June 7	+29.59	−29.35	+0.23	4	1.0	−140.0	+142.0	+2.0	4	1.5
1859, Nov. 19.7	− 1.83	+ 1.81	−0.02	13	2.0	− 9.9	+ 3.7	−6.2	12	3.0
Dec. 3.1	+ 0.50	− 0.66	−0.16	4	0.7	− 0.2	− 3.2	−3.4	4	1.0
1861, Feb. 7	. .	. .	−0.50	2	0.5	. .	. .	+0.6	2	0.5
1862, Mar. 28	. .	. .	+0.24	2	0.4	. .	. .	+4.8	2	0.5

TABLE VII.—*Mean corrections to normal Ephemeris given by observations*—Continued.

Date.		Right Ascension.					Declination.				
		Corr. of first Epheme-ris.	Diff. of Epheme-rides.	Corr. of normal Epheme-ris of El. V.	No. of obs.	Wt.	Corr. of first Epheme-ris.	Diff. of Epheme-rides.	Corr. of normal Epheme-ris of El. V.	No. of obs.	Wt.
		″	″	″			″	″	″		
1864, Dec.	4	− 7.59	+ 6.90	−0.69	2	0.4	− 8.3	+ 1.7	− 6.6	2	0.5
	24	− 7.44	+ 6.84	−0.60	5	1.0	− 8.8	+ 5.7	− 3.1	4	1.0
1868, June	19	− 5.45	+ 5.45	0.00	12	4.0	+ 1.0	+ 1.4	+ 2.4	12	4.0
1869, Dec.	9	− 1.64	+ 0.96	−0.68	3	1.0	+ 2.6	− 5.3	− 2.7	3	1.0
1873, July	17	− 8.82	+ 9.18	+0.36	6	3.0	− 25.1	+ 26.6	+ 1.5	6	3.0
	23	− 8.94	+ 9.35	+0.41	13	6.0	− 23.3	+ 24.8	+ 1.5	13	6.0
Aug.	31	− 8.87	+ 9.24	+0.37	2	1.0	− 22.1	+ 21.9	− 0.2	2	1.0
1875, Jan.	2	+ 1.08	− 1.16	−0.08	2	1.0	+ 8.1	− 0.5	+ 7.6	2	1.0
1877, Apr.	15	− 3.54	+ 3.89	+0.35	2	1.0	+ 20.4	− 19.5	+ 0.9	2	1.0
1878, Sept.	1	−16.45	+17.40	+0.95	4	2.0	− 97.8	+108.1	+10.3	4	2.0
	12	−16.37	+16.95	+0.58	3	1.5	−101.3	+102.2	+ 0.9	4	2.0
1880, Jan.	31	+ 1.28	− 1.36	−0.08	4	3.0	− 4.1	+ 3.3	− 0.8	4	3.0
1882, May	12			+0.55	3	1.0	. .	. .	+ 1.2	3	1.0
1883, Oct.	2	+ 1.73	− 1.20	+0.53	5	3.0	+ 8.3	− 8.4	− 0.1	5	2.0
Nov.	1	+ 1.67	− 0.94	+0.73	14	6.0	+ 7.0	− 7.3	− 0.3	11	4.0
1885, Feb.	1	+ 1.11	− 1.14	−0.03	9	4.0	− 5.3	+ 3.3	− 2.0	8	3.0
1886, Mar.	4	+ 0.80	− 0.48	+0.32	10	5.0	+ 0.3	+ 2.8	+ 3.1	10	5.0
1887, May	14	+ 0.48	− 0.09	+0.39	1	1.0	− 1.0	+ 0.5	− 0.5	1	1.0
1888, Oct.	9	+ 1.45	− 1.90	−0.45	8	4.0	+ 2.0	− 9.6	− 7.6	8	4.0
	27	+ 1.49	− 1.90	−0.41	22	8.0	+ 4.9	− 9.4	− 4.5	22	8.0
Nov.	8	+ 1.55	− 1.79	−0.24	17	6.0	+ 3.6	− 9.4	− 5.8	17	6.0

TABLE VIII.—*Equations of Condition; First Form.*

(α) In Right Ascension.

									″
1854, Nov.	9.5	2.783dM	−19012$d\mu$	+3.37$d\varphi$	+1.697$d\pi$	+0.0912 Sin $i d\Omega$	−0.426di	−33.719dm	=+ 6.0
Nov.	24.5	2.535	−17323	+3.09	+1.559	+ .0716	− .450	−30.646	+15.7
Dec.	14.5	2.128	−14539	+2.64	+1.327	+ .0335	− .439	−25.596	+14.6
1855, Jan.	12.5	1.612	−10982	+2.22	+1.067	− .0291	− .382	−18.953	+ 3.3
1856, Jan.	6.5	0.764	− 4860	+1.21	+1.267	− .0448	+ .338	− 6.062	− 2.8
Jan.	31.5	0.808	− 5158	+1.27	+1.326	− .0573	+ .302	− 6.416	− 6.8
1857, Mar.	17.0	0.665	− 3970	−0.52	+1.228	+ .2993	+ .059	− 4.175	− 0.7
Apr.	13.5	0.626	− 3754	−0.48	+1.165	+ .2849	+ .027	− 3.912	+ 6.0
1858, June	6.5	1.389	− 7714	−2.82	+1.433	− .0779	− .392	−13.957	+ 3.2
1859, Nov.	19.0	2.148	−10700	+3.54	+1.639	− .2023	− .299	−17.373	− 0.3
Dec.	2.6	2.134	−10638	+3.49	+1.630	− .2031	− .343	−17.278	− 2.3

TABLE VIII.—*Equations of Condition; First Form*—Continued.

(α) In Right Ascension—*Continued.*

1861,	Feb.	7.5	0.749 dM	− 3406 $d\mu$	+0.98 $d\varphi$	+1.295 $d\pi$	+ .0237 Sin $i d\Omega$	+ .303 di	− 2.680 dm	=	− 7.2″
1862,	Mar.	27.5	0.684	− 2831	−0.72	+1.231	+0.3069	− .013	− 2.129		+ 3.3
1864,	Dec.	4.0	1.778	− 5582	+3.32	+1.593	− .3257	− .137	− 0.511		− 9.3
	Dec.	24.0	1.708	− 5375	+3.14	+1.529	− .3139	− .193	− 0.529		− 8.1
1868,	June	18.5	2.449	− 4550	−3.79	+1.760	− .3899	− .082	+ 1.094		0.0
1869,	Dec.	9.5	1.384	− 1789	+2.82	+1.495	− .3368	+ .077	+ 1.045		− 9.1
1873,	July	17.0	3.518	+ 1	−3.39	+1.936	− .2826	+ .344	+ 0.000		+ 4.9
	July	23.5	3.544	+ 0	−3.40	+1.942	− .2773	+ .321	+ 0.000		+ 5.5
	Aug.	1.5	3.502	+ 0	−3.34	+1.909	− .2650	+ .286	+ 0.000		+ 5.0
1875,	Jan.	2.5	1.150	+ 621	+2.34	+1.450	− .2792	+ .183	+ 0.075		− 1.1
1877,	Apr.	15.5	0.951	+ 1213	−1.66	+1.308	+ .2187	− .237	− 0.255		+ 5.1
1878,	Sept.	1.0	4.232	+ 7918	−0.39	+1.963	+ .3471	+ .330	− 4.214		+10.4
	Sept.	12.5	4.129	+ 7729	−0.29	+1.912	+ .3581	+ .251	− 3.921		+ 8.6
1880,	Jan.	30.5	0.967	+ 2288	+1.85	+1.370	− .1935	+ .221	+ 0.159		− 1.1
1882,	May	9.5	1.081	+ 3453	−2.04	+1.339	+ .1358	− .327	+ 0.073		+ 8.1
1883,	Oct.	2.5	3.562	+13319	+2.61	+1.824	+ .3798	− .203	+ 3.093		+ 7.9
	Nov.	1.5	3.190	+11934	+2.45	+1.660	+ .3252	− .343	+ 2.138		+10.8
1885,	Feb.	1.5	0.867	+ 3646	+1.51	+1.343	− .1050	+ .280	+ 1.995		− 0.4
1886,	Mar.	4.5	0.651	+ 3013	−0.32	+1.218	+ .2894	+ .101	+ 1.360		+ 4.8
1887,	May	14.5	1.212	+ 6108	−2.48	+1.418	+ .2632	− .351	+ 4.244		+ 5.6
1888,	Oct.	9.5	2.687	+15031	+3.31	+1.630	+ .0737	− .296	+30.770		− 6.4
	Oct.	27.5	2.825	+15786	+3.47	+1.733	+ .0580	− .377	+32.244		− 5.8
	Nov.	8.5	2.774	+15506	+3.40	+1.711	+ .0493	−0.420	+31.616		− 3.4

(β) In Declination.

1854,	Nov.	9.5	+1.032 dM	−7045. $d\mu$	+1.28 $d\varphi$	+0.638 $d\pi$	−1.408 Sin $i d\Omega$	+1.139 di	−12.413 dm	=	− 1.9″
	Nov.	24.5	+0.980	−6686.	+1.23	+0.612	−1.226	+1.160	−11.739		+ 2.4
	Dec.	14.5	+0.838	−5720.	+1.08	+0.532	−0.966	+1.118	−10.003		− 1.0
1855,	Jan.	12.5	+0.608	−4140.	+ .86	+0.410	−0.653	+1.017	− 7.107		+ 1.5
1856,	Jan.	6.5	−0.252	+1608.	− .41	−0.413	+0.835	+1.041	+ 1.937		− 3.0
	Jan.	31.5	−0.238	+1523.	− .38	−0.384	+0.914	+1.036	+ 1.854		+ 2.9
1857,	Mar.	17.0	−0.313	+1871.	+ .24	−0.579	+1.352	+0.126	+ 1.898		+ 3.3
	Apr.	13.5	−0.292	+1748.	+ .23	−0.542	+1.300	+0.058	+ 1.764		+ 9.5
1858,	June	6.5	−0.456	+2527.	+ .91	−0.460	+0.875	−1.234	+ 4.605		+ 2.0
1859,	Nov.	19.0	+0.433	−2152.	+ .74	+0.340	−0.877	+1.432	− 3.432		− 6.2
	Dec.	2.6	+0.484	−2404.	+ .82	+0.382	−0.791	+1.473	− 3.832		− 3.4

Table VIII.—*Equations of Condition; First Form*—Continued.

(*b*) In Declination—*Continued.*

										″
1861,	Feb.	7.5	$-0.254dM$	$+1156.\ d\mu$	$-.34d\varphi$	$-0.435d\pi$	$+1.033\ \text{Sin}\ id\Omega$	$+0.897di$	$+0.856dm=$	$+0.6$
1862,	Mar.	27.5	−0.317	+1343.	+ .34	−0.584	+1.362	−0.027	+ 0.981	+ 4.8
1864,	Dec.	4.0	+0.144	− 448.	+ .29	+0.140	−0.402	+1.551	− 0.032	− 6.6
	Dec.	24.0	+0.209	− 651.	+ .40	+0.197	−0.283	+1.520	− 0.041	− 3.1
1868,	June	18.5	−0.135	+ 241.	+ .17	−0.083	+0.152	+1.755	− 0.079	+ 2.4
1869,	Dec.	9.5	−0.082	+ 111.	− .16	−0.079	+0.010	+1.504	− 0.065	− 2.7
1873,	July	17.0	+0.660	+ 12.	− .71	+0.376	−0.873	−1.767	0	+ 1.5
	July	23.5	+0.623	+ 12.	− .68	+0.354	−0.946	−1.741	0	+ 1.5
	Aug.	1.5	+0.563	+ 11.	− .61	+0.317	−1.033	−1.672	0	− 0.2
1875,	Jan.	2.5	−0.157	− 80.	− .32	−0.189	+0.405	+1.404	− 0.009	+ 7.6
1877,	Apr.	15.5	−0.345	− 527.	+ .72	−0.565	+1.315	−0.548	+ 0.115	+ 0.9
1878,	Sept.	1.0	+1.840	+3434.	− .23	+0.855	−2.003	−0.757	− 1.849	+10.3
	Sept.	12.5	+1.763	+3295.	− .17	+0.817	−2.022	−0.585	− 1.773	+ 0.9
1880,	Jan.	30.5	−0.178	− 418.	− .34	−0.247	+0.689	+1.220	− 0.019	− 0.8
1882,	May	9.5	+0.435	+1393.	− .84	+0.544	+1.199	−0.806	+ 0.034	+ 1.2
1883,	Oct.	2.5	+1.590	+5943.	+1.16	+0.814	+1.959	+0.457	+ 1.398	− 0.1
	Nov.	1.5	+1.467	+5488.	+1.16	+0.768	−1.694	+0.746	+ 1.313	− 0.3
1885,	Feb.	1.5	−0.223	− 934.	− .39	−0.341	+0.852	+1.103	− 0.493	− 2.0
1886,	Mar.	4.5	−0.306	−1413.	+ .14	−0.572	+1.328	+0.215	− 0.625	+ 3.1
1887,	May	14.5	−0.427	−2157.	+ .87	−0.493	+1.106	−1.013	− 1.570	− 0.5
1888,	Oct.	9.5	+0.841	+4707.	+1.05	+0.512	−1.512	+0.938	+ 9.692	− 7.6
	Oct.	27.5	+0.939	+5249.	+1.18	+0.581	−1.460	+1.119	+10.816	− 4.5
	Nov.	8.5	+0.968	+5416.	+1.22	+0.605	−1.366	+1.190	+11.148	− 5.8

The preceding equations are not in the form best adapted for the solution by least squares. The left-hand members have therefore been reduced so as to replace the unknown corrections to the elements by the following system of unknown quantities.

$$x=\delta L_0=\delta M_0+\delta\pi$$

$$y=5000\,\delta\mu-5\frac{\delta m}{m^2}$$

$$z=\delta\varphi$$

$$u=\frac{1}{2}\delta\pi$$

$$v=\sin i\ \delta\Omega$$

$$w=\delta i$$

$$t=-5\frac{\delta m}{m^2}$$

TABLE IX.—*Equations of Condition; Final Form.*

(α) In Right Ascension.

									″	Wt.
1854, Nov.	9.5	2.78x	−3.80y	+3.37z	−2.17u	+0.09v	−0.43w	−2.94t =	+ 6.0	2.0
Nov.	24.5	2.54	−3.46	+3.09	−1.95	+0.07	−0.45	−2.66	+15.7	0.2
Dec.	14.5	2.13	−2.90	+2.64	−1.60	+0.03	−0.44	−2.21	+14.6	0.6
1855, Jan.	12.5	1.61	−2.20	+2.22	−1.09	−0.03	−0.38	−1.59	+ 3.3	0.8
1856, Jan.	6.5	0.76	−0.97	+1.21	+1.01	−0.04	+0.34	−0.24	− 2.8	0.5
Jan.	31.5	0.81	−1.03	+1.27	+1.04	−0.06	+0.30	−0.25	− 6.8	1.0
1857, Mar.	17.0	0.66	−0.79	−0.52	+1.13	+0.30	+0.06	−0.04	− 0.7	1.0
Apr.	13.5	0.63	−0.75	−0.48	+1.08	+0.28	+0.03	−0.03	+ 6.0	0.5
1858, June	6.5	1.39	−1.54	−2.82	+0.09	−0.08	−0.39	−1.25	+ 3.2	1.0
1859, Nov.	19.0	2.15	−2.14	+3.54	−1.02	−0.20	−0.30	−1.33	− 0.3	2.0
Dec.	2.6	2.13	−2.13	+3.49	−1.01	−0.20	−0.34	−1.33	− 2.3	0.7
1861, Feb.	7.5	0.75	−0.68	+0.98	+1.09	+0.02	+0.30	+0.15	− 7.2	0.5
1862, Mar.	27.5	0.68	−0.57	−0.72	+1.09	+0.31	-0.01	+0.14	+ 3.3	0.4
1864, Dec.	4.0	1.78	−1.12	+3.32	−0.37	−0.33	−0.14	+1.01	− 9.3	0.4
Dec.	24.0	1.71	−1.08	+3.14	−0.36	−0.31	−0.19	+0.97	− 8.1	1.0
1868, June	18.5	2.45	−0.91	−3.79	−1.38	−0.39	−0.08	+1.13	0.0	4.0
1869, Dec.	9.5	1.38	−0.36	+2.82	+0.22	−0.34	+0.08	+0.57	− 9.1	1.0
1873, July	17.0	3.52	0	−3.39	−3.16	−0.28	+0.34	0	+ 4.9	3.0
July	23.5	3.54	0	−3.40	−3.20	−0.28	+0.32	0	+ 5.5	6.0
Aug.	1.5	3.50	0	−3.34	−3.19	−0.26	+0.29	0	+ 5.0	1.0
1875, Jan.	2.5	1.15	+0.12	+2.34	+0.60	−0.28	+0.18	−0.11	− 1.1	1.0
1877, Apr.	15.5	0.95	+0.24	−1.66	+0.71	+0.22	−0.24	−0.29	+ 5.1	1.0
1878, Sept.	1.0	4.23	+1.58	−0.39	−4.54	+0.35	+0.33	−2.43	+14.0	2.0
Sept.	12.5	4.13	+1.55	−0.29	−4.43	+0.36	+0.25	−2.33	+ 8.6	1.5
1880, Jan.	30.5	0.97	+0.46	+1.85	+0.80	−0.19	+0.22	−0.42	− 1.1	3.0
1882, May	9.5	1.08	+0.69	−2.04	+0.52	+0.14	−0.33	−0.68	+ 8.1	1.0
1883, Oct.	2.5	3.56	+2.66	+2.61	−3.48	+0.38	−0.20	−2.05	+ 7.9	3.0
Nov.	1.5	3.19	+2.39	+2.45	−3.06	+0.33	−0.34	−1.96	+10.8	6.0
1885, Feb.	1.5	0.87	+0.73	+1.51	+0.95	−0.10	+0.28	−0.33	− 0.4	4.0
1886, Mar.	4.5	0.65	+0.60	−0.32	+1.13	+0.29	+0.10	−0.33	+ 4.8	5.0
1887, May	14.5	1.21	+1.22	−2.48	+0.41	+0.26	−0.35	−0.37	+ 5.6	1.0
1888, Oct.	9.5	2.69	+3.00	+3.31	−2.11	+0.07	−0.30	+3.15	− 6.4	4.0
Oct.	27.5	2.82	+3.16	+3.47	−2.18	+0.06	−0.38	+3.29	− 5.8	8.0
Nov.	8.5	2.77	+3.10	+3.40	−2.12	+0.05	−0.42	+3.22	− 3.4	6.0

(β) In Declination.

									″	Wt.
1854, Nov.	9.5	+1.03x	−1.41y	+1.28z	−0.79u	−1.41v	+1.14w	−1.07t =	− 1.9	3.0
Nov.	24.5	+0.98	−1.34	+1.23	−0.74	−1.23	+1.16	−1.01	+ 2.4	0.6
Dec.	14.5	+0.84	−1.14	+1.08	−0.61	−0.97	+1.12	−0.86	− 1.0	1.0
1855, Jan.	12.5	+0.61	−0.83	+0.86	−0.40	−0.65	+1.02	−0.59	+ 1.5	2.0

TABLE IX.—*Equations of Condition; Final Form*—Continued.

(β) In Declination—*Continued.*

											Wt.
										$''$	
1856,	Jan.	6.5	$-0.25x$	$+0.32y$	$-0.41z$	$-0.32u$	$+0.84v$	$+1.04w$	$+0.07t =$	-3.0	0.6
	Jan.	31.5	-0.24	$+0.30$	-0.38	-0.29	$+0.91$	$+1.04$	$+0.07$	$+2.9$	2.0
1857,	Mar.	17.0	-0.31	$+0.37$	$+0.24$	-0.53	$+1.35$	$+0.13$	0	$+3.3$	1.0
	Apr.	13.5	-0.29	$+0.35$	$+0.23$	-0.50	$+1.30$	$+0.06$	0	$+9.5$	0.6
1858,	June	6.5	-0.46	$+0.51$	$+0.91$	-0.01	$+0.88$	-1.23	$+0.42$	$+2.0$	1.5
1859,	Nov.	19.0	$+0.43$	-0.43	$+0.74$	-0.18	-0.88	$+1.43$	-0.26	-6.2	3.0
	Dec.	2.6	$+0.48$	-0.48	$+0.82$	-0.20	-0.79	$+1.47$	-0.29	-3.4	1.0
1861,	Feb.	7.5	-0.25	$+0.23$	-0.34	-0.36	$+1.03$	$+0.90$	-0.06	$+0.6$	0.5
1862,	Mar.	27.5	-0.32	$+0.27$	$+0.34$	-0.53	$+1.36$	-0.03	-0.07	$+4.8$	0.5
1864,	Dec.	4.0	$+0.14$	-0.09	$+0.29$	-0.01	-0.40	$+1.55$	$+0.08$	-6.6	0.5
	Dec.	24.0	$+0.21$	-0.13	$+0.40$	-0.02	-0.28	$+1.52$	$+0.12$	-3.1	1.0
1868,	June	18.5	-0.14	$+0.05$	$+0.17$	$+0.10$	$+0.15$	-1.76	-0.06	$+2.4$	4.0
1869,	Dec.	9.5	-0.08	$+0.02$	-0.16	$+0.01$	$+0.01$	$+1.50$	-0.04	-2.7	1.0
1873,	July	17.0	$+0.66$	0	-0.71	-0.57	-0.87	-1.77	0	$+1.5$	3.0
	July	23.5	$+0.62$	0	-0.68	-0.54	-0.95	-1.74	0	$+1.5$	6.0
	Aug.	1.5	$+0.56$	0	-0.61	-0.49	-1.03	-1.67	0	-0.2	1.0
1875,	Jan.	2.5	-0.16	-0.02	-0.32	-0.06	$+0.40$	$+1.40$	$+0.01$	$+7.6$	1.0
1877,	Apr.	15.5	-0.34	-0.11	$+0.72$	-0.44	$+1.32$	-0.55	$+0.13$	$+0.9$	1.0
1878,	Sept.	1.0*	$+1.84$	$+0.69$	-0.23	-1.97	-2.00	-0.76	-1.06	$+10.3$	[2.0]
	Sept.	12.5	$+1.76$	$+0.66$	-0.17	-1.89	-2.02	-0.58	-1.01	$+0.9$	2.0
1880,	Jan.	30.5	-0.18	-0.08	-0.34	-0.14	$+0.69$	$+1.22$	$+0.08$	-0.8	3.0
1882,	May	9.5	$+0.44$	$+0.28$	-0.84	$+0.22$	$+1.20$	-0.81	-0.27	$+1.2$	1.0
1883,	Oct.	2.5	$+1.59$	$+1.19$	$+1.16$	-1.55	-1.96	$+0.46$	-0.91	-0.1	2.0
	Nov.	1.5	$+1.47$	$+1.10$	$+1.16$	-1.40	-1.69	$+0.75$	-0.84	-0.3	4.0
1885,	Feb.	1.5	-0.22	-0.19	-0.39	-0.24	$+0.85$	$+1.10$	$+0.09$	-2.0	3.0
1886,	Mar.	4.5	-0.31	-0.28	$+0.14$	-0.53	$+1.33$	$+0.22$	$+0.16$	$+3.1$	5.0
1887,	May	14.5	-0.43	-0.43	$+0.87$	-0.13	$+1.11$	-1.01	$+0.12$	-0.5	1.0
1888,	Oct.	9.5	$+0.84$	$+0.94$	$+1.05$	-0.66	-1.51	$+0.94$	$+1.00$	-7.6	4.0
	Oct.	27.5	$+0.94$	$+1.05$	$+1.18$	-0.72	-1.40	$+1.12$	$+1.11$	-4.5	8.0
	Nov.	8.5	$+0.97$	$+1.08$	$+1.22$	-0.73	-1.37	$+1.19$	$+1.15$	-5.8	6.0

*This equation has been omitted in forming the normal equations, owing to its discordance, which leads to a suspicion of abnormal error.